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A REMARK TO THE SCHAUDER FIXED POINT THEOREM

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Dedicated to the memory of J. Schauder on the occasion of the centenary of his birth

Abstract. In the paper some sufficient conditions are established in order that a continuous map have a fixed point. The results are related to those obtained by R. D. Nussbaum in [18], L. Górniewicz and D. Rozpłoch-Nowakowska in [12], S. Szufla in [21] and D. Bugajewski in [6].

The famous Schauder Fixed Point Theorem [20] has been generalized in various directions by using different methods. For references, see [4]–[6], [9], [10], [12], [18], [19], [21] and [24]. Closely related with a generalization of that theorem is a long-standing conjecture in the fixed point theory which was formulated by R. D. Nussbaum in 1972 in [18] and which reads as follows:

Let M be a closed, bounded convex set in a Banach space and $T : M \to M$ a continuous map. Assume that there exists an integer $n \geqq 1$ such that T^n is compact. Then T has a fixed point.

R. D. Nussbaum proved this conjecture with the additional assumption that T restricted to an appropriate open set is continuously Fréchet differentiable. Using algebraic topology methods, especially the generalized Lefschetz number, he proved a series of asymptotic fixed point theorems, that is, theorems in which

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the existence of fixed points of a map T is established with the aid of assumptions on the iterates T^k of T. Similarly F. E. Browder in [5] with the help of the Lefschetz Fixed Point Theorem for compact absolute neighbourhood retracts proved asymptotic fixed point theorems for mappings defined on subsets of absolute neighbourhood retracts. He proved the above conjecture under assumption that M is a compact absolute neighbourhood retract and $T^n(M)$ is homologically trivial in M. K. Deimling in [9, p. 214] recalled the above conjecture.

In [12] L. Górniewicz and D. Rozpłoch-Nowakowska using again algebraic topology methods proved the Schauder Fixed Point Theorem for several classes of mappings in a metric space X. They also formulated an open problem to prove these fixed point theorems in an elementary way, i.e. without using algebraic topology. The above conjecture was proved by adding assumption that T is locally compact.

In the paper several sufficient conditions have been established that a continuous self-map of a convex, closed set in a Banach space have a fixed point. Their proof is based solely on the Schauder Fixed Point Theorem and is of an elementary character. The above mentioned conjecture is proved under assumption that T is proper. We recall that T is proper when the preimage of each compact set is compact. In all these cases an approach is applied which is similar to that used by H. Mőnch (see [9, pp. 204–205]), S. Szufla in [21] and D. Bugajewski in [6].

First we introduce the following notations. Let $(X, \|\cdot\|)$ be a Banach space, M be a nonempty subset of X and let $T\colon M \to M$ be a continuous mapping. For $x \in M$ let

$$\gamma^+(x)\colon = \{T^k(x)\colon 1, 2, \dots\}, \quad T^0(x)\colon = x$$

be the *positive semiorbit of* x and

$$\omega(x) := \{w \in X : \exists k_l \to \infty \text{ such that } T^{k_l}(x) \to w \text{ as } l \to \infty\}$$

the *ω-limit set of x*. A point $x \in M$ is a *k-periodic point of T* ($k \geqq 2$) if $T^k(x) = x$, and $T^l(x) \neq x$, $l = 1, \dots, k-1$. A set A is called a *k-cycle* (*of T*) if $A = \gamma^+(x)$ for some k-periodic point x of T. The set $C_T = \bigcap_{k=0}^{\infty} T^k(M)$ is called the *center of T* (for the definition, see [12]).

In the whole paper the set inclusion $\subset$ will mean $\subseteqq$, $\overline{A}$ and ∂A will mean the closure and the boundary of a set A, respectively. Further $\operatorname{int}(A)$ and $\overline{\operatorname{co}}(A)$ will denote the interior and the convex hull of the set A, respectively. If $A \subset B$ are two sets in X, then $\operatorname{int}_B(A)$ will mean the interior of A with respect to B. The dimension of X will be denoted by $\dim(X)$. $U(a,r)$ will mean the r-neighbourhood of the point a.

If $K \subset M$ is a compact set and $T(K) \subset K$, then by the Cantor theorem [17, p. 5], $\bigcap_{k=0}^{\infty} T^k(K)$ is a nonempty compact subset of C_T.

In our considerations the following definition will play an important role.

Definition 1. A couple (K_1, K) will be called *admissible* (*with respect to the mapping* T) if

(i) $\emptyset \neq K_1 \subset K \subset M$,
(ii) the set K_1 is compact, and
(iii) the set K is convex and closed, $T(K) \subset K$.

Throughout the whole paper the following assumption will be used

(H1) *M is a nonempty convex and closed subset of a Banach space X and $T\colon M \to M$ is a continuous mapping.*

Some properties of admissible couples are collected in

Lemma 1. *Let assumption* (H1) *be satisfied. Then the following statements hold:*

(i) *If (K_1, K) is an admissible couple and $T(K_1) \subset K_1$, then so is the couple (K_0, K) where $K_0 = \bigcap_{k=0}^{\infty} T^k(K_1)$ has the property $T(K_0) = K_0$ and hence, K_0 is a subset of C_T.*
(ii) *If (K_1, K) is an admissible couple, then there exists the least convex closed set K_2 such that (K_1, K_2) is admissible.*

Proof. (i) As it was already mentioned, K_0 is a nonempty compact set such that $T(K_0) \subset K_0 \subset K_1$. If $x \in K_0$ is an arbitrary element, then there exists $y_k \in T^k(K_1)$ such that $T(y_k) = x$ and by the compactness of K_1 there exists a subsequence $y_l \in T^l(K_1)$ which converges to $y \in K_0$ as $l \to \infty$. $T(y) = x$ and hence $x \in T(K_0)$.

(ii) Let

$$G = \{F \in 2^X : K_1 \subset F \subset M,\ F \text{ is convex, closed and } T(F) \subset F\}. \tag{1}$$

Let $K_2 = \bigcap_{F \in G} F$. Then K_2 is the least element of G in the sense of the set inclusion. □

Definition 2. The admissible couple (K_1, K_2) will be called *minimal* if K_2 is the least convex closed set containing K_1.

We shall need some properties of convex sets in a linear normed space. By Theorem 2, [2, p. 19] and Proposition 1.11, [7, p. 102], the following proposition holds.

Proposition 1. *Let K be a convex set of a normed space X. Then:*

(i) *The closure $\overline{K}$ of K and the interior* $\operatorname{int}(K)$ *of K are convex.*

(ii) *If* $\operatorname{int}(K) \neq \emptyset$*, then* $\overline{K} = \overline{\operatorname{int}(K)}$ *and* $\operatorname{int}(K) = \operatorname{int}(\overline{K})$*.*

(iii) *If* $x \in \operatorname{int}(K)$ *and* $y \in \overline{K}$*, then* $[x, y) := \{(1-t)x + ty : 0 \leqq t < 1\} \subset \operatorname{int}(K)$*.*

Proposition 1 is completed by

Lemma 2. *Suppose that* K *is a convex set of a normed space* X *such that* $\operatorname{int}(K) \neq \emptyset$*. Then for the boundary* $\partial(\operatorname{int}(K))$ *of the interior* $\operatorname{int}(K)$ *of* K *we have*

$$\partial(\operatorname{int}(K)) = \partial(K).$$

Proof. In the case of an arbitrary set K we have the inclusion $\partial(\operatorname{int}(K)) \subset \partial(K)$ (see [8, p. 65]). To prove the converse inclusion, let us consider a point $x \in \operatorname{int}(K)$ and an arbitrary point $y \in \partial(K)$. Then by statement (iii), Proposition 1, $[x, y) \subset \operatorname{int}(K)$ and hence $y \in \partial(\operatorname{int}(K))$. This completes the proof of the lemma. □

Further we shall use the following property of relatively compact subsets of a linear normed space X with $\dim(X) = \infty$. By Proposition 7.1, [9, p. 40] we have the following result.

Proposition 2. *Every relatively compact subset* K *of a linear normed space* X *with* $\dim(X) = \infty$ *has no interior points, that is,* $\operatorname{int}(K) = \emptyset$*.*

The following version of the Baire Category Theorem (Theorem 15.8.2 in [8, p. 100]) will be stated here as

Proposition 3. *Let* $P \neq \emptyset$ *be a metric space which is homeomorphic to a complete metric space. Let a set* A *be of first category in* P*. Then the set* $P \setminus A$ *is dense in* P*.*

By [8, pp. 113, 110] we have

Proposition 4. *Let* P *be a metric space. Then:*

(i) *If* P *is compact, then it is separable.*

(ii) *If* A_k*,* $k = 1, 2, \dots,$ *are separable subsets of* P *and* $\bigcup_{k=1}^{\infty} A_k = P$*, then* P *is separable.*

(iii) *If* A *is a separable subset of* P*, then the closure* $\overline{A}$ *of* A *is separable, too.*

The Alexandroff lemma ([1, p. 86]) is given here as

Proposition 5. *Every* G_δ *in a complete metric space is completely metrizable.*

Now we state the fundamental lemma.

LEMMA 3. *Suppose that assumption* (H1) *is satisfied. Let* (K_1, K_2) *be a minimal admissible couple. Then:*

(i)

$$K_2 = \overline{\bigcup_{k=1}^{\infty} S_k} \tag{2}$$

where $\{S_k\}_{k=1}^{\infty}$ *is a nondecreasing sequence of convex compact subsets of* K_2 *which are defined by the relations*

$$S_1 := \overline{\mathrm{co}}(K_1), \tag{3}$$

$$S_{k+1} := \overline{\mathrm{co}}(S_k \cup T(S_k)), \quad k = 1, 2, \ldots. \tag{4}$$

(ii) K_2 *is separable.*

(iii) *If* $\bigcup_{k=1}^{\infty} S_k$ *is not closed, then* $\overline{\bigcup_{k=1}^{\infty} S_k} \setminus \bigcup_{k=1}^{\infty} S_k$ *is a* G_δ *set.*

(iv) *If* $K_1 \subset T(K_1)$, *then* $\overline{\mathrm{co}}(T(K_2)) = K_2$.

PROOF. (i) By the Mazur Theorem [7, p. 180], S_1 is convex and compact. Since $S_1 \subset K_2$, we also have that the compact set $S_1 \cup T(S_1) \subset K_2$ and the set $S_2 = \overline{\mathrm{co}}(S_1 \cup T(S_1)) \subset K_2$ is convex and compact. By mathematical induction we get that the sequence $\{S_k\}_{k=1}^{\infty}$ which is defined by (3) and (4) is a nondecreasing sequence of convex compact subsets of K_2. Clearly $\bigcup_{k=1}^{\infty} S_k \subset K_2$ is convex and by Proposition 1, $\overline{\bigcup_{k=1}^{\infty} S_k}$ is a convex and closed subset of K_2. Further, $T(\bigcup_{k=1}^{\infty} S_k) \subset \bigcup_{k=1}^{\infty} S_k$ and, on the basis of the continuity of T, we have that $T(\overline{\bigcup_{k=1}^{\infty} S_k}) \subset \overline{\bigcup_{k=1}^{\infty} S_k}$. Hence $(K_1, \overline{\bigcup_{k=1}^{\infty} S_k})$ is an admissible couple and since $\overline{\bigcup_{k=1}^{\infty} S_k} \subset K_2$, equality (2) follows.

(ii) In view of Proposition 4, (2) implies that K_2 is separable.

(iii) If $\overline{\bigcup_{k=1}^{\infty} S_k} \setminus \bigcup_{k=1}^{\infty} S_k \neq \emptyset$, then $\overline{\bigcup_{k=1}^{\infty} S_k} \setminus \bigcup_{k=1}^{\infty} S_k = \bigcap_{k=1}^{\infty}(\overline{\bigcap_{l=1}^{\infty} S_l} \setminus S_k)$ is a G_δ set.

(iv) Denote $K_3 = \overline{\mathrm{co}}(T(K_2))$. As $T(K_3) \subset K_3$, $K_1 \subset K_3$, we have that (K_1, K_3) is an admissible couple and thus, the minimality of (K_1, K_2) implies that $K_3 = K_2$. $\square$

LEMMA 4. *Suppose that assumption* (H1) *is satisfied. Let* K_1, $\emptyset \neq K_1 \subset M$, *be a compact set. Then there exists an admissible couple* (K_1, K) *such that* K *satisfies*

$$\overline{\mathrm{co}}(T(K)) = K. \tag{5}$$

PROOF. Let a be the cardinal number of the set G given by (1). By the Cantor Theorem [13, p. 93], the cardinal number $2^a > a$. Let b be the initial ordinal number of power 2^a. Then we define a transfinite sequence $\{F_\alpha\}$ of the type b with values in G (see [11, pp. 18–19]) in the following way:

$$(6)\qquad \begin{aligned} F_0 &= M, \\ F_\alpha &= \begin{cases} \overline{\mathrm{co}}(T(F_{\alpha-1})) & \text{if } \alpha - 1 \text{ exists}, \\ \bigcap_{\beta<\alpha} F_\beta & \text{in the other case } (\alpha \text{ is a limit number}), \end{cases} \end{aligned}$$

for $\alpha > 0$. The sequence $\{F_\alpha\}$ is nonincreasing with respect to the set inclusion and we claim: There exists an ordinal number $\delta < b$ such that $F_\delta = F_{\delta+1}$ which on the basis of (6) means that $K = F_\delta$ satisfies (5).

If (5) were not true for any $K = F_\delta$, then the sequence $\{F_\alpha\}$ would be injective and the cardinal number of G would be greater or equal to 2^a which, on the basis of the Cantor Theorem, is a contradiction with the properties of cardinal numbers. □

By means of the last two lemmas we prove the following theorem which is similar to the results obtained by S. Szufla in [21] and D. Bugajewski in [6].

Theorem 1. *Suppose that assumption* (H1) *is satisfied. Let* K_1, $\emptyset \neq K_1 \subset M$ *be a compact set. Then each of the following statements is a sufficient condition for the existence of a fixed point of* T*:*

(i) *For each set* K *the implication holds:*

(7) *If* $K_1 \subset K \subset M$ *and* (5) *is true, then* K *is compact.*

(ii) $K_1 \subset T(K_1)$ *and for each set* K *the implication holds:*

(8) *If* $K_1 \subset K \subset M$, K *is separable and* (5) *is true, then* K *is compact.*

Proof. (i) By Lemma 4 there exists an admissible couple (K_1, K) such that (5) is true. Then (7) yields that K is compact. By the Schauder Fixed Point Theorem there is a $u \in K$ such that $u = T(u)$.

(ii) If instead of Lemma 4 we apply Lemma 3 and (7) is replaced by (8), again we come to a fixed point u of T in K. □

The next theorem gives a statement on the alternative.

Theorem 2. *Suppose that assumption* (H1) *is satisfied. Let* (K_1, K_2) *be a minimal admissible couple and let* K_2 *be bounded. Then either* K_2 *is compact and hence, there exists a fixed point of* T *in* K_2 *or* $\dim(X) = \infty$, K_2 *is separable and all* S_k, $k = 1, 2, \dots,$ *defined by* (3) *and* (4) *are such that*

$$(9)\qquad \mathrm{int}_{K_2}(S_k) = \emptyset \quad \textit{for } k \in N.$$

Hence the set

$$(10)\qquad K_3 = \bigcup_{k=1}^{\infty} S_k$$

is convex, dense and of first category in K_2*,* $T(K_3) \subset K_3$*,* $K_1 \subset K_3 \subset K_2$ *and* $K_2 \setminus K_3$ *is dense and a* G_δ *set in* K_2*.*

Proof. By Lemma 3, K_2 is separable and there exists a sequence $\{S_k\}_{k=1}^{\infty}$ determined by (3) and (4) such that (2) is true. As to that sequence, we have to distinguish the following three cases:

(i) There exists an $l \in N$ such that $S_{l+1} = S_l$ and hence, $T(S_l) \subset S_l$. Since S_l is a convex compact subset of a Banach space, by the Schauder Fixed Point Theorem there is a $u \in S_l$ such that $u = T(u)$.

(ii) For each $k \in N$ $S_k \subsetneqq S_{k+1}$. Consider the set K_3 determined by (10). Again we have two possibilities. The first one is that there exists an $l \in N$ such that the interior of S_l with respect to K_2

$$\operatorname{int}_{K_2}(S_l) \neq \emptyset. \tag{11}$$

Then there exists an $x_0 \in S_l$ and $r > 0$ such that the $r-$neighbourhood of x_0 in K_2 $U(x_0, r) \cap K_2 \subset S_l$. Therefore

$$(U(x_0, r) \cap K_2) - x_0 \subset S_l - x_0 \tag{12}$$

and by the boundedness of K_2 there exists an $n \in N$ such that

$$\frac{1}{n}(K_2 - x_0) \subset U(x_0, r) - x_0 = U(0, r). \tag{13}$$

Further

$$(U(x_0, r) \cap K_2) - x_0 = (U(x_0, r) - x_0) \cap (K_2 - x_0). \tag{14}$$

Since $K_2 - x_0$ is convex and contains 0, we have that

$$\frac{1}{n}(K_2 - x_0) \subset K_2 - x_0. \tag{15}$$

Thus (12)–(15) imply that

$$\frac{1}{n}(K_2 - x_0) \subset (U(x_0, r) - x_0) \cap (K_2 - x_0) = (U(x_0, r) \cap K_2) - x_0 \subset S_l - x_0.$$

As the set $S_l - x_0$ is compact and $K_2 - x_0$ is closed, K_2 is compact and again, we have a fixed point u of T in K_2.

(iii) $S_k \subsetneqq S_{k+1}$ and (9) is true. If $\dim(X) < \infty$, then again K_2 is compact. Consider the case $\dim(X) = \infty$. Then K_3 given by (10) is convex and dense in K_2 satisfying (2), $T(K_3) \subset K_3$, K_3 is of first category in K_2 and by Proposition 3 and Lemma 3, $K_2 \setminus K_3$ is dense as well as a G_δ set in K_2. The proof of the theorem is complete. $\square$

Corollary 1. *Suppose that assumption* (H1) *is satisfied. Let there exist an admissible couple* (K_1, K) *such that*

(i) $\mathrm{int}_K(K_1) \neq \emptyset$,

(ii) K *is bounded.*

Then there exists a fixed point of T *in* K.

Proof. Consider the minimal admissible couple (K_1, K_2). Since $K_1 \subset K_2 \subset K$, both conditions (i), (ii) are satisfied also for $K = K_2$. Then Theorem 2 gives the statement of the corollary. □

Remark 1. Corollary 1 says that a convex, closed and bounded set K has a fixed point property if it contains a compact subset K_1 with nonempty interior with respect to K.

Corollary 2. *Suppose that assumption* (H1) *is satisfied,* M *is bounded,* T *is proper and there exists an open subset* U *of* M *such that* $\overline{T(U)}$ *is a compact in* M. *Then there exists a fixed point* u *of* T.

Proof. Since T is proper, $V = T^{-1}(\overline{T(U)})$ is compact. Further $V \supset U$ and thus, there exists a compact set with nonempty interior. Then Corollary 1 implies the statement of the corollary. □

Corollary 3. *Suppose that assumption* (H1) *is satisfied,* M *is bounded,* T *is proper and locally compact* (*i.e. for each point* $x_0 \in M$ *there is a neighbourhood* $U(x_0)$ *of* x_0 *such that* $T(U)$ *is relatively compact*). *Then* T *has a fixed point.*

Proof. Let K_1 be a compact subset of M. As T is locally compact, K_1 can be covered by a finite number of open neighbourhoods $U(x_k)$, $k = 1, \dots, j$, such that $T(U(x_k))$ is relatively compact. Hence $U = \bigcup_{k=1}^{j} U(x_k)$ is a neighbourhood of K_1 and $T(U)$ is relatively compact. Since T is proper, $T^{-1}(\overline{T(U)})$ is a compact set which contains U and by Corollary 1, there exists a fixed point of T. □

Now in our considerations we will start from the fact that if (K_1, K_2) is a minimal admissible couple, then the set K_3 determined by (10) is a convex subset of K_2 such that $T(K_3) \subset K_3$.

Corollary 4. *Assume that* (H1) *is satisfied and let* (K_1, K_2) *be a minimal admissible couple with the properties:*

(i) K_2 *is bounded,*

(ii) *for each convex set* K *such that* $K_1 \subset K \subset K_2$ *and* $T(K) \subset K$ *the implication holds:*

If K *is dense in* K_2, *then* $K_2 \setminus K$ *is not dense in* K_2.

Then T *has a fixed point in* K_2.

Proof. The implication above is in contradiction with the properties of the set K_3 in Theorem 2. □

The next result will be based on the following lemma.

Lemma 5. *Let $T : M \to M$ be a map. Then the following statements are true:*

(a) *Each point of a k-cycle of T is a fixed point of T^k.*
(b) *Each fixed point of T^k is either a fixed point of T or belongs to an l-cycle of T where l is a divisor of k.*

Proof. Only the statement (b) will be proved. Let $x = T^k(x)$ and let $x \neq T(x)$. Consider the sequence $\{x, T(x), \dots, T^{k-1}(x)\}$. Then two cases may occur. Either all terms $T^l(x)$, $l = 1, \dots, k-1$ are different from x and then the sequence $\{x, \dots, T^{k-1}(x)\}$ is injective and x belongs to a k-cycle of T, or there exists an l, $1 < l < k$ such that $T^l(x) = x$ and $T^m(x) \neq x$ for $m = 1, \dots, l-1$. In this case x belongs to an l-cycle of T and with respect to the fact that $T^k(x) = x$ we must have that l is a divisor of k. □

The following theorem gives a partial answer to the conjecture above.

Theorem 3. *Suppose that assumption* (H1) *is satisfied, T is proper and there exists an integer $n \geqq 2$ such that T^n is compact. Then T has a fixed point.*

Proof. By the Schauder Fixed Point Theorem, the assumption T^n is compact implies that there exists a point $u \in M$ such that $T^n(u) = u$. Then Lemma 5 gives that either u is a fixed point of T, or there is a natural $l \geqq 2$ such that $C = \{u, T(u), \dots, T^{l-1}(u)\}$ is an l-cycle of T whereby $T(C) = C$. Suppose that the latter case is true. Then there exists a unique minimal admissible couple (C, K_2). In view of Lemma 3, K_2 satisfies (5). As T is proper, from the compactness of $\overline{T^n(K_2)}$ it follows that $T^{-1}(\overline{T^n(K_2)})$ as well as $\overline{T^{n-1}(K_2)}$ are compact. Proceeding in this way, step by step we get that $\overline{T^{n-2}(K_2)}, \dots, \overline{T(K_2)}$ are compact and hence K_2 is compact, too. □

If T is not proper, then the following alternative holds.

Theorem 4. *Suppose that assumption* (H1) *is satisfied and there exists an integer $n \geqq 2$ such that T^n is compact. Then in the compact set $K = \overline{T^n(M)}$ either T has a fixed point or for each prime number $p \geqq n$ there exists a p-cycle of T. Moreover, each cycle of T lies in K.*

Proof. As $S = \overline{\mathrm{co}}(K)$ is a convex compact subset of M, and for each $k \geqq n$ $T^k(S) \subset K$, there exists a fixed point $x_k \in K$ of T^k. If T has no fixed point, then x_k belongs to an l-cycle of T where l is a divisor of k. In case $k = p$, l is p. Further $T(K) \subset K$ and hence, together with x_p, all elements of this p-cycle of T belong to K. The last statement follows from the fact that in each cycle of T there is an element in K.

Remark 2. Since the existence of a fixed point of T is proved in a convex compact subset K of a Banach space X, by F. E. Browder [4, 5] in the case of an infinite dimensional Banach space X, all existence statements throughout the paper guarantee the existence of a *non-ejective fixed point* u of T, i.e. such that each neighbourhood U of u contains a point $x \neq u$ for which $T^k(x)$ lies in U for all $k \geqq 1$.

Some properties of a proper mapping will be given in the following proposition which is a consequence of Proposition 11.14 in [24, p. 499], Corollary 19.18 ([25, p. 636]) and Examples 11.11, 11.12 in [24, p. 498].

Proposition 6. *Suppose that M is a nonempty set in a Banach space X and $T\colon M \subset X \to X$. Then the following statements hold:*

- (i) *The set $T(M)$ is closed if T is continuous and proper on the closed set M.*
- (ii) *If T is proper, then $T\colon S \subset M \to X$ is proper on each closed subset S of M.*
- (iii) *If S is compact, then each continuous map $A\colon S \subset X \to X$ is proper.*
- (iv) *If T is injective, has a continuous inverse operator T^{-1} and a closed range, then it is proper.*
- (v) *The operator $I - T$ is proper on M if T is condensing and the set M is closed and bounded.*

Remark 3. If T is a proper mapping, then we can modify the construction of the sequence $\{S_k\}_{k=1}^{\infty}$ defined by (3) and (4) without violating its main property to be a nondecreasing sequence of convex compact subsets of K_2. In the new construction (3) is retained and (4) is replaced by

$$S_{k+1} = \overline{\mathrm{co}}(S_k \cup T(S_k) \cup T^{-1}(S_k)), \quad k = 1, 2, \dots. \tag{16}$$

Then all statements of Lemma 3 and Theorem 2 are true if the sequence $\{S_k\}_{k=1}^{\infty}$ is defined by (3) and (16).

With help of this remark and Theorem 2 we prove

Theorem 5. *Suppose that assumption* (H1) *is satisfied, M is bounded, T is proper and C_T is a nonempty relatively compact set. Then C_T is compact and T has a fixed point.*

Proof. By Proposition 6, each set $T^k(M)$, $k = 1, 2, \dots$, is closed and therefore C_T is closed and hence, compact. Consider the minimal admissible couple (K_1, K_2) with $K_1 = C_T$ and the sequence $\{S_k\}_{k=1}^{\infty}$ determined by (3) and (16). Let K_3 be determined by (10). As $T(S_k) \subset S_{k+1}$ as well as $T_1(S_k) \subset S_{k+1}$, we have that $T(K_3) \subset K_3$ and

$$T^{-1}(K_3) \subset K_3. \tag{17}$$

Suppose that $K_2 \setminus K_3 \neq \emptyset$. Then (17) implies that $T(K_2 \setminus K_3) \subset K_2 \setminus K_3$. Hence $C_T \cap (K_2 \setminus K_3) \neq \emptyset$. Therefore $K_3 = K_2$, which on the basis of Theorem 2 and respecting Remark 3, implies that T has a fixed point. $\square$

Now we will study the sequence $\{S_k\}_{k=1}^{\infty}$ from the point of view of the Hausdorff distance. Let H be the collection of all nonempty bounded closed subsets of M. Then the distance function $d(.,A)$ is defined by $d(x,A) = \inf\{\|x-y\| : y \in A\}$ for $x \in M$, $A \in H$ and for each pair of the sets $A, B \in H$ the Hausdorff distance $d_H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}$.

Since M is a complete subset of X, by Theorems 3.51, 3.58 and 3.59 in [1, pp. 100, 105, 106] the folowing proposition holds.

Proposition 7. *The set H equipped with d_H is a metric space which is complete and the collection K of nonempty compact subsets of M is closed in (H, d_H).*

Let A, A_k, $k = 1, 2, \ldots$, be the subsets of X. We recall that the *ε-neighbourhood of A* is defined by

$$U(A,\varepsilon) = \{x \in X : d(x,A) < \varepsilon\}$$

and the *topological lim sup of the sequence* $\{A_k\}$, denoted by $Ls\ A_k$ (the *topological lim inf of the sequence* $\{A_k\}$, in notation $Li\ A_k$), is the set of all points x belonging to X such that for every neighbourhood $U(x)$ of x there are infinitely many k with $U(x) \cap A_k \neq \emptyset$ (there is an l such that $k \geqq l$ implies $U(x) \cap A_k \neq \emptyset$). Equivalently,

$$Ls\ A_k = \bigcap_{m=1}^{\infty} \overline{\bigcup_{k=m}^{\infty} A_k} = \bigcap_{\varepsilon>0} \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} U(A_k,\varepsilon),$$
$$Li\ A_k = \bigcap_{\varepsilon>0} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} U(A_k,\varepsilon),$$

see [1, p. 102–103]. Clearly $Li\ A_K \subset Ls\ A_K$. If $Li\ A_k = Ls\ A_k = A$, then the set A is called the *closed limit of the sequence* $\{A_k\}$.

On the basis of Theorems 3.55, 3.65 in [1, pp. 103, 109], the relation between the convergence in the space (H, d_H) and the closed limit is given by

Proposition 8. *Let A, A_k, $k = 1, 2, \ldots$, be closed bounded subsets of M. If $\lim_{k\to\infty} A_k = A$ in (H, d_H), then $A = Li\ A_k = Ls\ A_k$. Conversely, if A is compact and $Li\ A_k = Ls\ A_k = A$, then $\lim_{k\to\infty} A_k = A$ again in (H, d_H).*

Lemma 6. *Suppose that assumption* (H1) *is satisfied. Let (K_1, K_2) be a minimal admissible couple. Then K_2 is compact if and only if the sequence*

$\{S_k\}_{k=1}^{\infty}$ *defined by* (3) *and* (4) *is a Cauchy sequence in* (H, d_H)*, that is, for each* $\varepsilon > 0$ *there exists a* $k_0 = k_0(\varepsilon) \in N$ *such that*

$$\sup_{x \in S_l} d(x, S_k) < \varepsilon \quad \textit{for each } l > k \geqq k_0. \tag{18}$$

Proof. Since $\{S_k\}_{k=1}^{\infty}$ is a nondecresing sequence,

$$Ls\ S_k = \overline{\bigcup_{k=1}^{\infty} S_k} = Li\ S_k$$

and by (1), K_2 is the closed limit of the sequence $\{S_k\}_{k=1}^{\infty}$. If K_2 is compact, then by Proposition 8 $\lim_{k\to\infty} S_k = K_2$ in (H, d_H).

If, conversely, $\{S_k\}_{k=1}^{\infty}$ is a Cauchy sequence, then in view of Propositions 7 and 8 $\{S_k\}_{k=1}^{\infty}$ is convergent to K_2 as $k \to \infty$. □

Theorem 6. *Let assumption* (H1) *be satisfied. Let* K_1 *be a compact subset of* M*. Suppose that the sequence* $\{S_k\}_{k=1}^{\infty}$ *is defined by* (3) *and* (4)*. Then a sufficient condition for the existence of a fixed point* u *of* T *in* M *is that for each* $\varepsilon > 0$ *there exist a* $k_0 = k_0(\varepsilon) \in N$ *such that* (18) *is fulfilled.*

Proof. (K_1, M) is an admissible couple and by Lemma 1 there exists a unique minimal admissible couple (K_1, K_2). Lemma 6 implies that K_2 is compact and hence, by the Schauder Fixed Point Theorem, there exists a fixed point u of T in M. □

Somewhat more general result is given in the following theorem.

Theorem 7. *Let assumption* (H1) *be satisfied. Assume that there exists a sequence of positive real numbers* $\{a_k\}_{k=1}^{\infty}$ *such that* $\sum_{k=1}^{\infty} a_k < \infty$ *and a nondecreasing sequence of convex compact sets* $\{S_k\}_{k=1}^{\infty}$ *such that the following statements hold:*

(i) $T(S_k) \subset S_{k+1}$,
(ii) $S_{k+1} \subset \overline{U(S_k, a_k)} = \{x \in M : d(x, S_k) \leqq a_k\}$.

Then there exists a convex compact set K_2 *such that* $T(K_2) \subset K_2$ *and hence,* T *has a fixed point in* K_2*.*

Proof. Similarly as in the proof of Lemma 3 we get that $K_2 = \overline{\bigcup_{k=1}^{\infty} S_k}$ is a convex and closed subset of M and $T(K_2) \subset K_2$. In view of assumption (ii), the sequence $\{S_k\}_{k=1}^{\infty}$ is Cauchy in (H, d_H) and hence, using Lemma 6 which is applicable in this case, we get that K_2 is compact. □

References

[1] Ch. D. Aliprantis, N. C. Yanellis, *Infinite Dimensional Analysis*, A Hitchhiker's Guide, Springer-Verlag, Berlin-Heidelberg, 1994.

[2] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, Berlin-Heidelberg, 1984.

[3] F. E. Browder, *A new generalization of the Schauder fixed point theorem*, Math. Ann. **174** (1967), 285–290.

[4] ______, *A further generalization of the Schauder fixed point theorem*, Duke Math. J. **32** (1965), 575–578.

[5] ______, *Asymptotic fixed point theorems*, Math. Ann. **185** (1970), 38–60.

[6] D. Bugajewski, *Weak solutions of integral equations with weakly singular kernel in Banach spaces*, Comm. Math. **34** (1994), 49–58.

[7] J. B. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York-Berlin-Heidelberg, 1990.

[8] E. Čech, *Point Sets*, Academia, Prague, 1974.

[9] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin-Heidelberg, 1985.

[10] J. Dugundji and A. Granas, *Fixed Point Theory*, Polish Scientific Publishers, Warszawa, 1982.

[11] R. Engelking, *Outline of General Topology*, North-Holland Publ. Co.-PWN, Amsterdam, 1968.

[12] L. Górniewicz and D. Rozpłoch-Nowakowska, *On the Schauder Fixed Point Theorem*, Topology in Nonlinear Analysis, Banach Center Publications, vol. 35, Polish Academy of Sciences, Warszawa, 1996.

[13] P. R. Halmos, *Naive Set Theory*, Springer-Verlag, New York, Inc., 1974.

[14] P. Hess, *Periodic-parabolic boundary value problems and positivity*, Pitman Research Notes in Mathematics, Longman Sci. and Tech., Burnt Mill, Harlow, 1991.

[15] A. Kucia, *Two examples of sets without fixed point property*, Personal Communication.

[16] C. Kuratowski, *Topologie* I, Monografie Matematyczne, vol. 20, Warszawa-Wrocław, 1948.

[17] ______, *Topologie* II, Pol. Tow. Mat., Warszawa, 1952.

[18] R. D. Nussbaum, *Some asymptotic fixed point theorems*, Trans. Amer. Math. Soc. **171** (1972), 349–375.

[19] B. Sadovskiĭ, *On a fixed-point principle*, Analiz. i Priložen. **1** (1967), 74–76. (Russian)

[20] J. Schauder, *Der Fixpunktsatz in Funktionalräumen*, Studia Math. **2** (1930), 171–180.

[21] S. Szufla, *On the application of measure of noncompactness to existence theorems*, Rend. Sem. Math. Univ. Padova **75** (1986), 1–14.

[22] V. Šeda, *Fredholm mappings and the generalized boundary value problem*, Differential Integral Equations **8** (1995), 19–40.

[23] ______, *On condensing discrete dynamical systems*, Math. Bohem. (to appear).

[24] E. Zeidler, *Nonlinear Functional Analysis and its Applications* I, Fixed-Point Theorems, Springer-Verlag, New York-Berlin, 1993.

[25] ______, *Nonlinear Functional Analysis and its Applications* II/B, Nonlinear Monotone Operators, Springer-Verlag, New York-Berlin, 1990.

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