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TOTAL AND LOCAL TOPOLOGICAL INDICES FOR MAPS OF HILBERT AND BANACH MANIFOLDS

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Dedicated to the memory of Juliusz P. Schauder

Abstract. Total and local topological indices are constructed for various types of continuous maps of infinite-dimensional manifolds and ANR's from a broad class. In particular the construction covers locally compact maps with compact sets of fixed points (e.g. maps having a certain finite iteration compact or having compact attractor or being asymptotically compact etc.); condensing maps (k-set contraction) with respect to Kuratowski's or Hausdorff's measure of non-compactness on Finsler manifolds; maps, continuous with respect to the topology of weak convergence, etc.

The characteristic point is that all conditions are formulated in internal terms and the index is in fact internal while the construction is produced through transition to the enveloping space. Examples of applications are given

Introduction

In this paper we put together our results on constructing the topological index for various types of maps of infinite-dimensional manifolds, both published in [5], [12], [6] and [7], etc., and their new developments obtained with a new approach suggested in [15] and [16]. The topological characteristics, constructed here, are motivated by applications and so there are a lot of problems where they are applied. The characteristics are calculated so that existence theorems are obtained for those problems.

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We consider a Hilbert or Banach manifold M that can be embedded into some Hilbert or Banach (respectively) space E as a neighbourhood retract. Sometimes we do not use the manifold structure of M (and so the construction is valid for Banach ANR's) while some other cases involve the manifold structure of M. Unlike many other approaches our construction is not based on homologies of infinite-dimensional spaces, etc. The main idea is rather elementary and looks as follows.

Let M be embedded into E and U be its neighbourhood such that there exists a retraction $R : U \to M$. Let $f : M \to M$ be a continuous map. Consider the map $F : U \to U$ defined by the formula $F = f \circ R$. Let there exist a compact set $B \in M$ containing the set $\mathrm{Fix}(f)$ of all fixed points of f (i.e., of F). Then on the boundary $\dot{\Omega}$ of a certain neighbourhood Ω of B in U we may consider the index (rotation) of (vector field) $I - F$, if it is well-posed for F, and introduce the total index (or Lefschetz number) of f on M as the value of index (rotation) for $I - F$ on $\dot{\Omega}$. The local index can be defined quite analogous.

Then we show that the obtained index does not depend on the space E, the method of embedding, the choice of Ω and so on and that it has usual properties.

Our approach is valid for all classes of operators on infinite-dimensional manifolds and ANR's, for which the index was constructed in the literature that we know (see [8], [10], [9]), and for many others that were not considered previously. Namely, it covers the following cases:

(I) M is a topological space that can be embedded as a neighbourhood retract into a Banach space or into a normed space or into a locally convex space. The map f is locally compact and its set $\mathrm{Fix}(f)$ of fixed points is compact.

Remark 1. Particular cases are locally compact maps with at least one of the following additional conditions:

(a) a certain finite iteration $f^n(M)$ is compact,
(b) f has a compact attractor,
(c) f is asymptotically compact, i.e., $f^\infty(M)$ is compact.

(II) M is a Finsler manifold that can be isometrically embedded into a Banach space. f is condensing (k-set contraction) with respect to Kuratowski's or Hausdorff's measure of non-compactness defined in terms of the internal distance on M generated by the Finsler metric.

Remark 2. Notice that in spite of the fact that the formulations in internal terms sound reasonably, the direct internal construction of index fails since it involves the notion of convex closure absent in nonlinear manifolds.

(III) M is as in (II), f is locally condensing with respect to Kuratowski's and Hausdorff measures of non-compactness and $\mathrm{Fix}(f)$ is compact.

Remark 3. Recall that the latter is satisfied if at least one of additional conditions (a), (b) or (c) from Remark 1 is fulfilled.

(IV) M is a manifold and f is continuous with respect to the topology of weak convergence.

Remark 4. Notice that the usual weak topology is ill-posed on manifolds as even a chart is not open in weak topology. However the topology of weak convergence is well-posed and we use it instead of the former. Here M should be embedded into a reflexive Banach space as a neighbourhood retract and the embedding of M should be continuous with respect to both ordinary Banach topology and topology of weak convergence.

The characteristic point of (II)–(IV) is that all conditions are formulated in internal terms and the index is in fact internal while the construction is produced through transition to the enveloping space. This must mean that probably it is possible to find an internal construction however (as we know) this has never been done. For (I) (to be exact, for special cases of (a), (b) and (c)) the internal construction in homological terms is known (see [8], [10] and [9]).

Our first paper [5] in this subject was originated by investigation of a certain integral-type operator [11], defined in terms of Riemannian parallel translation on the Banach manifold of C^1-curves on a compact Riemannian manifold $\mathcal{M}$. It is locally compact and its second iteration is compact. At that moment we knew only Browder's paper [8] (notice that [10] and [9] had not been published yet) where the Lefschetz number was constructed for compact maps of Banach ANRs. Since our operator was not the case, it was a demand for our construction. The study of shift operator along the solutions of functional-differential equations on manifolds made us generalise the construction onto condensing operators [12]. It should be pointed out that in the above cases the total index is proved to be equal to the Euler characteristic $\chi_{\mathcal{M}}$ of M that allowed us to establish the existence of a periodic solution if $\chi_{\mathcal{M}} \neq 0$. Recently new applications are obtained for differential equations of Carathéodory type on the Hilbert manifold of Sobolev diffeomorphisms of $\mathcal{M}$, connected with the modern Lagrangian approach by Arnold, Ebin and Marsden to hydrodynamics [15], [16]. We cannot mention all the applications here and refer the reader to the bibliography in [6], [7], [13], [14], [15] and [16]. We present only one example of applications in Section 2.

The structure of paper is as follows. In Section 1 we describe the main construction with some details on the example of locally compact maps with compact sets of fixed points. As compared with previous publications, we deal with this general problem from the very beginning considering (a), (b) and (c) from Remark 1 above as particular cases.

Section 2 is devoted to condensing and locally condensing maps. Here we modify and use the new approach suggested in [15] and [16] for studying differential equaitons on Hilbert manifolds. This approach allows us to obtain deeper results than before. We illustrate the construction by considering the shift operator for a fucntional-differential equation on a compact manifold $\mathcal{M}$. The shift operator acts on the Banach manifold of continuous curves on $\mathcal{M}$. We show that its total index is well-posed and is equal to the Euler characteristic of $\mathcal{M}$.

In Section 3 we present additional points necessary to extend the construction on weakly continuous maps.

1. Locally compact maps with compact sets of fixed points

Let X be a topological space that can be embedded into a Banach space as a neighbourhood retract (usually such spaces are called BANR, Banach absolute neighbourhood retracts). Let $f : X \to X$ be a continuous locally compact map such that $\mathrm{Fix}(f)$, the set of fixed points of f, is compact. In this section we construct the total index for f.

Various classes of maps have the above property. Among them we can mention the maps whose certain iteration is compact, i.e., there exists a number n such that $f^n(X) \in X$ is a compact set, a map with compact attractor or a limit compact map (in both cases the infinite iteration $f^\infty(X)$ is a compact set), etc. It is obvious that $f^n(X)$ and $f^\infty(X)$ contain $\mathrm{Fix}(f)$ and so the latter is compact.

Embed X into a Banach space E and denote by $R : U \to X$ a neighbourhood and a retraction existing by our general assumption. Define the composition map $F : U \to E$ by the formula $F = f \circ R$. Obviously F is locally compact since such is f and $\mathrm{Fix}(F) = \mathrm{Fix}(f)$. Since $\mathrm{Fix}(f)$ is compact, there exist a finite number of open sets Ω_α in U such that the image $F(\Omega_\alpha)$ is relatively compact for each α and $\Omega = \cup_\alpha \Omega_\alpha$ contains $\mathrm{Fix}(f)$. Thus $F(\Omega)$ is relatively compact. Note that there are no fixed points of F on the boundary $\dot{\Omega}$. Hence the index (rotation) $\gamma(I - F, \dot{\Omega})$ of (the vector field) $I - F$ on $\dot{\Omega}$ is well-posed.

Definition 1. The index $\gamma(I-F, \dot{\Omega})$ is called the *total index* or *the Lefschetz number* of f on X and is denoted by Λ_f.

Notice that in particular cases, mentioned above, we needn't know $\mathrm{Fix}(f)$ from the very beginning since $\mathrm{Fix}(f)$ (if it is not empty) belongs to another compact set: $f^n(X)$ or $f^\infty(X)$ or to the compact attractor, see above. We should consider a neighbourhood Ω of that compact set.

Evidently Λ_f takes the same values for all possible Ω by the usual properties of index.

Lemma 2. *Λ_f is independent of the choice of U and R.*

Proof. Suppose that there are two neighbourhoods U_1 and U_2 with retractions R_1 and R_2, respectively. For $U = U_1 \cap U_2$ define $F_1 = f \circ R_1 : U \to X$ and $F_2 = f \circ R_2 : U \to X$. Let $\Omega \subset U$ be a neighbourhood of $\mathrm{Fix}(f)$ such that both $F_1(\Omega)$ and $F_2(\Omega)$ are compact. Since $\mathrm{Fix}(f)$ is compact, for a certain $\varepsilon > 0$, small enough, the ε-neighbourhood Ω_ε of $\mathrm{Fix}(f)$ belongs to Ω and (since in addition the continuous map $F_1 - F_2$ is equal to zero on $\mathrm{Fix}(f) \subset X$) there exists $\delta > 0$ such that for any u from the δ-neighbourhood Ω_δ of $\mathrm{Fix}(f)$ we get $\|F_1(u) - F_2(u)\| < \varepsilon$, i.e., a straight segment γ, joining $F_1(u)$ and $F_2(u)$, belongs to Ω_ε. Retracting γ on X, we get the path that joins $F_(u)$ and $F_2(u)$. It is easy to see that we obtain a homotopy that is compact on $\Omega_\delta \times [0,1]$ and has no fixed points on the boundary $\dot\Omega_\delta$. Thus $\gamma(I - F_1, \dot\Omega_\delta) = \gamma(I - F_2, \dot\Omega_\delta)$. □

Lemma 3. *Λ_f is independent of the choice of E and embedding.*

Proof. Let E_1 and E_2 be Banach spaces with embeddings $i_1 : X \to E_1$, $i_2 : X \to E_2$ and retractions $R_1 : U_1 \to i_1X$, $R_2 : U_2 \to i_2X$ of the corresponding neighbourhoods. Introduce $F_1 = f \circ R_1$ and $F_2 = f \circ R_2$ as above. Consider the Banach space $E = E_1 \times E_2$ with the norm equal to the sum of norms of the factors. Then $i = (i_1, i_2) : X \to E$ is an embedding and $R = (R_1, R_2) : U \to i\,X$ is a retraction of $U = U_1 \times U_2$ onto $i\,X$. Introduce $F = f \circ R$. Denote by Ω, Ω_1 and Ω_2 the neighbourhoods of $\mathrm{Fix}(f)$ in E, E_1 and E_2, respectively, that define $\Lambda_f = \gamma(I - F, \dot\Omega)$, $\Lambda_f^1 = \gamma(I - F_1, \dot\Omega_1)$ and $\Lambda_f^2 = \gamma(I - F_2, \dot\Omega_2)$ for $\Omega = \Omega_1 \times \Omega_2$. Using Krasnosel'skiĭ's product of rotations theorem (see [17], p. 134) one can easily get that $\Lambda_f^1 = \Lambda_f$ and $\Lambda_f^2 = \Lambda_f$. See details, e.g., in [5], [6], [7]. □

Our total index Λ_f has usual properties that we formulate in the following

Theorem 4.

(1) *If f has only isolated fixed points in X then Λ_f is the sum of their indices.*
(2) *If f_1 and f_2 from the class of locally compact maps with compact sets of fixed points are homotopic to each other in the same class, then $\Lambda_{f_1} = \Lambda_{f_2}$.*
(3) *If $\Lambda_f \neq 0$, there exists a fixed point of f in X.*
(4) *If X is contractible, then f is homotopic to a constant map so that $\Lambda_f = 1$ and there exists a fixed point of f in X.*

The proof of theorem is routine. It is based on the analogous properties for completely continuous operators. For example, if $\Lambda_f = \gamma(I - F, \dot\Omega) \neq 0$, there exists a fixed point x^* of F in Ω. By the construction $x^* \in X$ and $f(x^*) = x^*$.

Assume in addition that X is a Banach manifold. Let the image $f(X)$ belong to a closed submanifold Y in X. This means that f can be restricted to $f_{|Y} : Y \to Y$ and the construction of Λ is valid for $f_{|Y}$.

Theorem 5. $\Lambda_f = \Lambda_{f_{|Y}}$.

To prove Theorem 5 one should consider a tubular neighbourhood U_Y of Y in X with retraction R_Y and then show that $F = f \circ R$ and $F_Y = f \circ R_Y \circ R$ are homotopic as maps of $R^{-1}(U_Y) \subset U$.

The above construction is generalised to introduce the local index. Let X be as above and $f : \overline{W} \to X$ sends the closure of open set W into X without fixed points on the boundary $\dot{W}$. Let f be continuous, locally compact and have compact $\mathrm{Fix}(f)$ (as well as above the last condition is satisfied if, e.g., a finite iteration $f^k(\overline{W})$ is compact or $f^\infty(\overline{W})$ is compact or f on $\overline{W}$ has compact attractor). Here we should consider $F = f \circ R$ on a certain neighbourhood Ω of $\mathrm{Fix}(f)$ in W such that $f(\overline{\Omega})$ is compact. We introduce the local index $\gamma(f, \dot{\Omega})$ by the equality $\gamma(f, \dot{W}) = \gamma(I - F, \dot{\Omega})$. All properties of $\gamma(f, \dot{W})$ are similar to those of ordinary local index (and analogous to those of Λ_f). In particular, $\gamma(f, \dot{W})$ is constant under homotopies in the above class of maps without fixed points on $\dot{W}$ and if $\gamma(f, \dot{W}) \neq 0$ there exists a fixed point of f in W.

2. The case of condensing maps

Let M be a smooth infinite-dimensional Finsler manifold. This means that there is a structure of smooth Banach manifold on M, that in any tangent space T_mM at a point $m \in M$ the norm $\|\cdot\|_m$, equivalent to that of the model space of M, is given and that the norm $\|\cdot\|_m$ smoothly depends on $m \in M$. The family of the norms $\|\cdot\|_m$, $m \in M$, is called the Finsler metric on M.

We shall follow the usual notations and omit the index m since it is obvious at what point a certain vector is applied. For instance, let $m(t)$ be a smooth curve on M. Then evidently the velocity vector $\dot{m} = dm(t)/dt$ is applied at $m(t)$.

The length of a smooth curve $m(t)$ in M for $t \in [a, b]$ is determined by the formula $\int_a^b \|\dot{m}(t)\|\, dt$.

Recall the standard notion of Finsler geometry:

Definition 6. For $m^0, m^1 \in M$ ($X^0, X^1 \in TM$) the number equal to the infimum of the lengths of curves connecting them, is called the *intrinsic distance* $d(m^0, m^1)$ between them in M.

It is a well-known fact that $d(m^0, m^1)$ satisfies the axioms of a metric and so M becomes a metric space.

Everywhere below we suppose that M is isometrically embedded in a certain Banach space E with the norm $\|\cdot\|_E$. This means that for any $X \in T_mM$, $m \in M$, the equality $\|X\| = \|X\|_E$ holds. That is why we shall omit E in the notations for norm in the space.

Proposition 7. *For any $m^0, m^1 \in M$ the inequality*

$$d(m^0, m^1) \geq \|m^0 - m^1\|$$

holds.

The proof of Proposition 7 is routine.

We also shall suppose that there exist a neighbourhood U of M in E and a smooth retraction $R : U \to M$. Denote by TR the tangent map to R. Recall that TR sends a vector $X \in T_xE$ (tangent to E at $x \in E$) into the vector $d_xR(X) \in T_{R(x)}M$ where d_xR denotes the (linear operator of) derivative of R at x.

Proposition 8. *For any point $m \in M \subset U$ and for any $Q > 1$ there exists a neighbourhood V_m^Q of m in U such that for any $x \in V_m^Q$ the inequality $Q > \|d_xR\| > 1/Q$ holds where $\|d_mR\|$ is the norm of operator d_mR.*

Proof. Since R is smooth and for any $m \in M \subset U$ by definition $R(m) = m$, at $m \in M$ the equality $\|d_mR\| = 1$ holds and consequently in a certain neighbourhood V_m^Q of m in U we get $Q > \|d_xR\| > 1/Q$ as $\|d_xR\|$ is continuous in $x \in U$. □

Specify an arbitrary point $\overline{m} \in M$ and a number $Q > 1$. Since $V_{\overline{m}}^Q$ is open, there exists a ball $B \subset V_{\overline{m}}^Q$ of some radius ρ with the centre at $\overline{m}$.

Theorem 9. *The retraction R on B is Lipschitz continuous with respect to the norm $\|\cdot\|$ of E in B and the intrinsic metric d on M with the Lipschitz constant Q*

Proof. Consider two points u_0 and u_1 in B and denote by $u(s)$, $s^0 \leq s \leq s^1$, the line interval connecting u_0 and u_1. Since B is convex, the complete interval $u(s)$ belongs to B and thus the map R is well-posed at its points. Consider the curve $m(s) = R(u(s))$ in M connecting $m_0 = R(u_0) \in M$ with $m_1 = R(u_1) \in M$. Let s be the natural parameter (the length) on $u(s)$ (if it is not so, change the parameter). Thus $\|\dot{u}(s)\| = 1$ for $s \in [s^0, s^1]$ where $\dot{u} = du/ds$. Evidently the length $\int_{s^0}^{s^1} \|\dot{u}(s)\|\, ds$ of $u(s)$ from u_0 to u_1 is equal to the distance $\|u_0 - u_1\|$ between those points in E and is equal to $|s^1 - s^0|$ since $\|\dot{u}(s)\| = 1$. The length of $m(s)$, $s^0 \leq s \leq s^1$, between m_0 and m_1 is equal to $\int_{s^0}^{s^1} \|\dot{m}(s)\|\, ds$ and is not shorter than the intrinsic distance $d(m_0, m_1)$ between m_0 and m_1.

Notice in addition that by definition $\dot{m}(s) = TR\dot{u}(s)$. Then applying Proposition 8 and the above arguments we get the following estimates:

$$\begin{aligned} d(m_0, m_1) &\leq \int_{s^0}^{s_1} \|\dot{m}(s)\|\, ds = \int_{s^0}^{s^1} \|d_{u(s)}R\dot{u}(s)\|\, ds \leq \int_{s^0}^{s^1} \|d_{u(s)}R\|\|\dot{u}(s)\|\, ds \\ &\leq \int_{s^0}^{s^1} Q\|\dot{u}(s)\|\, ds = Q\int_{s^0}^{s^1} ds = Q(s^1 - s^2) = Q\|u_0 - u_1\|. \end{aligned}$$

□

COROLLARY 10. *R is Lipschitz continuous on B with constant Q with respect to the norm* $\|\cdot\|$ *in both B and M*

Indeed, by Proposition 7, $d(m_0, m_1)$ is not shorter than the distance $\|m_0 - m_1\|$ between those points in E, and the last formula in the proof of Theorem 9 leads to $\|m_0 - m_1\| < Q\|u_0 - u_1\|$.

Recall (see details in [1], [20]) the notions of Hausdorff and Kuratowski measures of non-compactness in a Banach space. Let $\Omega \subset E$ be a bounded subset in E.

DEFINITION 11. $\chi(\Omega) = \inf\{\varepsilon > 0 \mid \Omega$ has in E a finite ε-net with respect to the norm $\|\cdot\|\}$ is called the *Hausdorff measure of non-compactness* of Ω.

DEFINITION 12. $\alpha(\Omega) = \inf\{\mathbf{d} > 0 \mid \Omega$ permits its partition in E into a finite number of subsets with diameters less than $\mathbf{d}$ with respect to the norm $\|\cdot\|\}$ is called the *Kuratowski measure of non-compactness* of Ω.

In what follows in this section by *measure of non-compactnes* ψ we denote either χ or α.

Notice some properties of χ and α such as:

(i) its invariance with respect to convex closure: $\psi(\operatorname{co}\Omega) = \psi(\Omega)$ for any bounded set Ω,
(ii) (monotonicity) if $\Omega_0 \subseteq \Omega_1$ then $\psi(\Omega_0) \leq \psi(\Omega_1)$,
(iii) (non-singularity) $\psi(\{h\} \cup \Omega) = \psi(\Omega)$ for any bounded $\Omega \subset E$ and any point $h \in E$.

Both χ and α can be defined in any metric space, in particular, in M with the distance d taken as metric. Namely, denote by $\chi_I(\Omega)$ the *internal Hausdorff measure of non-compactness* of a bounded set in M defined by the formula $\chi_I(\Omega) = \inf\{\varepsilon > 0 \mid \Omega$ has in M a finite ε-net with respect to the distance $d\}$. Similarly, the *internal Kuratowski measure of non-compactness* is $\alpha_I(\Omega) = \inf\{\mathbf{d} > 0 \mid \Omega$ permits its partition in M into a finite number of subsets with diameters less than $\mathbf{d}$ with respect to the distance $d\}$.

An important difference between ψ and ψ_I is that property (i) is not valid for the latter at least because there is no analogue of convex closure in nonlinear spaces. Properties (ii) and (iii) remain true for ψ_I.

PROPOSITION 13. *In any metric space* $\psi(\Omega) = 0$ *if and only if* Ω *is compact.*

Proposition 13 follows from well-known facts of the theory of metric spaces.

DEFINITION 14. A continuous map $F : E \to E$ ($f : M \to M$) is called *condensing with respect to the measure of non-compactness* ψ (ψ_I, respectively) with constant q if for any bounded set Ω the inequality $\psi(F(\Omega)) < q\psi(\Omega)$ ($\psi_I(f(\Omega)) < q\psi_I(\Omega)$, respectively) holds.

Recall that property (i) is essentially involved into the construction of index for condensing maps in Banach linear spaces so that the direct analogue of classical scheme for introducing index for condensing maps with respect to ψ_I on M fails. However it is possible to construct the index by the general scheme of this paper.

Let $f : M \to M$ be condensing with respect to ψ_I with $q < 1$ and let M has finite diameter. Consider the set $f^\infty(M) = \bigcap_{k=1}^\infty f^k(M)$.

Lemma 15. *f^∞ is compact.*

Proof. Assume it is not so. Let $\psi_I(f^\infty(M)) = a > 0$. Choose k such that $q^k\psi_I(M) < a$. Then $\psi_I(f^k(M)) < \psi_I(f^\infty(M))$ that contradicts the inclusion $f^\infty(M) \subset f^k(M)$. □

Notice that $f^\infty(M)$ contains all fixed points of f.

Introduce $F : U \to M \in U$ by the formula $F = f \circ R$ as above.

Theorem 16. *There exists a neighbourhood $V \supset M$ in U such that F is locally condensing on V with respect to ψ with a certain constant $\overline{q} < 1$.*

Proof. Specify $Q > 1$ such that $qQ < 1$. Determine V as the union of balls B mentioned in Theorem 9 with the above Q. Specify a point $x \in V$ belonging to a certain B. By Theorem 9 R is Lipschitz continuous on the neighbourhood B of x with constant Q with respect to the norm $\|\cdot\|$ in B and the intrinsic distance d in M. Let a set $A \subset B$ has a finite ε-net N_ε for some $\varepsilon > 0$ with respect to the norm $\|\cdot\|$. Then the set $R(N_\varepsilon) \subset M$ is a finite (at least) $Q\varepsilon$-net of $R(A)$ with respect to d. This means that $\chi_I(R(A)) \le Q\chi(A)$. Since $\chi_I(f(R(A))) < q\chi_I(R(A))$, we get $\chi_I(f(R(A))) < Qq2\chi(A)$. From Proposition 7 it follows that $\chi(f(R(A))) < \chi_I(f(R(A)))$. Thus $\chi(F(A)) < Qq\chi(A)$. Define $\overline{q} = qQ$. Notice that $\overline{q} < 1$ by the choice of Q. □

As well as in Section 1 there exists a neighbourhood Ω of the compact set $f^\infty(M)$ (containing all fixed points of f and so of F, see above) in V such that $F : \overline{\Omega} \to U$ is condensing with respect to ψ with constant $\overline{q} < 1$. By the construction there are no fixed points of F on the boundary $\dot{\Omega}$. Thus the index $\gamma(I - F, \dot{\Omega})$ is well-posed (see [1], [20]).

Definition 17. $\gamma(I - F, \dot{\Omega})$ is called the *total index* or the Lefschetz number of f on M and is denoted by Λ_f.

As well as the total index in Section 1, this Λ_f has the usual properties that we summarise in the following

Theorem 18.

(1) *If f has only isolated fixed points in X then Λ_f is the sum of their indices.*

(2) *If* f_1 *and* f_2 *from the class of condensing maps with* $q < 1$ *are homotopic to each other in the same class, then* $\Lambda_{f_1} = \Lambda_{f_2}$.
(3) *If* $\Lambda_f \neq 0$, *there exists a fixed point of* f *in* X.
(4) *If* X *is contractible, then* f *is homotopic to a constant map so that* $\Lambda_f = 1$ *and there exists a fixed point of* f *in* X.
(5) *Let the image* $f(M)$ *belong to a closed submanifold* M_0 *in* M. *Then* $\Lambda_f = \Lambda_{f_{|M_0}}$.
(6) *If* f *is as in Section* 1, *then the total indices of* f *constructed in Section* 1 *and here coincide.*

The construction of local index $\gamma(f, \dot{W})$ is a simple modification of that from Section 1.

Notice that the constructions of Λ_f and of $\gamma(f, \dot{W})$ can be applied directly for a map f that is locally condensing with some constant $q < 1$ and such that its set $\mathrm{Fix}(f)$ (in M or in W, respectively) is compact. In particular, for a locally condensing map with constant $q < 1$ whose certain finite iteration or f^∞ is compact (of M or of W, respectively), or that has a compact attractor (in M or in W, respectively). The properties of total and local indices are the same as above.

Now we present an example of applications. It is influenced by a construction of M. I. Kamenskiĭ for equations of neutral type in linear spaces.

Let $\mathcal{M}$ be a compact Riemannian manifold with a Riemannian metric $\langle \cdot, \cdot \rangle$ and let $J = [-\tau, 0]$, $\tau > 0$, be an interval of R. Consider the smooth Banach manifold $C^0(J, \mathcal{M})$ of continuous curves (mappings) in $\mathcal{M}$ defined on J. Denote by $\pi : T\mathcal{M} \to \mathcal{M}$ the natural projection of tangent bundle to $\mathcal{M}$.

Definition 19. A functional-differential equation (FDE) is a map $X : R \times C^0(J, \mathcal{M}) \to T\mathcal{M}$ such that $\pi(X(t, \varphi)) = \varphi(0)$ for any $\varphi \in C^0(J, \mathcal{M})$.

For a continuous curve $x : [-\tau, a] \to \mathcal{M}$, $a > 0$, denote by $x_t : [-\tau, 0] \to \mathcal{M}$, $t \in [0, a]$, the curve from $C^0(J, \mathcal{M})$ defined by the formula $x_t(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0]$. Let $\varphi(t) \in C^0(J, \mathcal{M})$.

Definition 20. A C^1-curve $x^\varphi(t)$, $t \in [-\tau, a]$, $a > 0$, such that for $t \in [-\tau, 0]$ $x^\varphi(t) = \varphi(t)$, is called a *solution* of FDE X with initial condition φ if for any $t \in [0, a]$ the equality $dx^\varphi(t)/dt = X(t, x_t^\varphi)$ holds.

We shall suppose that for any $\varphi(\theta)$ there exists a unique solution $x^\varphi(t)$ of FDE X and that this solution is continuous in initial value φ. It is known, that this assumption is fulfilled, e.g., if X is locally Lipschitz continuous (for example, smooth). We also shall suppose that $\|X(t, \varphi)\|_\mathcal{M}$ is uniformly bounded for all $t \in R$, $\varphi \in C^0(J, \mathcal{M})$ where $\| \cdot \|_\mathcal{M}$ is the norm in tangent space generated by the Riemannian metric on $\mathcal{M}$. Under this assumption all solutions exist for t up to ∞.

Let X be periodical with period $\omega > 0$, i.e., $X(t,\varphi) = X(t+\omega,\varphi)$ for any φ.

Definition 21. The operator $u_\omega : C^0(J,\mathcal{M}) \to C^0(J,\mathcal{M})$ sending a curve φ into the curve x^φ_ω is called the *shift operator* along solutions of FDE X.

From the above assumptions it follows that u_ω is continuous and that its fixed points correspond to ω-periodic solutions of the FDE X.

The tangent space $T_\varphi C^0(J,\mathcal{M})$ to $C^0(J,\mathcal{M})$ at $\varphi \in C^0(J,\mathcal{M})$ is the set of all continuous vector fields $\{X(\theta)\}$ along $\varphi(\theta)$ (i.e., $X(\theta) \in T_{\varphi(\theta)}\mathcal{M}$ for each $\theta \in J$). Consider the following Finsler metric on $C^0(J,\mathcal{M})$ generated by the Riemannian metric $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathcal{M}$: $\|X(\,\cdot\,)\| = \max_{\theta\in J} e^\theta \|X(\theta)\|_{\mathcal{M}}$ for $X(\,\cdot\,) \in T_\varphi C^0(J,\mathcal{M})$, where $\|X(\theta)\|_{\mathcal{M}} = \langle X(\theta), X(\theta)\rangle^{1/2}$.

Immediately from the definition of the above Finsler metric we derive the following

Lemma 22. *The internal distance d on $C^0(J,\mathcal{M})$ of the above Finsler metric is represented by the formula: $d(\varphi_0,\varphi_1) = \max\limits_{\theta\in J} e^\theta \rho(\varphi_0(\theta),\varphi_1(\theta))$ where ρ is the internal distance on $\mathcal{M}$ corresponding to Riemannian metric $\langle\,\cdot\,,\,\cdot\,\rangle$.*

Consider the Hausdorff measure of non-compactness χ_I on $C^0(J,\mathcal{M})$ with respect to d (see above). Let Ω be a bounded set in $C^0(J,\mathcal{M})$. For a point $r \in J$ consider restrictions of curves from Ω on the intervals $[-\tau, r]$ and $[r, 0]$. Those restrictions form Banach manifolds where we can define Finsler metrics and their internal distances in analogy with the above Finsler metric and d. The corresponding Hausdorff measures of non-compactness are denoted by $\chi_{[-\tau,r]}$ and $\chi_{[r,0]}$, respectively. The notations $\chi_{[-\tau,r]}(\Omega)$ and $\chi_{[r,0]}(\Omega)$ mean the application of $\chi_{[-\tau,r]}$ and $\chi_{[r,0]}$ to the sets of corresponding restrictions.

Lemma 23. $\chi_I(\Omega) \geq \max(\chi_{[-\tau,r]}(\Omega), \chi_{[r,0]}(\Omega))$ *for any $r \in J$.*

The proof of Lemma 23 is routine.

Theorem 24. $\chi_I(u_\omega(\Omega)) < e^{-\omega}\chi_I(\Omega)$.

Proof. Since $\|X(t,\varphi)\|_{\mathcal{M}}$ is bounded and Ω is a bounded set in $C^0(J,\mathcal{M})$, all curves from $u_\omega(\Omega)$ are smooth with uniformly bounded derivatives for $t \geq 0$, hence they are uniformly bounded and equicontinuous, i.e., compact. This means that if $\tau \leq \omega$, $\chi_I(u_\omega(\Omega) = 0$ and the Theorem is proved.

Let $\omega \leq \tau$. Then $\chi_{[-\omega,0]}(u_\omega(\Omega)) = 0$ and so $\chi_I(u_\omega(\Omega)) = \chi_{[-\tau,-\omega]}(u_\omega(\Omega))$. From the construction of u_ω and χ_I it follows that

$$\chi_I(u_\omega(\Omega)) = \chi_{[-\tau,\omega]}(u_\omega(\Omega)) = e^{-\omega}\chi_{[\omega-\tau,0]}(\Omega).$$

Hence, by Lemma 23,

$$e^{-\omega}\chi_I(\Omega) \ge e^{-\omega}\max(\chi_{[-\tau,\omega-\tau]}(\Omega), \chi_{[\omega-\tau,0]}(\Omega)) \ge e^{-\omega}\chi_{[\omega-\tau,0]}(\Omega) = \chi_I(u_\omega(\Omega)).$$

□

Thus, in order to show that the total index for u_ω is well-posed we need to embed $C^0(J,\mathcal{M})$ isometrically into a certain Banach space as a neighbourhood retract.

By classical Nash's theorem $\mathcal{M}$ can be isometrically embedded into some Euclidean space R^K for K large enough. There exists a tubular neighbourhood Θ of $\mathcal{M}$ in R^K with retraction $R : \Theta \to \mathcal{M}$. Then $C^0(J,\Theta)$ is a neighbourhood of $C^0(J,\mathcal{M})$ in $C^0(J,R^K)$ such that $R : C^0(J,\Theta) \to C^0(J,\mathcal{M})$ is a retraction. If we in addition introduce the norm in $C^0(J,R^K)$ by the formula $\|\varphi(\,\cdot\,)\| = \max\limits_{t\in J} e^t\|\varphi(t)\|_{R^K}$, $C^0(J,\mathcal{M})$ turns out to be isometrically embedded into $C^0(J,R^K)$.

So, we have proved the following

Theorem 25. *Λ_{u_ω} is well-posed.*

Theorem 26. *Λ_{u_ω} is equal to the Euler characteristic $\chi_{\mathcal{M}}$.*

Proof. For $s \in [0,1]$ denote by $\varphi^s \in C^0(J,\mathcal{M})$ the curve such that

$$\varphi^s(\theta) = \begin{cases} \varphi(\theta) & \text{for } \theta \in [-s\tau, 0], \\ \varphi(-s\tau) & \text{for } \theta \in [-\tau, -s\tau]. \end{cases}$$

Consider the FDE X_s defined for $\varphi \in C^0(J,\mathcal{M})$ by the formula $X_s(\varphi) = X(\varphi^s)$. Denote by $x^{s,\varphi}$ the solution of X_s with initial condition φ and by $u_{s\omega}$ the shift operator sending φ into $x^{s,\varphi}_{s\omega}$. The homotopy $u_{s\omega}$ obviously satisfy the conditions of Theorem 18(2). In addition $u_{s\omega}$ for $s = 1$ coincides with u_ω and u_0 sends $C^0(J,\mathcal{M})$ into the submanifold of constant curves that is isomorphic to $\mathcal{M}$ (denote it also by $\mathcal{M}$) where it coincides with the shift operator along the solutions of ordinary differential equation X_0. By Theorem 18(5) we can restrict u_0 on $\mathcal{M}$. But on $\mathcal{M}$ the operator is homotopic to identical map and so its total index is equal to $\chi_{\mathcal{M}}$. □

Corollary 27. *If $\chi_{\mathcal{M}} \neq 0$,* FDE *$X$ has an ω-periodic solution.*

3. Weakly continuous maps

Weakly continuous maps form another class of infinite-dimensional operators, for which the index is well-posed. The starting points of this theory in linear Banach spaces can be seen in [2] and [3], for further references see, e.g., the bibliography in [7]. Note that dealing with weakly continuous maps is in some sense closer to the usual finite dimensional case. The significant difference here is that the index must be considered on the boundary of a strongly open set that is not open in weak topology. This difficulty is overcome by applying the idea of relative rotation (index).

Before extending this index theory onto nonlinear Banach manifolds we should solve another serious problem: an analogue of ordinary weak topology is ill-posed on a manifold. Indeed, the chart on a manifold is regarded as an open disk in model Banach space and it is not an open set in weak topology. That is why we instead consider the topology of weak convergence, well-posed on manifolds.

Let E be a Banach space. Determine on E a topology ω as follows. The open sets from ω are subsets of E such that their intersections with any bounded set $B \subset E$ are open sets in B with respect to the topology in B induced from the weak topology in E. Evidently ω is stronger than the weak topology of E. If E is a reflexive Banach space, ω coincides with the topology of weak convergence in E. We refer the reader to [19] for the proof of this statement. Some other cases when ω coincides with the topology of weak convergence are also described in [19].

Let now E be a reflexive Banach space. Denote its strong topology (topology of the norm) by τ.

Definition 28. Bimanifold modelled on E is a manifold with the following additional properties:

(i) the charts are open sets in the topology ω (and so also in τ) on E,
(ii) the changes of coordinates (i.e., the transition maps from one chart to another) are homeomorphisms both with respect to τ and with respect to ω.

Among examples of bimanifolds there are several manifolds of maps, Hilbert space spheres, etc.

In order to realise our scheme for introducing the total index here, we make the following

Assumption 29. *A bimanifold M is such that:*

(i) *M can be embedded in a reflexive Banach space E_1 so that the embedding is continuous with respect to topology ω both in M and in E_1,*
(ii) *there is a neighbourhood U of M in E_1 with respect to ω that is retracted onto M by a retraction r continuous with respect to ω.*

Below in this section M denotes a bimanifold satisfying Assumption 29.

Definition 30. A map $f : M \to M$ is called *weakly compact* if it is continuous with respect to ω and the image $f(M)$ is compact with respect to ω.

As before, define the map $F : U \to U$ by the formula $F = f \circ r$. Obviously F is continuous in ω and $F(U)$ is compact in ω. By the Eberlein–Shmulyan theorem, $F(U)$ is weakly compact in E_1 and so it is weakly closed and bounded by the norm in E_1 since E_1 is reflexive.

Let $\overline{B} \subset E_1$ be a closed ball in E_1 containing the bounded set $F(U)$. By the definition of ω we have $\overline{B} \cap U = \overline{B} \cap V$ where V is a certain weakly open set in E_1. Obviously there exists a weakly open set $V_1 \supset F(U)$ whose weak closure $\overline{V}_1$ is contained in V. Consider the set $\overline{B} \cap \overline{V}_1 \subset \overline{B} \cap U$. The operator F is defined on $\overline{B} \cap \overline{V}_1$ and there are no fixed points of F on the relative boundary $(\overline{B} \cap \overline{V}_1)^\bullet$. Thus the relative rotation $\gamma(I - F, (\overline{B} \cap \overline{V}_1)^\bullet, \overline{B})$, (see [4]) known as the rotation of weakly compact vector fields ([2], [3], etc., see above) is well-posed.

DEFINITION 31. $\gamma(I - F, (\overline{B} \cap \overline{V}_1)^\bullet, \overline{B})$ is called the *total index* or *the Lefschetz number* of the weakly continuous map f on the bimanifold M and is denoted by Λ_f.

In the same manner as in previous sections it is shown that Λ_f is independent of all choices in its construction (cf. Section 1) and so it is in fact an internal topological characteristic in M.

Λ_f is constant under homotopies in the same class of maps, if it is not equal to 0, f has a fixed point in M, etc.

The local index $\gamma(f, \dot{W})$ is constructed in complete analogy with Section 1 and has usual properties.

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