Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 14, 1999, 371–384

A NOTE ON BOUNDED SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS AT RESONANCE

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Dedicated to the memory of Juliusz P. Schauder

Abstract. In the paper we study the existence of bounded solutions for differential equations of the form: $x'' - Ax = f(t,x)$, where $A \in L(H)$, $f : \mathbb{R} \times H \to H$ (H — a Hilbert space) is a continuous mapping. Using a perturbation of the equation, the Leray–Schauder topological degree and fixed point theory, we overcome the difficulty that the linear problem is non-Fredholm in any resonable Banach space.

1. Introduction

The first papers on nonlinear boundary value problems at resonance appeared nearly thirty years ago. Since that time, nonlinear techniques have been very successful in proving the existence of bounded solutions for nonlinear differential equations at resonance. In this paper we employ two techniques: fixed point theory and Leray–Schauder degree theory (compare [15], [13], [6]). We consider a nonlinear operator on a Banach space, which is completely continuous and maps some ball into itself in the first case, and we create a homotopy between a solvable linear problem and the nonlinear problem in the second one (compare [11]).

Consider the abstract functional equation of the form:

$$Lx = N(x), \tag{1.1}$$

1991 *Mathematics Subject Classification.* Primary 34C11; Secondary 34G20, 47H10, 47H11.
Key words and phrases. Differential equations, resonance, boundedness, unbounded domain.

where L is a linear Fredholm operator of index 0 and N — a superposition operator in appropriate function spaces. We have no problem if the kernel of the linear part of this equation contains only zero because then L is surjective. It can be reduced to the fixed point problem for $L^{-1}N$, which is usually compact (or contractive or monotone or A-proper or $\ldots$) mapping and a suitable topological degree theory works and gives a large number of results (compare [13], [4]–[6], [14], [23], [12]). But, if kernel L is nontrivial, it has a finite dimension equal to the codimension of its image $L(Y)$. Then the equation (1.1) is said to be at resonance and one can deal with the problem by using the coincidence degree ([15], [9]) in that case. When the domain is unbounded (the half-line: for example, a boundary value problem with $\lim_{t\to\infty} x(t) = 0$; the line: x is bounded on $\mathbb{R}$, or other cases) (see: [8], [21]–[23], [7], [2], [1], [19]), the operator is usually non-Fredholm. For instance, two-point boundary value problems on the half-line like:

$$\begin{cases} x'' = f(x, x', t), \\ x(0) = \alpha, \\ \lim_{t\to\infty} x(t) = 0, \end{cases}$$

where f is continuous or Carathéodory function and satisfies some other conditions, are considered in [21], [22], [7], [23]. When the third condition of the problem is replaced by: x is bounded on $[0,\infty)$, we can see [22], [23], [2] (moreover in [2] a unique bounded solution on $[0,\infty)$ is obtained). We have also some results for the existence and uniqueness of solutions of boundary value problems of the following type:

$$\begin{cases} x'' = f(x, t), \\ x(t) \text{ bounded on } (-\infty, \infty), \end{cases}$$

(see [2]).

One can see a variety of existence results for bounded solutions of first and second order equations at resonance in the papers of Mawhin. There are bounded on $(-\infty,\infty)$ solutions for the nonautonomous equations

$$x' = f(t, x)$$

in [17] and the problem

$$x'' + Cx' + f(t, x) = p(t)$$

in [16] (where the existence conditions are of Landesman–Lazer type). Some inetersting results for the system of differential equations of Duffing's type:

$$x''(t) + [b(t)I + B(t)]x'(t) = F(t, x(t))$$

are obtained in [3] and [18] and, under some assumptions, there exists a unique solution in appropriate function spaces.

Some other boundary value problems on an infinite or noncompact intervals are considered also in [8], [24], [10], [1], [19].

We shall study the existence of a solution bounded on $\mathbb{R}$ of the differential equation

$$x'' - Ax = f(t,x) \tag{1.2}$$

which is of type (1.1) with $x'' - Ax$ denoted by L (linear part) and the nonlinear part, the operator $x \to f(\,\cdot\,, x(\,\cdot\,))$, denoted by N (in the appropriate function spaces).

In this equation $A \in L(H)$, $f : \mathbb{R} \times H \to H$ is continuous (H is a Hilbert space). We shall work in the Banach space $BC(\mathbb{R}, H)$ — the space of all bounded continuous functions $x : \mathbb{R} \to H$ with the norm

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} \|x(t)\|.$$

The operator L for equation (1.2) is actually non-Fredholm. We could easily observe that the range of L is not a closed subspace in any reasonable Banach space, if, for $A = 0$, we shall ask about the conditions for which the equation $x'' = h(t)$, $h : \mathbb{R} \to \mathbb{R}$, has a solution bounded on $\mathbb{R}$. It is the case when we cannot use the Green function (like in [4], [5], [12]) and we cannot apply the scheme of Mawhin (compare [15], [9]). Therefore for the existence of solutions we need additional or stronger assumptions.

That problem is solved in [11], but only for the first order equation:

$$x' - Ax = f(t,x) \tag{1.3}$$

(with A, f, x as above). As one can easily observe, the existence of solutions which are bounded on $\mathbb{R}$ for equation (1.3) is not equivalent to the existence of bounded solutions for (1.2). It means that we cannot consider the equation (1.2) as the system of two first order equations.

Our technique (see [11]) involves a family of equations dependent on a real parameter $\lambda \in [0, \lambda_1]$, namely we shall use the perturbation of the linear part L

$$x'' - Ax + \lambda Px = f(t,x), \tag{1.4}$$

where P is a linear projector, in such a way that, for $\lambda = 0$, we get the studied equation with a non-Fredholm operator, and, for $\lambda > 0$, the linear part is invertible (we can find a fixed point by using fixed point theory or Leray–Schauder degree theory).

2. Some preliminaries

We shall study the differential equation

$$x'' - Ax = f(t,x) \tag{2.1}$$

where $A \in L(H)$ is a bounded, selfadjoint operator (hence the spectrum $\operatorname{Sp} A$ is purely real), $f : \mathbb{R} \times H \to H$ is a continuous mapping (and satisfies some other conditions). We shall look for bounded (on $\mathbb{R}$) solutions of equation (2.1). Next, we shall need some notation.

Suppose that $\operatorname{Sp} A = \sigma_- \cup \sigma_0 \cup \sigma_+$, where $\beta \in \sigma_-$ means that $\beta < 0$, $\beta \in \sigma_+$ means that $\beta > 0$ and $\sigma_0 = \{0\}$ and are closed sets. Then we obtain the decomposition $H = H_- \oplus H_0 \oplus H_+$ in the A-invariant and orthogonal subspaces, such that $\operatorname{Sp}(A|_{H_i}) = \sigma_i$, $i = \pm, 0$. Denote by P_-, P_0, P_+ the projectors onto corresponding subspaces. Then $P_- + P_0 + P_+ = I$. Next, denote by f_0 the composition $P_0 \circ f$ and, analogously, f_- and f_+.

We shall use estimates for the norm of the exponential function of the operator A. Let $A \in L(H)$ and $\operatorname{Sp} A \subset \{\lambda : \operatorname{Re} \lambda < 0\}$. Then, for any $\nu > 0$ such that $\operatorname{Re} \lambda < -\nu$ for all $\lambda \in \operatorname{Sp} A$, we find $N > 0$ for which

$$\|e^{At}\| < Ne^{-\nu t}, \quad t \geq 0, \tag{2.2}$$

(cf. [4]). Then we have that $\exp tA|_{H_-} \to 0$ for $t \to \infty$, $\exp tA|_{H_+} \to 0$ for $t \to -\infty$ (with an exponential rate of convergence).

In the next section, we shall apply

Theorem 1 ([20]). *Let E be a Banach space and let $BC(\mathbb{R}, E)$ be the space of all bounded continuous functions $x : \mathbb{R} \to E$ with the norm*

$$\|x\|_\infty = \sup_{t \in \mathbb{R}} \|x(t)\|.$$

Let $S : BC(\mathbb{R}, E) \to BC(\mathbb{R}, E)$ be a nonlinear integral operator given by

$$Sx(t) = \int_{-\infty}^{\infty} G(t,s) f(s, x(s))\, ds$$

where:

(1) *there exist the finite limits $\lim_{s \to t^+} G(t,s)$, $\lim_{s \to t^-} G(t,s)$,*
(2) *$\|G(t,s)\| \leq Ne^{-\alpha|t-s|}$ for all t, $s \in \mathbb{R}$, where N and α are some positive constants,*
(3) *$G : \mathbb{R} \times \mathbb{R} \to L(E)$ is a continuous mapping for $t \neq s$,*
(4) *$f : \mathbb{R} \times E \to E$ is uniformly continuous on bounded sets,*
(5) *$f(t, \cdot) : E \to E$ is completely continuous for any $t \in \mathbb{R}$,*
(6) *there exists a bounded continuous function $b : \mathbb{R} \to E$ such that, for any $M > 0$, $\varepsilon > 0$, there is $T > 0$ such that $\|f(t,x) - b(t)\| \leq \varepsilon$ where $\|x\| < M$ and $|t| \geq T$.*

Then S is completely continuous.

We shall use also the following compactness criterion:

THEOREM 2 ([26]). *Let Y be a metric, locally compact space countable at ∞ and let X be a Banach space. Then the relative compactness of the set $F \subset BC(Y,X)$ is equivalent to the conjunction of three conditions:*

(i) *The set $\{x(t) : x \in F\}$ is relatively compact in X for each $t \in Y$.*
(ii) *For each compact K, $K \subset Y$, the functions in $F_K = \{x|_K : x \in F\}$ are equicontinuous.*
(iii) *For each $\varepsilon > 0$, there exist $\delta > 0$ and compact $K \subset Y$ such that, for any $x, y \in F$, if $\|x|_K - y|_K\|_\infty \le \delta$, then $\|x - y\|_\infty \le \varepsilon$.*

We shall use these theorems for a Hilbert space.

LEMMA 1. *If $g : \mathbb{R} \times H \to H$ is a continuous function which satisfies condition (5) of Theorem 1 and*

$$\forall r > 0\ \exists H_r : \mathbb{R} \to H\ \|g(t,x)\| \le H_r(t) \quad \forall t \in \mathbb{R},\ \|x\| \le r,$$

where H_r is a continuous and integrable function, then the operator

$$T : BC(\mathbb{R}, H) \to BC(\mathbb{R}, H)$$

given by

$$Tx(t) = \int_{-\infty}^{t} g(s, x(s))\, ds$$

is completely continuous.

PROOF. Operator T transforms bounded sets into bounded ones because for $\|x\|_\infty \le r$ (for some $M > 0$) we have

$$\|Tx(t)\| = \left\| \int_{-\infty}^{t} g(s, x(s))\, ds \right\| \le \left| \int_{-\infty}^{\infty} H_r(t)\, dt \right| < \infty.$$

It is also continuous. By using assumptions (i)–(iii) of Theorem 2 we will prove that for any $r > 0$ the set $T(B(0,r))$ is relatively compact.

To check (i), fix $t_0 \in \mathbb{R}$ and $\varepsilon > 0$. We shall find a finite ε-net for $A := \{Tx(t_0) : \|x\| \le r\}$. Choose a compact set $K \subset \mathbb{R}$ such that

$$\int_{\mathbb{R}\setminus K} H_r(t)\, dt < \frac{\varepsilon}{2}.$$

Set $x_0 = \int_{\mathbb{R}\setminus K} H_r(t)\, dt$. Due to [24, Lemma 3] and, by assumption (5) of Theorem 1, the set $Z = \{g(s,x) : \|x\| \le r,\ s \in K\}$ is relatively compact in H. But the integrals from set $B = \{\int_K g(s, x(s))\, ds$ for $\|x\| \le r\}$ belong to the convex hull $\mu(K)\overline{\mathrm{conv}} Z$, so there exists a finite $\varepsilon/2$-net $x_1, \ldots, x_p$ of B. Then

$$\|Tx(t_0) - (x_0 + x_j)\| \le \left\| \int_K g(s, x(s))\, ds - x_j \right\| + \|x_0\| \le \varepsilon.$$

To prove condition (ii), take $\varepsilon > 0$ and a compact set $K \subset \mathbb{R}$. Then, for $t \in K$ and $\|x\|_\infty \le r$, we have

$$\|(Tx)'(t)\| = \|g(t, x(t))\| \le H_r(t) \le K_1,$$

for some $K_1 > 0$. Therefore, for $t_1, t_2 \in K$, such that $|t_1 - t_2| < \varepsilon/K_1$, $\xi \in (\min\{t_1, t_2\}, \max\{t_1, t_2\})$ we have

$$\|Tx(t_1) - Tx(t_2)\| \le |t_1 - t_2| \|(Tx)'(\xi)\| \le \frac{\varepsilon}{K_1} \cdot K_1 = \varepsilon.$$

To prove condition (iii) we take $\varepsilon > 0$ and a compact set K which satisfies the condition:

$$2 \int_{\mathbb{R} \setminus K} H_r(s)\, ds \le \frac{\varepsilon}{3}.$$

Let $\|x\|_\infty \le r$ and $\|y\|_\infty \le r$. Then, if $t < \inf K$,

$$\|Tx(t) - Ty(t)\| \le \int_{-\infty}^{t} \|g(s, x(s)) - g(s, y(s))\|\, ds \le 2 \int_{-\infty}^{t} H_r(s)\, ds < \frac{\varepsilon}{3} < \varepsilon.$$

If $t \in [\inf K, \sup K]$, then

$$\begin{aligned} \|Tx(t) - Ty(t)\| &\le 2 \int_{\mathbb{R} \setminus K} H_r(s)\, ds + \int_{[\inf K, t]} \|g(s, x(s)) - g(s, y(s))\| \\ &\le \frac{\varepsilon}{3} + \operatorname{diam} K \cdot \sup_{s \in K} \|g(s, x(s)) - g(s, y(s))\| \le \varepsilon, \end{aligned}$$

because by equicontinuity of g on the set K and for $\|x - y\| \le \delta = \varepsilon/3$, we have

$$\|g(s, x(s)) - g(s, y(s))\| \le \frac{\varepsilon}{2 \operatorname{diam} K}.$$

If $t > \sup K$, then

$$\|Tx(t) - Ty(t)\| \le \int_K \|g(s, x(s)) - g(s, y(s))\|\, ds + \int_{\mathbb{R} \setminus K} H_r(s)\, ds \le \varepsilon,$$

which ends the proof. □

Lemma 2. *If K_1, K_2 are compact sets, $\lambda_1, \lambda_2 \in \mathbb{R}$, then the set*

$$Z = \{\lambda_1 u_1 + \lambda_2 u_2 : u_i \in K_i\}$$

is compact.

The lemma is obvious since Z is a continuous image of the compact set $K_1 \times K_2$ under the mapping $K_1 \times K_2 \ni (u_1, u_2) \to \lambda_1 u_1 + \lambda_2 u_2 \in Z$.

3. Existence of solutions

Theorem 3. *If $A \in L(H)$ is a selfadjoint operator which has finite-dimensional kernel, 0 is an isolated eigenvalue of the operator A and $f : \mathbb{R} \times H \to H$ is a continuous function satisfying conditions* (4)–(6) *of Theorem* 1 *and:*

(i) $\exists M_+ > 0\ \forall t \in \mathbb{R},\ \|x_+\| \geq M_+\ (x_+, f_+(t,x)) \geq 0,$
(ii) $\exists M_- > 0\ \forall t \in \mathbb{R},\ \|x_-\| \geq M_-\ (x_-, Ax_- + f_-(t,x)) > 0,$
(iii) $\exists M_0 > 0\ \forall t \in \mathbb{R},\ \|x_0\| \geq M_0\ (x_0, f_0(t,x)) > 0,$
(iv) $\forall r > 0\ \exists H_r : \mathbb{R} \to H\ \|f_-(t,x)\| \leq H_r(t) \quad \forall t \in \mathbb{R},\ \|x\| \leq r,$

where H_r is a continuous and integrable function, then the equation

$$x'' - Ax = f(t,x) \tag{3.1}$$

has a solution bounded on $\mathbb{R}$.

Proof. Equation (3.1) can be written down in the following form:

$$\begin{cases} x_+'' = Ax_+ + f_+(t,x), \\ x_0'' = f_0(t,x), \\ x_-'' = Ax_- + f_-(t,x), \end{cases}$$

where $x = (x_+, x_0, x_-)$. Moreover, we remark (like before) that

$$\mathrm{Sp}\,(A|_{H_-}) \subset (-\infty, 0), \quad \mathrm{Sp}\,(A|_{H_+}) \subset (0, \infty), \quad \mathrm{Sp}\,(A|_{H_0}) = \{0\}.$$

Notice that if A is selfadjoint and $\mathrm{Sp}\,(A|_{H_0}) = \{0\}$, then $A|_{H_0} = 0$.

Step 1. Take $\lambda > 0$ and consider the perturbed equation:

$$\begin{cases} x_+'' = Ax_+ + f_+(t,x), \\ x_0'' = \lambda x_0 + f_0(t,x), \\ x_-'' = Ax_- + f_-(t,x). \end{cases} \tag{3.2}$$

We use this perturbation because for $\lambda > 0$ the linear part is invertible. Now if we embed this equation in the family continuously depending on the real parameter $\mu \in [0,1]$:

$$\begin{cases} x_+'' = Ax_+ + \mu f_+(t,x), \\ x_0'' = \lambda x_0 + \mu f_0(t,x), \\ x_-'' = Ax_- + \mu f_-(t,x), \end{cases} \tag{3.3}$$

then the system of corresponding integral equations (3.3) has the form

$$\begin{cases} x_+(t) = \dfrac{-\mu}{2\sqrt{A|_{H_+}}} \displaystyle\int_{-\infty}^{\infty} \exp\Big[-\sqrt{A|_{H_+}}\,|t-s| \Big] f_+(s, x(s))\,ds, \\ x_0(t) = \dfrac{-\mu}{2\sqrt{\lambda}} \displaystyle\int_{-\infty}^{\infty} \exp\Big[-\sqrt{\lambda}\,|t-s| \Big] f_0(s, x(s))\,ds, \\ x_-(t) = \dfrac{\mu}{\sqrt{-A|_{H_-}}} \displaystyle\int_{-\infty}^{t} \sin\Big[\sqrt{-A|_{H_-}}(t-s) \Big] f_-(s, x(s))\,ds, \end{cases} \tag{3.4}$$

(for the meaning of $A^{-1/2}$ one can see in [4], [27]). The integrals are well defined as a consequence of estimates (2.2) and assumption (iv). It is sufficient to prove that there exist solutions for system (3.4). For $\mu \in [0,1]$, define a function

$$h_\mu : BC(\mathbb{R}, H_+ \oplus H_0 \oplus H_-) \to BC(\mathbb{R}, H_+ \oplus H_0 \oplus H_-)$$

by the formula

$$h_\mu(x_+, x_0, x_-)(t) = \begin{cases} x_+(t) + \dfrac{\mu}{2\sqrt{A|_{H_+}}} \displaystyle\int_{-\infty}^{\infty} \exp\Big[-\sqrt{A|_{H_+}}\,|t-s|\Big] f_+(s, x(s))\,ds, \\ x_0(t) + \dfrac{\mu}{2\sqrt{\lambda}} \displaystyle\int_{-\infty}^{\infty} \exp\Big[-\sqrt{\lambda}\,|t-s|\Big] f_0(s, x(s))\,ds, \\ x_-(t) - \dfrac{\mu}{\sqrt{-A|_{H_-}}} \displaystyle\int_{-\infty}^{t} \sin\Big[\sqrt{-A|_{H_-}}\,(t-s)\Big] f_-(s, x(s))\,ds. \end{cases}$$

Then $h_\mu = I - \mu S_\lambda$ where $S_\lambda : BC(\mathbb{R}, H_+ \oplus H_0 \oplus H_-) \to BC(\mathbb{R}, H_+ \oplus H_0 \oplus H_-)$ is defined by the right-hand side of system (3.4) and is completely continuous according to Theorem 1, Lemmas 1 and 2.

Now, we show that $h_\mu(x_+, x_0, x_-) = 0$ has no solution for $\mu \in [0,1]$ and $x = (x_+, x_0, x_-)$ belonging to the boundary of the product of the balls $B = B(0, M_+) \times B(0, M_0) \times B(0, M_-)$, that is:

$$\begin{aligned} \partial B = \partial B(0, M_+) \times \overline{B}(0, M_0) \times \overline{B}(0, M_-) \cup \overline{B}(0, M_+) \times \partial B(0, M_0) \times \overline{B}(0, M_-) \\ \cup\, \overline{B}(0, M_+) \times \overline{B}(0, M_0) \times \partial B(0, M_-) \end{aligned}$$

where M_+, M_0, M_- are the positive constants from conditions (i)–(iii) and ε_+ is the smallest positive eigenvalue of the operator A, $-\varepsilon_-$ is the largest negative eigenvalue of A.

Suppose that there exists a solution of the equation $h_\mu(x_+, x_0, x_-) = 0$ with $\mu \in [0,1]$ and $x \in \partial B$. Then $x_+ \in \partial B(0, M_+)$ or $x_0 \in \partial B(0, M_0)$, or $x_- \in \partial B(0, M_-)$.

Consider the case when $x_+ \in \partial B(0, M_+)$. If we get $h_0(x_+, x_0, x_-) = 0$ for $\mu = 0$, then $x_+(t) \equiv 0$, which is imposible because $x_+ \in \partial B(0, M_+)$. For $\mu \in (0,1]$, let $\varphi_+ : \mathbb{R} \to \mathbb{R}$ be defined by $\varphi_+(t) = \|x_+(t)\|^2$. Then, for some $t_0 \in \mathbb{R}$, we have $\varphi_+(t_0) = M_+^2$ and at the point t_0 function φ_+ has a maximum (which can be weak). From condition (i) we get

$$\begin{aligned} \varphi_+''(t) &= 2(x_+(t), x_+''(t)) + 2\|x_+'(t)\|^2 \\ &= 2\mu(x_+(t), f_+(t, x(t))) + 2(x_+(t), Ax_+(t)) + 2\|x_+'(t)\|^2 \\ &\ge 2\varepsilon_+\varphi_+(t) + 2\mu(x_+(t), f_+(t, x(t))) + 2\|x_+'(t)\|^2 \\ &\ge 2\varepsilon_+\varphi_+(t) + 2\|x_+'(t)\|^2 \ge 2\varepsilon_+\varphi_+(t) \end{aligned}$$

for $t = t_0$. For this t_0, we have

$$\varphi_+''(t_0) \geq \varepsilon_+ M_+^2 > 0,$$

which contradits $\varphi_+''(t_0) \leq 0$.

Our reasoning is similar when proving, that the equation $h_\mu(x) = 0$ has no solution for $x_0 \in \partial B(0, M_0)$ (using condition (iii)). For $x_- \in \partial B(0, M_-)$, define the function $\varphi_- : \mathbb{R} \to \mathbb{R}$ by $\varphi_-(t) = -\|x_-(t)\|^2$. Analogously, for some $t_0 \in \mathbb{R}$, we have $\varphi_-(t_0) = -M_-^2$ and at the point t_0 function φ_- has a minimum (which can be not strong). By condition (ii) we get

$$\begin{aligned}\varphi_-''(t) &= -2(x_-(t), x_-''(t)) - 2\|x_-'(t)\|^2 \\ &= -2(x_-(t), Ax_-(t) + f_-(t, x(t))) - 2\|x_-'(t)\|^2 < -2\|x_-'(t)\|^2\end{aligned}$$

and, for $t = t_0$, we have $\varphi_-''(t) < 0$, which contradits $\varphi_-''(t_0) \geq 0$. Therefore, by the properties of the Leray–Schauder topological degree,

$$d(h_1, B, 0) = d(h_\mu, B, 0) = d(h_0, B, 0) = 1,$$

i.e. system (3.2) has a solution in B for any $\lambda > 0$.

Step 2. Now let $\lambda_n \to 0$ and let $x_n = (x_{n+}, x_{n0}, x_{n-})$ denote a bounded solution on $\mathbb{R}$ of system (3.2) with $\lambda = \lambda_n$. Let (x_n) be a bounded sequence. Then there exists a positive constant M such that $\|x_n\|_\infty \leq M$. Fix $t \in \mathbb{R}$ and $\varepsilon > 0$. We shall find a finite ε-net for the set

$$F_t^+ := \bigg\{ z_{n+}(t) = \frac{\mu}{2\sqrt{A|_{H_+}}} \int_{-\infty}^{\infty} \exp\Big[-\sqrt{A|_{H_+}}\,|t-s|\Big] f_+(s, x_n(s))\, ds : n \in \mathbb{N} \bigg\}.$$

Choose $T > 0$ such that (by asumption (6) of Theorem 1):

$$\|f_+(s, x_n)\| \leq \frac{\nu\varepsilon\sqrt{\varepsilon_+}}{2N} \tag{3.5}$$

for $|s| \geq T$. Set

$$x^0(t) = -\frac{1}{2\sqrt{A|_{H_+}}} \int_{|s|\geq T} \exp\Big[-\sqrt{A|_{H_+}}\,|t-s|\Big]\, ds.$$

Then $x^0 \in BC(\mathbb{R}, H)$. By (2.2) and (3.5)

$$\begin{aligned}&\bigg\|\frac{-1}{2\sqrt{A|_{H_+}}} \int_{|s|\geq T} \exp\Big[-\sqrt{A|_{H_+}}|t-s|\Big] f_+(s, x_n(s))\, ds\bigg\| \\ &\qquad\qquad \leq \frac{1}{2\sqrt{\varepsilon_+}} \cdot \frac{\nu\varepsilon\sqrt{\varepsilon_+}}{2N} \int_{\mathbb{R}} N e^{-\nu|t-s|}\, ds = \frac{\varepsilon}{2}.\end{aligned}$$

Consider the set

$$B^+ := \bigg\{ -\frac{1}{2\sqrt{A|_{H_+}}} \int_{|s|<T} \exp\Big[-\sqrt{A|_{H_+}}\,|t-s|\Big] f_+(s, x_n(s))\, ds : n \in \mathbb{N} \bigg\}.$$

It is easy to see that the set $\{f_+(s,x_n) : n \in \mathbb{N},\ |s| < T\}$ is relatively compact (by (5) of Theorem 1 and the continuity of f). The set

$$Z^+ := \left\{ \exp\left[-\sqrt{A|_{H_+}}\,|t-s|\right](f_+(s,x_n) : n \in \mathbb{N},\ |s| < T \right\}$$

is also relatively compact by the properties of function exp. But the integrals in B^+ belong to the convex hull $(T/\sqrt{\varepsilon_+})\overline{\mathrm{conv}}Z^+$, so there exists a finite $\varepsilon/2$-net of B^+: $x^1,\ldots,x^p$. Now, we easily see that $x^0+x^1,\ldots,x^0+x^p$ constitute an ε-net of F_t^+:

$$\begin{aligned}
&\|z_{n+}(t) - (x^0+x^j)\| \\
&\quad\le \left\| -\frac{1}{2\sqrt{A|_{H_+}}}\int_{|s|<T}\exp\left[-\sqrt{A|_{H_+}}\,|t-s|\right] f_+(s,x_n(s))\,ds - x^j\right\| \\
&\qquad + \left\| -\frac{1}{2\sqrt{A|_{H_+}}}\int_{|s|\ge T}\exp\left[-\sqrt{A|_{H_+}}\,|t-s|\right] f_+(s,x_n(s))\,ds\right\| \\
&\quad\le 2\frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence F_t^+ is relatively compact in H_+.

We can also prove that the set

$$F_t^- := \left\{ z_{n-}(t) = \frac{\mu}{\sqrt{-A|_{H_-}}}\int_{-\infty}^{t}\sin\left[\sqrt{-A|_{H_-}}\,(t-s)\right] f_-(s,x_n(s))\,ds : n\in\mathbb{N}\right\}$$

is relatively compact in H_- (using the assumption (iv) and Lemmas 1 and 2).

The set

$$F_t^0 := \{x_{n0}(t) : n \in \mathbb{N}\}$$

is relatively compact in H_0 because (x_{n0}) is a bounded sequence and the kernel of A is finite-dimensional. Then $F_t := \{x_n(t) : n \in \mathbb{N}\}$ is relatively compact in H.

The equicontinuity of the family $\{x_n|_{[-a,a]} : n \in \mathbb{N}\}$, for each $a > 0$, is the consequence of the boundedness of (x_n), (x_n') and (x_n'').

Now, by the Arzéla–Ascoli criterion, which can be applied due to the above arguments, we can choose a subsequence (x_n^1) uniformly convergent on the interval $[-1,1]$. The same reasoning can be repeated inductively to get a sequence $(x_n^k)_{n\in\mathbb{N}}$ uniformly convergent on the interval $[-k,k]$ for any $k \in \mathbb{N}$. The diagonal subsequence $(x_k^k)_{k\in\mathbb{N}}$ is, obviously, uniformly convergent on any compact subset of $\mathbb{R}$ to some function x. It is easy to see that x is a solution of equation (3.1), that is bounded on $\mathbb{R}$ by M.

Step 3. Suppose that the sequence (x_n) is unbounded. Then at least one of the sequences (x_{n+}), (x_{n0}), (x_{n-}) is unbounded.

If a sequence (x_{n+}) is unbounded, then, for some n, we have $\|x_{n+}\| > M_+$. Define the function $\varphi_+ : \mathbb{R} \to \mathbb{R}$ by $\varphi_+(t) = \|x_{n+}(t)\|^2$. From condition (i) (like in Step 1) we get

$$\begin{aligned}\varphi_+''(t) &= 2(x_{n+}(t), x_{n+}''(t)) + 2\|x_{n+}'(t)\|^2 \\ &= 2(x_{n+}(t), f_+(t, x_n(t)) + 2(x_{n+}(t), Ax_{n+}(t)) + 2\|x_{n+}'(t)\|^2 \\ &\geq 2\varepsilon_+\varphi_+(t) + 2(x_{n+}(t), f_+(t, x_n(t)) + 2\|x_{n+}'(t)\|^2 \\ &\geq 2\varepsilon_+\varphi_+(t) + 2\|x_{n+}'(t)\|^2.\end{aligned}$$

If for some $t = t_0$ such that $\varphi_+(t) > M_+^2$ function φ_+ has a maximum at that point then $\varphi_+''(t_0) \geq 0$, which contradicts the fact, that $\varphi_+''(t_0) < 0$ (from the inequalities above). If the function does not have any maximum, then it is strongly convex in the sense that $\varphi_+''(t) > 0$. Then it cannot be bounded, which contradicts the fact, that the function x_{n+} is bounded.

In this way (by condition (ii) and by the definition $\varphi_-(t) = -\|x_{n-}(t)\|^2$), we get a contradiction if we assume, that the sequence (x_{n-}) is unbounded (we get $\varphi_-''(t) < 0$). Then we get the contradiction with the fact that this function has a minimum or that it is strongly concave (in the sence that $\varphi_-''(t) < 0$).

If the sequence (x_{n0}) is unbounded, then, for some n, we have $\|x_{n0}\| > M_0$. Define the function $\varphi_0 : \mathbb{R} \to \mathbb{R}$ by $\varphi_0(t) = \|x_{n0}(t)\|^2$. From condition (iii) we get

$$\begin{aligned}\varphi_0''(t) &= 2(x_{n0}(t), x_{n0}''(t)) + 2\|x_{n0}'(t)\|^2 \\ &\geq 2\lambda\varphi_0(t) + 2(x_{n0}(t), f_0(t, x_n(t)) + 2\|x_{n0}'(t)\|^2 > 2\lambda\varphi_0(t) + 2\|x_{n0}'(t)\|^2.\end{aligned}$$

Then for some $t = t_0$ (where t_0 is the least t such that $\varphi_0(t) = M_0^2$) we have $\varphi_0''(t_0) > 0$ and $\varphi_0''(t) > 0$ for $t > t_0$. Then $\varphi_0'(t_0) \geq 0$ and φ_0 cannot be bounded. □

Theorem 4. *Let A be a bounded selfadjoint linear operator on a Hilbert space H which has finite-dimensional kernel and 0 is an isolated point of the spectrum* $\operatorname{Sp} A$. *Let $f : \mathbb{R} \times H \to H$ be a bounded continuous function satisfying conditions* (4)–(6) *of Theorem* 1 *and*

(i) $\exists M > 0\ \forall t \in \mathbb{R},\ \|x_0\| \geq M,\ x_\pm \in H_\pm\ (x_0, f_0(t, x)) > 0,$
(ii) $\forall r > 0\ \exists H_r : \mathbb{R} \to H\ \|f_-(t, x)\| \leq H_r(t) \quad \forall t \in \mathbb{R},\ \|x\| \leq r,$

where H_r is a continuous and integrable function, then the equation

$$x'' - Ax = f(t, x) \tag{3.6}$$

has a solution $x : \mathbb{R} \to H$ bounded on $\mathbb{R}$.

Remark 1. We see that for the function b from condition (6) of Theorem 1 one has $b_0 = 0$, but b_+ and b_- do not have to vanish.

Proof of Theorem 4. Equation (3.6) can be perturbed and written as system (3.2) with $\lambda > 0$. Since $\operatorname{Sp}(A|_{H_-}) \subset (-\infty, -\varepsilon_-]$, $\operatorname{Sp}(A|_{H_+}) \subset [\varepsilon_+, \infty)$, system (3.2) is equivalent to integral system (3.4) with $\mu = 1$. Then the operator S_λ defined in Step 1 of Theorem 3 is completely continuous by Theorem 1 and Lemma 1. From the boundedness of f $\|f(t,x)\| \leq R$, for some $R > 0$, and by assumption (ii)

$$\left\| \int_{-\infty}^{\infty} f_-(s, x(s))\, ds \right\| \leq W,$$

for some $W > 0$, we obtain

$$\begin{aligned}
\|(S_\lambda(x))_+(t)\| &\leq \frac{N}{2\sqrt{\varepsilon_+}} R \frac{2}{\sqrt{\varepsilon_+}} = \frac{NR}{\varepsilon_+},\\
\|(S_\lambda(x))_0(t)\| &\leq \frac{1}{2\sqrt{\lambda}} R \frac{2}{\sqrt{\lambda}} = \frac{R}{\lambda},\\
\|(S_\lambda(x))_-(t)\| &\leq \frac{W}{\sqrt{-\varepsilon_-}},
\end{aligned}$$

which means that S_λ maps the whole space into the ball $\overline{B}_\lambda$ with radius

$$\max\{NR, R, W\} \cdot \left(\frac{1}{\lambda} + \left(\frac{1}{\varepsilon_+} + \frac{1}{\sqrt{-\varepsilon_-}}\right)\right).$$

Due to Schauder Fixed Point Theorem, S_λ has a fixed point $x_\lambda \in \overline{B}_\lambda$ which is a solution of (3.2).

We cannot expect a priori that $x_\lambda : \lambda > 0$ is a bounded family because the radius of the ball $\overline{B}_\lambda$ tends to infinity as $\lambda \to 0^+$. However, $\{x_{\lambda+}\}$ and $\{x_{\lambda-}\}$ are bounded by NR/ε_+ and $W/\sqrt{-\varepsilon_-}$ respectively. The unboundedness of $\{x_{\lambda 0}\}$ contradicts assumption (i) as in the proof of Theorem 3.

The final part of the proof is the same as in Theorem 3. □

3. Examples of applications

Example 1. If $H = H_+ \oplus H_0 \oplus H_-$, $\dim H_0 < \infty$ and $A = \begin{bmatrix} \mathbb{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbb{B} \end{bmatrix}$ where, for example, $\mathbb{A} : H_+ \to H_+$ is a positive definite (i.e. $(\mathbb{A}x_+, x_+) \geq \varepsilon_+ \|x_+\|^2$) selfadjoint operator and $\mathbb{B} : H_- \to H_-$ is a negative definite (i.e. $(\mathbb{B}x_-, x_-) \leq -\varepsilon_- \|x_-\|^2$) selfadjoint operator, $h : \mathbb{R} \to H$ is a continuous and integrable function with a constant sign, satisfying the condition

(i) $\lim_{t\to\pm\infty} h(t) = 0$,

$f = (f_+, f_0, f_-) : H \to H$ is a completely continuous and bounded function satisfying

(ii) $\liminf_{\|x_0\|\to\infty} \inf_{x_+,x_-} (x_0, f_0(x_+, x_0, x_-)) > 0$

then the equation

$$x'' - Ax = h(t) f(x) \tag{4.1}$$

has a solution $x : \mathbb{R} \to H$ bounded on $\mathbb{R}$ (as a consequence of Theorem 4).

In particular, if $H = \mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ and $A = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -b \end{bmatrix}$ where $a, b > 0$ and $f = (f_1, f_2, f_3) = (f_+, f_0, f_-)$ is bounded continuous and

$$\liminf_{|x_0| \to \infty} \inf_{x_+, x_-} x_0 f_0(x_+, x_0, x_-) > 0,$$

then (4.1), i.e.

$$\begin{cases} x'_+ = ax_+ + h(t)f_+(x), \\ x'_0 = h(t)f_0(x), \\ x'_- = bx_- + h(t)f_-(x), \end{cases}$$

has a bounded solution.

Example 2. Let A, h, f be as in Example 1, but only f_- is supposed to be bounded. Instead of (ii) from Ex. 1, we assume:

(ii) $\exists M_+ > 0\ \forall t \in \mathbb{R},\ \|x_+\| \geq M_+\ (x_+, f_+(x_+, x_0, x_-)) \geq 0$,
(iii) $\exists M_- > 0\ \forall t \in \mathbb{R},\ \|x_-\| \leq M_-\ (x_-, Ax_- + h(t)f_-(x_+, x_0, x_-)) > 0$,
(iv) $\liminf_{\|x_0\| \to \infty} \inf_{x_+, x_-}\ (x_0, f_0(x_+, x_0, x_-)) > 0$.

Then (4.1) has a bounded solution (as a consequence of Theorem 3).

Remark 2. In the above examples, we deal with the nonlinearity of f having asymptotic behaviour $b = 0$. It is easy to formulate the corresponding conditions for the equation $x'' - Ax = h(t)f(x) + b(t)$ where b is bounded and continuous.

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Manuscript received November 11, 1999

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