Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 14, 1999, 327–343

ON THE ACYCLICITY OF FIXED POINT SETS OF MULTIVALUED MAPS

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Dedicated to the memory of Juliusz P. Schauder

Abstract. The paper is devoted to studying a topological structure of sets of fixed points (or equilibria) of some classes of single-valued or multivalued maps. In our opinion, this paper contains all main results connected with this problem. We deal with Browder–Gupta type results and the inverse systems approach.

1. Introduction

Well known fixed point theorems (Brouwer, Schauder and Lefschetz) give us the existence of fixed points of some classes of single-valued and multivalued maps. One knows that a topological characterization of fixed point sets is also weighty and has important applications. Namely, it performs a vital role in investigating the structure of the solution sets of differential equations and inclusions (see [30], [23], [21], [4], [7], [35], [34], [28], [5], [13], [18], and [14] for further references), when the uniqueness condition is not satisfied. It can be also applied to some existence results making use of the method described for inclusions in [2].

1991 *Mathematics Subject Classification.* Primary: 54B99, 54C60, 47H10, 54H25, 55M20; Secondary: 34A60, 46A04, 55M25.

Key words and phrases. Fixed points, multivalued maps, inverse systems, acyclicity, topological structure, limit map, topological degree, admissible maps, differential inclusions, Fréchet spaces.

The paper collects known results on the topological structure of fixed point sets of maps in metric spaces and contains some of their new multivalued generalizations. It is organized as follows.

Section 2 is devoted to studying the acyclicity (or R_δ property) of fixed point sets (or sets of equilibria) by the use of some generalizations of results which were obtained by Browder and Gupta in [7]. We give some remarks in a single-valued case and prove some multivalued generalizations. We also include some references to other papers concerning this topic.

In Section 3 we deal with the inverse systems approach, which was initiated in [3], and study the topological structure of fixed point sets of limit maps induced by multivalued maps of inverse systems of topological spaces.

Section 4 shows how the abstract results from Section 3 can be applied to function spaces. We suggest seeing [3] for applications in differential inclusions on noncompact intervals.

2. The Browder–Gupta theorem and its generalizations

In 1942 Aronszajn showed that the set of all local solutions to single-valued Cauchy problem in finite dimensional space is an R_δ set. He used the following characterization of R_δ-sets: K is compact R_δ if there is a sequence of compact absolute retracts A_n such that $K \subset A_n$ for every $n \geq 1$ and K is a limit of $\{A_n\}$ in the sense of the Hausdorff metric. In 1969 Browder and Gupta proved a theorem (see [7, Theorem 7]), giving an easy way to obtain Aronszajn's result. One can formulate this theorem as follows:

Theorem 2.1. *Let X be a metric space, E a Banach space, and let $f : X \to E$ be a (continuous) proper map*[1]. *Assume that there is a sequence of proper maps $f_k : X \to E$ such that*

(i) $||f_k(x) - f(x)|| < 1/k$ *for every* $x \in X$,
(ii) *for every* $k \geq 1$ *and every* $u \in E$ *with* $||u|| \leq 1/k$ *the equation* $f_k(x) = u$ *has a unique solution.*

Then the set $\mathcal{S} = f^{-1}(0)$ *is compact* R_δ.

Remark 2.2. The above theorem is still true under weaker assumptions (Palais–Smale type conditions) on maps f_k, f (see e.g. [28], [31], [14]). For another single-valued generalizations see e.g. [35], [34], [12] and [11].

The following lemma was the key to the proof by Browder and Gupta of the above theorem.

[1] It means that preimages of compact subsets of E are compact.

Lemma 2.3 (Browder–Gupta, [6, Lemma 5]). *Let X be a metric space, and $\{R_n\}$ be a sequence of absolute retracts in X. Assume that $M \subset X$ is such that the following hold:*

(i) *$M \subset R_n$ for every n,*
(ii) *M is the set-theoretic limit of the sequence $\{R_n\}$,*
(iii) *For each open neighbourhood V of M in X there is an infinite subsequence $\{R_{n_i}\}$ of $\{R_n\}$ such that $R_{n_i} \subset V$ for every i.*

Then M is R_δ.

We show that, under the assumption on compactness of the sets R_n, we only need that R_n are R_δ. At first, we give a clarifying remark concerning the above lemma.

Remark 2.4. Following the proof of Lemma 2.3 in [7], assumption (i) and the fact that the set-theoretic upper limit[2] of any sequence of sets is contained in the topological upper limit[3] of it, we can assume in (ii) that M is a topological limit of $\{R_n\}$.

Example 2.5. It is easy to see that the intersection M ($\emptyset \neq M \subset X$) of a decreasing sequence of closed subsets of X is its topological limit.

Proposition 2.6. *Let X be a metric space, and $\{R_n\}$ be a sequence of compact R_δ-sets in X. Assume that $M \subset X$ is such that the following conditions hold:*

(i) *$M \subset R_n$ for every n,*
(ii) *M is the topological limit of the sequence $\{R_n\}$,*
(iii) *For each open neighbourhood V of M in X there is an infinite subsequence $\{R_{n_i}\}$ of $\{R_n\}$ such that $R_{n_i} \subset V$ for every i.*

Then M is R_δ.

Proof. Let $R_n = \bigcap_{k=1}^{\infty} R_n^k$ for every $n \geq 1$, where each R_n^k is a compact absolute retract and $R_n^{k+1} \subset R_n^k$ for all $n, k \geq 1$. For every n, by the compactness of R_n^k, there exists $W_n = R_n^{k_n}$ such that $R_n^{k_n} \subset N_{1/n}(R_n)$, where $N_{1/n}(R_n)$ denotes the $1/n$-neighbourhood of the set R_n.

Notice that

(i)' $M \subset R_n \subset R_n^{k_n} = W_n$ for every n.
(ii)' From (i) it follows that $M \subset \operatorname{Li} W_n$. Let $x \in \operatorname{Ls} W_n$ be an arbitrary point. Then there are a subsequence $n_1 < n_2 < \dots$ and points $x_{n_i} \in$

[2]Let us recall that $\operatorname{Liminf} R_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} R_k$, $\operatorname{Limsup} R_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} R_k$ and, if $\operatorname{Liminf} R_n = \operatorname{Limsup} R_n$, then this set is called the set-theoretic limit $\operatorname{Limes} R_n$ of $\{R_n\}$.

[3]Recall that a lower topological limit is the set $\operatorname{Li} R_n = \{x \in X \mid \exists \{x_n\} : x_n \in R_n \text{ and } x_n \to x\}$, an upper topological limit of $\{R_n\}$ is the set $\operatorname{Ls} R_n = \{x \in X \mid \exists n_1 < n_2 < \dots : x_{n_i} \in R_{n_i} \text{ and } x_{n_i} \to x\}$ and, if $\operatorname{Li} R_n = \operatorname{Ls} R_n$, then this set is called a topological limit $\operatorname{Lim} R_n$ of $\{R_n\}$.

W_{n_i} such that $x = \lim_{i\to\infty} x_{n_i}$. By the definition of W_{n_i}, for every $i \geq 1$ there exists $z_{n_i} \in R_{n_i}$ such that $d(x_{n_i}, z_{n_i}) < 1/n_i$. Thus $z_{n_i} \to x$, which implies that $x \in \operatorname{Ls} R_n$. From assumption (ii) we obtain $\operatorname{Ls} W_n \subset M$ which gives $M = \operatorname{Lim} W_n$.

(iii)' Let U be an arbitrary open neighbourhood of M in X. By the compactness of M, there is $n_0 \geq 1$ such that $N_{2/n_0}(M) \subset U$. By assumption (iii), we can find a subsequence $\{R_{n_i}\}$, $n_0 < n_1 < n_2 < \ldots$, such that $R_{n_i} \subset N_{1/n_i}(M)$, for every $i \geq 1$. Since $W_{n_i} \subset N_{1/n_i}(R_{n_i})$, one can easily obtain that $W_{n_i} \subset N_{2/n_0}(M) \subset U$.

The Browder–Gupta Lemma 2.3 and Remark 2.4 end the proof. □

Among several remarks on Browder–Gupta results the following should be added.

Remark 2.7. In the case when X is a metric absolute neighbourhood retract ($X \in$ ANR), Theorem 2.1 can be proved without using Lemma 2.3. Instead of it we can use the characterization below (Lemma 2.9) of R_δ-sets due to Hyman (see [22]). Although this case is not so general, it seems to be sufficient for applications.

Lemma 2.8 ([26]). *Let X, Y be metric spaces. If $\varphi : X \multimap Y$ is an u.s.c.*[4]*, proper*[5] *map, then the map $\psi = \varphi_+^{-1} : \varphi(X) \multimap X$ is u.s.c. with compact values. Especially, for every $y \in \varphi(X)$ and every open neighbourhood U of $\varphi_+^{-1}(y)$ in X there exists $\eta > 0$ such that $\varphi_+^{-1}(N_\eta(y)) \subset U$.*

Lemma 2.9 ([22]). *Let M be a compact subset of $Y \in$* ANR. *Then the following conditions are equivalent:*

(i) *M is contractible in every open neighbourhood in Y;*

(ii) *M is R_δ.*

Alternative proof of Theorem 2.1 with $X \in$ ANR. In a standard way one can check that $\mathcal{S}$ is compact and nonempty. Define $Y_k = f_k^{-1}(B(0,1/k))$, where $B(0,1/k)$ stands for the ball in E with center at 0 and radius $1/k$. It is easily seen that, for every k, $\mathcal{S} \subset Y_k$ and the restriction $f_k : Y_k \to B(0,1/k)$ is a homeomorphism. This implies that each Y_k is a contractible set. Moreover, $Y_{2k} \subset f^{-1}(B(0,1/k))$.

Let U be an arbitrary open neighbourhood of $\mathcal{S}$ in X. From Lemma 2.8 it follows that there is $k \geq 1$ such that $f^{-1}(B(0,1/k)) \subset U$. Thus $\mathcal{S} \subset Y_{2k} \subset f^{-1}(B(0,1/k)) \subset U$. Since Y_{2k} is contractible, we obtain that $\mathcal{S}$ is contractible in U. Lemma 2.9 implies that $\mathcal{S}$ is R_δ. □

[4] A multivalued map $\varphi : X \multimap Y$ is *upper semicontinuous* (*u.s.c.*) if $\varphi^{-1}(U) = \{x \in X \mid \varphi(x) \subset U\}$ is open in X, for every open subset U of Y.

[5] We say that a multivalued map $\varphi : \multimap Y$ is *proper* if the preimage $\varphi_+^{-1}(A) = \{x \in X \mid \varphi(x) \cap A \neq \emptyset\}$ is compact, for every compact subset A of Y.

There are possible multivalued generalizations of Theorem 2.1. We describe this below. All the results (cf. Theorems 2.16, 2.19, 2.20) imply the classical Krasnosel'skiĭ–Perov theorem, (see [24] or [25]). At first we recall some notions.

We say that a multivalued map $\varphi : X \multimap Y$ is *admissible* (in the sense of Górniewicz, [17]), if there are a metric space Z and a pair of single-valued continuous maps $X \xleftarrow{p} Z \xrightarrow{q} Y$ (a *selected pair* of φ) such that p is a Vietoris map[6] and $q(p^{-1}(x)) \subset \varphi(x)$, for every $x \in X$. Every admissible map induces the set $\{\varphi\}^*$ of linear maps from $H^*(Y)$ to $H^*(X)$, where H^* stands for the Čech–Alexander cohomology, $\{\varphi\}^* = \{(p^{-1})^* q^* \mid (p,q) \text{ is a selected pair of } \varphi\}$. A multivalued map $H : X \times [0,1] \multimap Y$ is called an *admissible homotopy* joining φ and ψ, if H is an admissible map and $H(x,0) \subset \varphi(x), H(x,1) \subset \psi(x)$, for every $x \in X$.

We will use the following two lemmas.

Lemma 2.10. *Let K be a compact subset of a space X. If, for every open neighbourhood U of K in X, there exists an acyclic set Z such that $K \subset Z \subset U$, then K is acyclic.*

The proof is an immediate consequence of the continuity of the cohomology functor.

Lemma 2.11. *Let Y be an acyclic space, and $\varphi : X \multimap Y$ be an u.s.c. map with compact acyclic values* (*this implies that φ is admissible*) *and such that in the selected pair $X \xleftarrow{p} \Gamma_\varphi \xrightarrow{q} Y$ the map q is a Vietoris map. Then X is acyclic.*

The assertion immediately follows from the fact that

$$(p^{-1})^* q^* : H^*(Y) \to H^*(X)$$

is an isomorphism.

Corollary 2.12. *Let Y be an acyclic space, and $\varphi : X \multimap Y$ be an u.s.c. surjective map with compact acyclic values and such that $\varphi_+^{-1}(y)$ is acyclic, for every $y \in Y$. Then X is acyclic.*

Using basic properties of admissible maps one can also easily obtain the following lemma.

Lemma 2.13. *Let A be a compact subset of a metric space X, and $H : A \times [0,1] \multimap X$ be an admissible homotopy. Assume that for every $\varepsilon > 0$ there exists $n_0 \geq 1$ such that, for every $n \geq n_0$,*

(i) *$H_n(A \times [0,1]) \subset N_\varepsilon(A)$,*
(ii) *$i_n \subset H_n(0, \cdot)$, where $i_n : A \to N_\varepsilon(A)$ is the inclusion map,*

[6] e.i., p is a proper surjection such that $p^{-1}(x)$ is acyclic, for every $x \in X$.

(iii) $a_n \subset H_n(1,\cdot)$, *where* $a_n : A \to N_\varepsilon(A)$ *is a constant map,*
(iv) $\{H_n(0,\cdot)\}^*$ *and* $\{H_n(1,\cdot)\}^*$ *are singletons.*

Then A *is acyclic.*

Proof. It is obvious that for each $\varepsilon > 0$ and $n \geq n_0$ we have $\{H_n(0,\cdot)\}^* = i_n^* = a_n^* = \{H_n(1,\cdot)\}^* = 0$. By the continuity of the theory of cohomology, we can easily obtain that all the reduced cohomology groups of A are trivial. □

Remark 2.14. From the above lemma one can prove Lemma 4.2 in [16] and, consequently, all the results in [16] on acyclicity of solution sets of functional inclusions. We note one of them (cf. Theorem 2.19).

Remark 2.15. Let Y be a contractible subset of a Fréchet space, and $\varphi : X \multimap Y$ be a u.s.c. surjective map with compact convex values and such that $\varphi_+^{-1}(y)$ is acyclic, for every $y \in Y$. Then one can check that the following multivalued homotopy $H : X \times [0,1] \multimap X$, $H(x,t) = \varphi_+^{-1}(h(\varphi(x) \times \{t\}))$, where h is a homotopy contracting Y to a point, is an admissible homotopy satisfying assumption (iv) of Lemma 2.13.

We are in a position to prove a multivalued generalization of Theorem 2.1.

Theorem 2.16. *Let* X *be a metric space,* E *a Fréchet space,* $\{U_k\}$ *a base of open convex symmetric neighbourhoods of the origin in* E*, and let* $\varphi : X \multimap E$ *be an u.s.c. proper map with compact values. Assume that there is a sequence of compact convex valued u.s.c. proper maps* $\varphi_k : X \to E$ *such that*

(i) $\varphi_k(x) \subset \varphi(N_{1/k}(x)) + U_k$, *for every* $x \in X$,
(ii) *if* $0 \in \varphi(x)$, *then* $\varphi_k(x) \cap \overline{U_k} \neq \emptyset$,
(iii) *for every* $k \geq 1$ *and every* $u \in E$ *with* $u \in U_k$ *the inclusion* $u \in \varphi_k(x)$ *has an acyclic set of solutions.*

Then the set $\mathcal{S} = \varphi^{-1}(0)$ *is compact and acyclic.*

Proof. We show that $\mathcal{S}$ is nonempty. To this end, notice that for every $k \geq 1$ we can find $x_k \in X$ such that $0 \in \varphi_k(x_k)$. Assumption (i) implies that there are $z_k \in N_{1/k}(x_k)$, $y_k \in \varphi_k(z_k)$ and $u_k \in U_k$ such that $0 = y_k + u_k$. Thus $y_k \to 0$. Consider the compact set $K = \{y_k\} \cup \{0\}$. Since φ is proper, the set $\varphi_+^{-1}(K)$ is compact. Moreover, $\{z_k\} \subset \varphi_+^{-1}(K)$. Thus we can assume, without loss of generality, that $\{z_k\}$ converges to some point $x \in X$. By the upper semicontinuity of φ, we have $0 \in \varphi(x)$ and, what follows, $\mathcal{S} \neq \emptyset$.

Since φ is proper, the set $\mathcal{S}$ is compact. We show that it is acyclic.

By assumption (ii), the set $A_k = \varphi_{k+}^{-1}(\overline{U_k})$ is nonempty. Consider the map $\psi_k : A_k \multimap \overline{U_k}$, $\psi_k(x) = \varphi_k(x) \cap \overline{U_k}$. Since $\overline{U_k}$ is contractible and ψ_k is u.s.c. convex valued surjection (see (iii)), we can apply Corollary 2.12 to obtain that A_k is acyclic.

Now we show that for every open neighbourhood U of $\mathcal{S}$ in X there exists $k \geq 1$ such that $A_k \subset U$. Indeed, assume on the contrary that there is an open neighbourhood U of $\mathcal{S}$ in X such that $A_k \not\subset U$, for every $k \geq 1$. It means that there are $x_k \in A_k$ with $x_k \notin U$ and, consequently, there are $y_k \in \varphi_k(x_k)$ such that $y_k \in \overline{U_k}$. Assumption (i) implies that there are $z_k \in N_{1/k}(x_k)$, $v_k \in \varphi_k(z_k)$ and $u_k \in U_k$ such that $y_k = v_k + u_k$. Therefore, $v_k = y_k - u_k \in 2U_k$ which implies that $v_k \to 0$. Consider the compact set $K_0 = \{v_k\} \cup \{0\}$. Since φ is proper, we can assume that $\{z_k\}$ and, consequently, $\{x_k\}$ converges to some point $x \in X$. Thus $x \in \mathcal{S}$. On the other hand, $x \notin U$, a contradiction.

Using Lemma 2.10 we obtain that $\mathcal{S}$ is acyclic. □

Remark 2.17. It is easy to see that in the above result we can assume that X is a subset of a Fréchet space. Then, instead of ε-neighbourhoods, we can consider the sets $x + V_k$, where $\{V_k\}$ is the base of open convex symmetric neighbourhoods of the origin.

As a consequence of Theorem 2.16 and properties of a topological degree of u.s.c. compact convex valued maps (see e.g. [29]) one can obtain the following theorem generalizing result of Czarnowski in [11].

Theorem 2.18. *Let Ω be an open subset of a Fréchet space E, $\{U_k\}$ the base of open convex symmetric neighbourhoods of the origin in E, and $\Phi : \overline{\Omega} \multimap E$ a compact u.s.c. map with compact convex values. Suppose that $x \notin \Phi(x)$ for every $x \in \partial\Omega$, and $\operatorname{Deg}(j - \Phi, \Omega, 0) \neq 0$, where $j : \overline{\Omega} \to E$ is an inclusion. Assume that there exists a sequence $\{\Phi_k : \Omega \multimap E\}$ of compact u.s.c. maps with compact convex values such that*

- (i) *$\Phi_k(x) \subset \Phi(x + U_k) + U_k$, for every $x \in \overline{\Omega}$,*
- (ii) *if $x \in \Phi(x)$, then $x \in \Phi_k(x) + U_k$,*
- (iii) *for every $u \in \overline{U_k}$ the set $\mathcal{S}^k_u$ of all solutions to the inclusion $x - \Phi_k(x) \ni u$ is acyclic or empty, for every $n > 0$.*

Then the fixed point set $\operatorname{Fix}(\Phi)$ *of Φ is compact and acyclic.*

Proof. Define the maps $\varphi, \varphi_k : \overline{\Omega} \multimap E$, $\varphi = j - \Phi$, $\varphi_k = j - \Phi_k$. One can check that φ, φ_k are proper maps. To apply Theorem 2.16, it is sufficient to show that, for sufficiently big k and for every $u \in \overline{U_k}$, the set $\mathcal{S}^k_u$ is nonempty.

For each $k \geq 1$ define the map $\Psi_k : \overline{\Omega} \multimap E$, $\Psi_k(x) = \Phi_k(x) + u$, for every $x \in \overline{\Omega}$. We prove that, for sufficiently big k, $\operatorname{Deg}(j - \Psi_k, \Omega, 0) \neq 0$ which implies, by the existence property of a degree, a nonemptiness of $\mathcal{S}^k_u$.

Since φ is a closed[7] map (see e.g. [29]), we can find, for sufficiently big k, a neighbourhood U_k of the origin such that $\varphi(\partial\Omega) \cap \overline{U_k} = \emptyset$.

[7]i.e., for every closed subset $A \subset \overline{\Omega}$ the set $\varphi(A)$ is closed in E.

Consider the following homotopy $H_k : \overline{\Omega} \times [0,1] \multimap E$, $H(x,t) = (1-t)\Phi(x) + t\Psi_k(x)$. We show that

$$Z_k = \{x \in \partial\Omega \mid x \in H_k(x,t) \text{ for some } t \in [0,1]\} = \emptyset,$$

for sufficiently big k. Suppose, on the contrary, that there are a subsequence of $\{H_k\}$ (we denote it also by $\{H_k\}$), points $x_k \in \partial\Omega$ and numbers $t_k \in [0,1]$ such that $x_k \in H_k(x_k, t_k)$, that is $x_k = (1-t_k)y_k + t_k s_k + t_k u$, for some $y_k \in \Phi(x_k)$ and $s_k \in \Phi_k(x_k)$. Assumption (i) implies that there are $z_k \in x_k + U_k$ and $v_k \in \Phi(z_k)$ such that $s_k \in v_k + U_k$. By the compactness of Φ, we can assume that $y_k \to y$ and $v_k \to v$. Therefore $s_k \to v$. Moreover, we can assume that $t_k \to t \in [0,1]$. This implies that $x_k \to x_0 = (1-t)y + tv + tu$ or, equivalently, that $0 = (1-t)(x_0 - y) + t(x_0 - v) - tu$. But by the upper semicontinuity of φ, we obtain that $x_0 - y \in \varphi(x_0)$ and $x_0 - v \in \varphi(x_0)$. Since φ is convex valued, $0 \in (1-t)\varphi(x_0) + t\varphi(x_0) - tu \subset \varphi(x_0) - tu$. This implies that $\varphi(x_0) \cap \overline{U_k} \neq \emptyset$, a contradiction.

Now, by the homotopy property of a topological degree, one obtains

$$\operatorname{Deg}(\Psi_k, \Omega, 0) = \operatorname{Deg}(\Phi, \Omega, 0) \neq 0,$$

which ends the proof of the theorem. □

For comparison, the following result, which is a consequence of Lemma 2.13, has been obtained in [16].

Theorem 2.19 ([16, Theorem 4.6]). *Let E be a Banach space, U an open bounded subset of E, and $F : \overline{U} \multimap E$ a compact u.s.c. map with compact convex values satisfying the following conditions:*

(i) $\operatorname{Deg}(j - F, U, 0) \neq 0$,
(ii) *for every $\varepsilon > 0$ there is a compact ε-approximation*[8] $f_\varepsilon : \overline{U} \to E$ *of F such that $\|x - f_\varepsilon(x)\| \leq \varepsilon$, for any $x \in \operatorname{Fix}(F)$,*
(iii) *there is a unique solution of $x = f_\varepsilon(x) + v$, for every $\varepsilon > 0$ and $\|v\| \leq \varepsilon$.*

Then the set $\operatorname{Fix}(F)$ *is acyclic.*

To enrich the list of results concerning the topological structure of fixed point sets of multivalued maps, it is worthwhile adding the following (see [8, Corollary 2.6)].

Theorem 2.20. *Let Y be a normed space, B its bounded closed subset, and $F : B \multimap Y$ a compact u.s.c. map with compact acyclic values. Assume that, for every $n \geq 1$, there exists a compact map $g_n : B \to Y$ such that $\|g_n(y)\| \leq 1/n$.*

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[8] We say that $f : X \to Y$ is an ε-approximation of a multivalued map $F : X \multimap Y$ if the graph of f is contained in an ε-neighbourhood of the graph of F in $X \times Y$.

If, for any $n \geq 1$ *and* $||z|| \leq 1/n$, *the set* $\{y \in B \mid y \in F(y) + g_n(y) + z\}$ *is acyclic, then* $\mathrm{Fix}(F)$ *is acyclic.*

Finally, we will show that in Theorem 2.16 one can weaken the assumption on a regularity of maps instead of strengthening a connection between φ and φ_k.

Theorem 2.21. *Let* X *be a metric space,* E *a Fréchet space,* $\{U_k\}$ *a base of open convex symmetric neighbourhoods of the origin in* E, *and let* $\varphi : X \multimap E$ *be an u.s.c. proper map with compact values. Assume that there is a sequence of compact valued u.s.c. proper maps* $\varphi_k : X \to E$ *such that*

(i) $\varphi_k(x) \subset \varphi(N_{1/k}(x)) + U_k$, *for every* $x \in X$,
(ii) $\varphi(x) \subset \varphi_k(x)$, *for every* $x \in X$,
(iii) *for every* $k \geq 1$ *the set* $\mathcal{S}_k = \varphi_{k+}^{-1}(0)$ *is acyclic.*

Then the set $\mathcal{S} = \varphi^{-1}(0)$ *is compact and acyclic.*

Proof. The main idea of the proof is similar to the one in Theorem 2.4, [8]. At first, one can easily prove that $\mathcal{S}$ is nonempty and compact (cf. the proof of Theorem 2.16).

Consider, for every $n \geq 1$, the following open sets $W_n = N_{2/n}(\mathcal{S})$ and $V_n = N_{1/n}(\mathcal{S})$. We show that there is $k_0 = k_0(n) \geq 1$ such that $1/k_0 < 1/n$ and $\varphi(x) \cap U_k = \emptyset$, for every $k \geq k_0$ and $x \in X \setminus V_n$.

Suppose, on the contrary, that there are sequences $k_1 < k_2 < \ldots$ and $\{x_{k_i}\} \subset X \setminus V_n$ such that $\varphi(x_{k_i}) \cap U_{k_i} \neq \emptyset$. This implies that there is a sequence $\{y_{k_i}\} \subset E$ such that $y_{k_i} \in \varphi(x_{k_i}) \cap U_{k_i}$. Thus $y_{k_i} \to 0$. Consider the set $K = \{y_{k_i}\} \cup \{0\}$ and take its preimage $\varphi_+^{-1}(K)$. Since $\{x_{k_i}\} \subset \varphi_+^{-1}(K)$ and φ is proper, we can assume that $x_{k_i} \to x$, for some $x \in X \setminus V_n$. On the other hand, since φ is u.s.c., it follows that $0 \in \varphi(x)$, which means that $x \in \mathcal{S}$, a contradiction.

Notice that $\mathcal{S}_k \subset W_n$, for every $k \geq k_0$. Indeed, suppose that there exists $x \notin W_n$ such that $x \in \mathcal{S}_k$ for some $k \geq k_0$. Then $0 \in \varphi_k(x) \subset \varphi(N_{1/k}(x)) + U_k$, which implies that there exist $z_k \in N_{1/k}(x)$ and $y_k \in \varphi(z_k)$ such that $y_k \in U_k$. But $z_k \in N_{1/k}(X \setminus W_n) \subset X \setminus V_n$ and hence, $\varphi(z_k) \cap U_k = \emptyset$, a contradiction.

Thus we have $\mathcal{S}_k \subset W_n$. Assumption (ii) implies that $\mathcal{S} \subset \mathcal{S}_k$, for each $k \geq 1$. Using Lemma 2.10 we obtain that $\mathcal{S}$ is acyclic, as required. □

3. Topological structure of fixed point sets of limit maps

In 1987 B. Ricceri [32] showed that if X is a nonempty, convex and closed subset of a Banach space E and $\varphi : E \multimap E$ is a contractive multivalued map with convex closed values, then the fixed point set $\mathrm{Fix}(\varphi)$ is an absolute retract. Some generalizations for maps with decomposable values in the space of integrable functions were also obtained (see [6]). The most general results for multivalued contractions have been proved in [20] and [19] (cf. Theorem 3.10).

An attempt to generalize the results mentioned above to the case of Fréchet spaces brings some troubles because of the topology of such spaces. Namely, they arise in checking a contractivity of operators. Even an operator which is a contraction in every seminorm with the same constant of contractivity, may not be a contraction with respect to a metric in the space (cf. Example 4.3).

In [3] authors presented a technique which allows us to overcome these troubles by the use of inverse systems of topological spaces and by studying a topological structure of fixed point sets of limit maps induced by maps of these systems.

Recall that by an *inverse system* of topological spaces we mean a family $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$, where Σ is a set directed by the relation $\leq$, X_α is a topological (Hausdorff) space for every $\alpha \in \Sigma$ and $\pi_\alpha^\beta : X_\alpha \to X_\beta$ is a continuous mapping for each two elements $\alpha, \beta \in \Sigma$ such that $\alpha \leq \beta$. Moreover, for each $\alpha \leq \beta \leq \gamma$ the following conditions should hold: $\pi_\alpha^\alpha = id_{X_\alpha}$ and $\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$.

A subspace of the product $\prod_{\alpha\in\Sigma} X_\alpha$ is called a *limit of the inverse system* S and it is denoted by $\varprojlim S$ or $\varprojlim \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ if

$$\varprojlim S = \left\{ (x_\alpha) \in \prod_{\alpha\in\Sigma} X_\alpha \,\middle|\, \pi_\alpha^\beta(x_\beta) = x_\alpha \text{ for all } \alpha \leq \beta \right\}.$$

An element of $\varprojlim S$ is called a *thread* or a *fibre* of the system S. One can see that if we denote by $\pi_\alpha : \varprojlim S \to X_\alpha$ a restriction of the projection $p_\alpha : \prod_{\alpha\in\Sigma} X_\alpha \to X_\alpha$ onto the α-th axis, then we obtain $\pi_\alpha = \pi_\alpha^\beta \pi_\beta$ for each $\alpha \leq \beta$.

Now we summarize some useful properties of limits of inverse systems.

Proposition 3.1 ([14]). *Let $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ be an inverse system.*

(3.1.1) *The limit $\varprojlim S$ is a closed subset of $\prod_{\alpha\in\Sigma} X_\alpha$.*

(3.1.2) *If, for every $\alpha \in \Sigma$, X_α is*

- (i) *compact, then $\varprojlim S$ is compact;*
- (ii) *compact and nonempty, then $\varprojlim S$ is compact and nonempty;*
- (iii) *a continuum, then $\varprojlim S$ is a continuum;*
- (iv) *compact and acyclic*[9]*, then $\varprojlim S$ is compact and acyclic;*
- (v) *metrizable, Σ is countable, and $\varprojlim S$ is nonempty, then $\varprojlim S$ is metrizable.*

The following further information is useful for applications.

Proposition 3.2. *Let $S = \{X_n, \pi_n^p, \mathbb{N}\}$ be an inverse system. If each X_n is a compact R_δ-set, then $\varprojlim S$ is R_δ, too.*

[9] with respect to any continuous theory of cohomology.

Proof. The assertion follows from Example 2.5 and Proposition 2.6. Indeed, define

$$Q_n = \left\{ (x_i) \in \prod_{i=1}^{\infty} X_i \;\middle|\; x_i = \pi_i^n(x_n) \text{ for } i \leq n \right\}.$$

It is easy to see that each Q_n is homeomorphic to the R_δ-set $\prod_{i=n}^{\infty} X_i$. Notice that

$$\bigcap_{n=1}^{\infty} Q_n = \left\{ (x_i) \in \prod_{i=1}^{\infty} X_n \;\middle|\; x_i = \pi_i^n(x_n) \text{ for every } n \geq 1 \text{ and } i \leq n \right\} = \varprojlim S.$$

This implies (comp. Example 2.5) that $\varprojlim S = \operatorname{Lim}\ Q_n$, and by Proposition 2.6, it is an R_δ-set, as required. □

The following example shows that a limit of an inverse system of absolute retracts does not have to be an absolute retract.

Example 3.3. Consider a family $\{X_n\}_{n=1}^{\infty}$ of subsets of $\mathbb{R}^2$ defined as follows:

$$X_n = ([0, 1/n\pi] \times [-1, 1]) \cup \{(x, y) \mid y = \sin 1/x \text{ and } 1/n\pi < x \leq 1\}.$$

One can see that for each $m, n \geq 1$ such that $m \geq n$ we have $X_m \subset X_n$.

Define the maps $\pi_n^m : X_m \to X_n$, $\pi_n^m(x) = x$. Therefore $S = \{X_n, \pi_n^m, \mathbb{N}\}$ is an inverse system of compact absolute retracts. It is evident that $\varprojlim S$ is homeomorphic to the intersection of all X_n. On the other hand

$$X = \bigcap_{n=1}^{\infty} X_n = \{(0, y) \mid y \in [-1, 1]\} \cup \{(x, y) \mid y = \sin 1/x \text{ and } 0 < x \leq 1\},$$

and X is not an absolute retract since, for instance, X is not locally connected.

Note that in [1] the following information on a limit of an inverse system of absolute retracts has been given.

Proposition 3.4. *Let $S = \{X_n, \pi_n^p, \mathbb{N}\}$ be an inverse system of compact absolute retracts such that $X_n \subset X_p$ and π_n^p be a retraction for all $n \leq p$. Then $\varprojlim S$ has the fixed point property, i.e. every continuous map $f : \varprojlim S \to \varprojlim S$ has a fixed point.*

Example 3.5. Consider the inverse system $S = \{X_n, \pi_n^p, \mathbb{N}\}$ such that $X_n = [n, \infty)$ and $\pi_n^p : X_p \hookrightarrow X_n$ are inclusion maps for $n \leq p$. It is obvious that $\varprojlim S$ is homeomorphic to the intersection of all X_n which is an empty set.

This shows that the compactness assumption in Proposition 3.1.2 is important in obtaining a nonemptiness of the limit $\varprojlim S$.

Now we introduce the notion of multivalued maps of inverse systems. Suppose that two systems $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ and $S' = \{Y_{\alpha'}, \pi_{\alpha'}^{\beta'}, \Sigma'\}$ are given.

Definition 3.6. By a *multivalued map of the system S into the system S'* we mean a family $\{\sigma,\varphi_{\sigma(\alpha')}\}$ consisting of a monotone function $\sigma:\Sigma'\to\Sigma$, that is $\sigma(\alpha')\le\sigma(\beta')$ for $\alpha'\le\beta'$, and of multivalued maps $\varphi_{\sigma(\alpha')}:X_{\sigma(\alpha')}\multimap Y_{\alpha'}$ with nonempty values, defined for every $\alpha'\in\Sigma'$ and such that

$$\pi^{\beta'}_{\alpha'}\varphi_{\sigma(\beta')}=\varphi_{\sigma(\alpha')}\pi^{\sigma(\beta')}_{\sigma(\alpha')}, \tag{1}$$

for each $\alpha'\le\beta'$. A map of systems $\{\sigma,\varphi_{\sigma(\alpha')}\}$ induces a *limit map* $\varphi:\varprojlim S\multimap\varprojlim S'$ defined as follows:

$$\varphi(x)=\prod_{\alpha'\in\Sigma}\varphi_{\sigma(\alpha')}(x_{\sigma(\alpha')})\cap\varprojlim S'.$$

In other words, a limit map is a map such that, for every $\alpha'\in\Sigma'$,

$$\pi_{\alpha'}\varphi=\varphi_{\sigma(\alpha')}\pi_{\sigma(\alpha')}. \tag{2}$$

Since a topology of a limit of an inverse system is the one generated by the base consisting of all sets of the form $\pi_\alpha(U_\alpha)$, where α runs over an arbitrary set cofinal in Σ and U_α are open subsets of the space X_α, it is easy to prove the following continuity property for limit maps:

Proposition 3.7 ([3, Proposition 2.7]). *Let $S=\{X_\alpha,\pi^\beta_\alpha,\Sigma\}$ and $S'=\{Y_{\alpha'},\pi^{\beta'}_{\alpha'},\Sigma'\}$ be two inverse systems, and $\varphi:\varprojlim S\multimap\varprojlim S'$ be a limit map induced by the map $\{\sigma,\varphi_{\sigma(\alpha')}\}$. If, for every $\alpha'\in\Sigma'$, $\varphi_{\sigma(\alpha')}$ is*

(i) *u.s.c. with compact values, then φ is u.s.c.,*
(ii) *l.s.c.[10], then φ is l.s.c.;*
(iii) *continuous[11], then φ is continuous.*

The following crucial result allows us to study a topological structure of fixed point sets of limit maps.

Theorem 3.8. *Let $S=\{X_\alpha,\pi^\beta_\alpha,\Sigma\}$ be an inverse system, and $\varphi:\varprojlim S\multimap\varprojlim S$ be a limit map induced by a map $\{\mathrm{id},\varphi_\alpha\}$, where $\varphi_\alpha:X_\alpha\multimap X_\alpha$. If fixed point sets of φ_α are compact acyclic, then the fixed point set of φ is compact acyclic, too.*

Proof. Denote by $\mathcal{F}_\alpha$ the fixed point set of φ_α, for every $\alpha\in\Sigma$, and by $\mathcal{F}$ the fixed point set of φ. We will show that $\pi^\beta_\alpha(\mathcal{F}_\beta)\subset\mathcal{F}_\alpha$.

Let $x_\beta\in\mathcal{F}_\beta$. Then $x_\beta\in\varphi_\beta(x_\beta)$ and $\pi^\beta_\alpha(x_\beta)\in\pi^\beta_\alpha\varphi_\beta(x_\beta)\subset\varphi_\alpha\pi^\beta_\alpha(x_\beta)$, which implies that $\pi^\beta_\alpha(x_\beta)\in\mathcal{F}_\alpha$.

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[10] A multivalued map $\varphi:X\multimap Y$ is *lower semicontinuous* (*l.s.c.*) if $\varphi^{-1}_+(U)=\{x\in X\mid\varphi(x)\cap U\ne\emptyset\}$ is open in X, for every open subset U of Y.

[11] A multivalued map $\varphi:X\multimap Y$ is continuous if φ is u.s.c. and l.s.c.

Similarly we show that $\pi_\alpha(\mathcal{F}) \subset \mathcal{F}_\alpha$. Denote by $\overline{\pi}_\alpha^\beta : \mathcal{F}_\beta \to \mathcal{F}_\alpha$ the restriction of π_α^β. One can see that $\overline{S} = \{\mathcal{F}_\alpha, \overline{\pi}_\alpha^\beta, \Sigma\}$ is an inverse system.

By Proposition 3.1, the set $\mathcal{F}$ is acyclic and the proof is complete. □

By Proposition 3.2 and the above proof we immediately obtain:

Theorem 3.9. *Let $S = \{X_n, \pi_n^p, \mathbb{N}\}$ be an iverse system, and $\varphi : \varprojlim S \multimap \varprojlim S$ be a limit map induced by a map $\{\mathrm{id}, \varphi_n\}$, where $\varphi_n : X_n \multimap X_n$. If fixed point sets of φ_n are compact R_δ, then the fixed point set of φ is R_δ, too.*

Following [19] recall that a lower semicontinuous map $\varphi : X \multimap X$, where X is a metric space, has the *selection property with respect to a subclass $\mathcal{D}$* of the class of metric spaces, if, for any $Y \in \mathcal{D}$, any pair of continuous functions $f : Y \to X$ and $h : Y \to (0, \infty)$ such that

$$\psi(y) = \mathrm{cl}[\varphi(f(y)) \cap N_{h(y)}(f(y))] \neq \emptyset, \quad y \in Y,$$

and any nonempty closed set $Y_0 \subset Y$, every continuous selection of $\psi|_{Y_0}$ admits a continuous extension g over Y fulfilling $g(y) \in \psi(y)$ for all $y \in Y$. If $\mathcal{D}$ is a class of all metric spaces, then we say that φ has the *selection property* ($\varphi \in SP(X)$).

Note that, for example, every closed convex valued l.s.c. map from a Fréchet space E into itself (and more generally, with values in any Michael family of subsets of E) has the selection property. Moreover, if X is a closed subset of $L^1(T, E)$, where E is a Banach space and $\varphi : X \multimap X$ is a l.s.c. map with closed decomposable values, then $\varphi \in SP(X)$.

Theorem 3.10 ([19, Theorem 3.1]). *Let X be a complete absolute retract and let $\varphi : X \multimap X$ be a multivalued contraction, i.e. a Lipschitz*[12] *map with a constant $0 \leq k < 1$. Suppose that $\varphi \in SP(X)$. Then the set $\mathrm{Fix}(\varphi)$ is a complete absolute retract.*

The above result gives us the following applications.

Corollary 3.11. *Let $S = \{X_n, \pi_n^p, \mathbb{N}\}$ be an iverse system, and $\varphi : \varprojlim S \multimap \varprojlim S$ be a limit map induced by a map $\{\mathrm{id}, \varphi_n\}$, where $\varphi_n : X_n \multimap X_n$. If all X_n are complete absolute retracts and all φ_n are compact valued contractions having the selection property, then $\mathrm{Fix}(\varphi)$ is compact R_δ.*

Proof. By Theorem 3.10, all the fixed point sets $\mathcal{F}_n$ of φ_n are absolute retracts. Since every map φ_n has compact values, Theorem 1 in [33] implies the compactness of $\mathcal{F}_n$. Therefore our assertion follows from Proposition 3.2. □

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[12] A multivalued map $\varphi : X \multimap Y$ is a *Lipschitz map*, if there exists a constant $k \geq 0$ such that $d_H(\varphi(x), \varphi(y)) \leq kd(x, y)$, for every $x, y \in X$, where d_H stands for the Hausdorff distance.

Corollary 3.12. *Let* $S = \{X_n, \pi^p_n, \mathbb{N}\}$ *be an iverse system, and* $\varphi : \varprojlim S \multimap \varprojlim S$ *be a limit map induced by a map* $\{\mathrm{id}, \varphi_n\}$*, where* $\varphi_n : X_n \multimap X_n$*. If all* X_n *are Fréchet spaces and all* φ_n *are contractions with convex compact values, then* $\mathrm{Fix}(\varphi)$ *is compact* R_δ*.*

4. Applications to function spaces

We start with the following important examples of inverse systems.

Example 4.1. Let, for every $m \in \mathbb{N}$, $C_m = C([0,m], \mathbb{R}^n)$ be a Banach space of all continuous functions of the closed interval $[0,m]$ into $\mathbb{R}^n$ with the usual sup norm, and $C = C([0,\infty), \mathbb{R}^n)$ be an analogous Fréchet space of continuous functions with the family of seminorms $p_m(x) = \sup\{|x(t)| \mid t \in [0,m]\}$.

Consider the maps $\pi^p_m : C_p \to C_m$, $\pi^p_m(x) = x|_{[0,m]}$. It is easy to see that C is isometrically homeomorphic to a limit of the inverse system $\{C_m, \pi^p_m, \mathbb{N}\}$. The maps $\pi_m : C \to C_m$, $\pi_m(x) = x|_{[0,m]}$ correspond with suitable projections.

Remark 4.2. In the same manner as above we can show that Fréchet spaces $C(J, \mathbb{R}^n)$, where J is an arbitrary interval, $L^1_{\mathrm{loc}}(J, \mathbb{R}^n)$ of all locally integrable functions, $AC_{\mathrm{loc}}(J, \mathbb{R}^n)$ of all locally absolutely continuous functions and $C^k(J, \mathbb{R}^n)$ of all continuously differentiable functions up to the order k can be considered as limits of suitable inverse systems.

More generally, every Fréchet space is a limit of some inverse system of Banach spaces.

Using the inverse system described in Example 4.1 we can give an example of a limit map induced by a map (of an inverse system) consisting of contractions (even with the same constant of contractivity) which is not a contraction with respect to the metric in a limit of this system.

Example 4.3. Consider the map $f : C([0,\infty), \mathbb{R}) \to C([0,\infty), \mathbb{R})$, $f(x) = x/2$. This map is a contraction (with $1/2$ as a constant of contractivity) with respect to each seminorm p_m.

Suppose that there is k, $0 \le k < 1$, such that

$$d(f(x), f(y)) \le k\, d(x,y), \quad \text{for any } x, y \in C([0,\infty), \mathbb{R}).$$

Take L, $\max\{1/2, k\} < L < 1$. We show that there are functions $x, y \in C([0,\infty), \mathbb{R})$ such that $d(f(x), f(y)) \ge L d(x,y)$.

Indeed, let $y \equiv 0$ and $x \equiv 2L/(1-L)$. Then $p_m(x-y) = 2L/(1-L)$, for every $m \ge 1$. One can easily check that

$$L \frac{p_m(x-y)}{1 + p_m(x-y)} = \frac{2L^2}{1+L},$$

and
$$\frac{p_m(f(x)-f(y))}{1+p_m(f(x)-f(y))} = \frac{p_m(x-y)/2}{1+p_m(x-y)/2} = L > \frac{2L^2}{1+L}.$$
Hence
$$\begin{aligned} Ld(x,y) &= L\sum_{m=1}^{\infty}\frac{1}{2^m}\frac{p_m(x-y)}{1+p_m(x-y)} \\ &\leq \sum_{m=1}^{\infty}\frac{p_m(f(x)-f(y))}{1+p_m(f(x)-f(y))} = d(f(x),f(y)). \end{aligned}$$
This implies that f is not a contraction with respect to the metric in $C([0,\infty),\mathbb{R})$.

Now we formulate the results obtained in the previous section in a special case of function spaces (to some examples described in Example 4.1 and Remark 4.2).

Corollary 4.4. *Let $\varphi_m : C([0,m],\mathbb{R}^n) \multimap C([0,m],\mathbb{R}^n)$ (respectively, $\varphi_m : L^1([0,m],\mathbb{R}^n) \multimap L^1([0,m],\mathbb{R}^n)$), $m \geq 1$, be compact valued contractions having the selection property, and such that $\varphi_p(x)|_{[0,m]} = \varphi_m(x|_{[0,m]})$, for every $x \in C([0,p],\mathbb{R}^n)$ (respectively, $L^1([0,p],\mathbb{R}^n)$), $p \geq m$. Define $\varphi : C([0,\infty),\mathbb{R}^n) \multimap C([0,\infty),\mathbb{R}^n)$ (respectively, $\varphi : L^1_{\mathrm{loc}}([0,\infty),\mathbb{R}^n) \multimap L^1_{\mathrm{loc}}([0,\infty),\mathbb{R}^n)$), $\varphi(x)|_{[0,m]} = \varphi_m(x|_{[0,m]})$, for every $x \in C([0,\infty),\mathbb{R}^n)$ (respectively, $L^1([0,\infty),\mathbb{R}^n)$). Then* $\mathrm{Fix}(\varphi)$ *is compact R_δ.*

Remark 4.5. From the above corollary one can infer that, if each φ_m has convex values (or decomposable — in the case of L^1 spaces), then $\mathrm{Fix}(\varphi)$ is compact R_δ.

The inverse system approach describing above gives us an easy way to study a topological structure of solution sets of differential problems on noncompact intervals. Namely, the suitable operator with solutions as fixed points can be often considered as a limit map induced by maps of Banach spaces of functions defined on compact intervals (see [3] for examples).

Aknowledgement. The author thanks professor W. Kryszewski for valuable comments and the alternative proof of Theorem 2.1.

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Manuscript received November 2, 1999

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