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TOPOLOGICAL DEGREE FOR A CLASS OF ELLIPTIC OPERATORS IN $\mathbb{R}^n$

Cristelle Barillon — Vitaly A. Volpert

Dedicated to the memory of Juliusz P. Schauder

Abstract. A class of elliptic operators in $\mathbb{R}^n$ is considered. It is proved that the operators are Fredholm and proper. The topological degree is constructed. Existence of solutions for a reaction-diffusion system is studied.

1. Introduction

Consider the elliptic operator

$$A(u) = a(x)\Delta u + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + F(x,u), \tag{1.1}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $u = (u_1, \ldots, u_p)$, $F(x,u) = (F_1(x,u), \ldots, F_p(x,u))$ is a vector-valued function, $a(x)$, $b_i(x)$ are $p \times p$ matrices, Δ denotes the Laplace operator. Conditions on the matrices $a(x)$, $b_i(x)$ and on the function $F(x,u)$ will be specified below.

The operator A is considered as acting from the weighted Hölder space $C^{2+\delta}_{\mu}(\mathbb{R}^n)$ to the space $C^{\delta}_{\mu}(\mathbb{R}^n)$. The norm in the space $C^{k+\delta}_{\mu}(\mathbb{R}^n)$ where k is an integer and $0 < \delta < 1$is defined by the equality

$$\|u\|^{k+\delta}_{\mu} = \|u\mu\|_{k+\delta}.$$

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Here $\mu(x)$ is a weight function and $\|\cdot\|_{k+\delta}$ is the usual Hölder norm:

$$\|v\|_{k+\delta} = \|v\|_k + [v]_{k+\delta}, \quad \|v\|_k = \sum_{j=0}^{k} \max_{|\alpha|=j} \|D^\alpha v\|_0, \quad \|v\|_0 = \sup_{x\in\mathbb{R}^n} |v(x)|,$$

$$[v]_{k+\delta} = \sup_{x,y\in\mathbb{R}^n} \frac{|D^k v(x) - D^k v(y)|}{|x-y|^\delta},$$
$$D^k = \frac{\partial^k}{\partial x_1^{k_1}\dots\partial x_n^{k_n}}, \quad k_1 + \dots + k_n = k.$$

The weight function $\mu(x)$ is a sufficiently smooth positive function, $\mu(x) \to \infty$ as $|x| \to \infty$, such that the functions

$$\mu_i(x) = \frac{1}{\mu(x)} \frac{\partial\mu(x)}{\partial x_i}, \quad i = 1, \dots, n, \quad \mu_\Delta(x) = \frac{\Delta\mu(x)}{\mu(x)}$$

are bounded in $C^\delta(\mathbb{R}^n)$ and tend to 0 as $|x| \to \infty$. As example we can take $\mu(x) = 1 + |x|^2$.

In this work we define the topological degree for the class of operators (1.1). We note that construction of the topological degree for elliptic operators in unbounded domains is essentially different in comparison with the operators in bounded domains. In the latter case the corresponding vector field can be reduced to a completely continuous one and the classical Leray–Schauder theory [6] can be used. In the case of unbounded domains it cannot be done and other approaches should be employed. In [3], [4], [11], [10] the degree is constructed in the one-dimensional spatial case and in [12] for the elliptic operators in unbounded cylinders. This construction is based on the estimations of the operators from below and on the approach developed in [9]. In [15] a wider class of elliptic operators in cylinders is considered. In this case the construction uses the degree theory for Fredholm operators [5]. In this work we apply the approach developed in [15] to define the degreefor elliptic operators in $\mathbb{R}^n$ (see also [1]).

We recall that the choice of function spaces is important for the degree construction. We use weighted spaces and we give an example when the degree cannot be constructed in spaces without weight.

The contents of the paper are as follows. In Section 2 we study linear operators and discuss the Fredholm property and the index. In Section 3 we obtain the conditions when the nonlinear operators are proper. The degree construction for Fredholm and proper elliptic operators is discussed in Section 4. We study existence of solutions for a reaction-diffusion system of equations in Section 5 and give the example where the degree cannot be defined in Section 6.

2. Linear operators

In this section we consider the linear elliptic operator

$$Lu = a(x)\Delta u + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u, \tag{2.1}$$

acting from the space $C^{2+\delta}(\mathbb{R}^n)$ to $C^{\delta}(\mathbb{R}^n)$. The matrices $a(x)$, $b_i(x)$, and $c(x)$ belong to $C^{\delta}(\mathbb{R}^n)$ and the ellipticity condition

$$(a(x)\xi, \xi) \geq \sigma > 0, \quad |\xi| = 1$$

is supposed to be satisfied. We suppose moreover that the coefficients of the operator have directional limits at infinity. To define the directional limits, consider the unit sphere S in $\mathbb{R}^n$ and consider $a(x)$ as a function of the variables $\phi \in S$ and $r \geq 0$. Assume, that for any $\phi \in S$, there exists the limit

$$a_\phi = \lim_{r\to\infty} a(r, \phi),$$

and that this limit is a continuous function of ϕ. Similarly we define $b_{i\phi}$ and c_ϕ and assume that $b_{i\phi}$ and c_ϕ are continuous functions of ϕ.

We introduce the limiting operators

$$L_\phi u = a_\phi \Delta u + \sum_{i=1}^{n} b_{i\phi}\frac{\partial u}{\partial x_i} + c_\phi u$$

which are operators with constant coefficients. The following conditions ensure that they are Fredholm operators.

Condition 1. For any $\phi \in S$ the problem $L_\phi u = 0$ does not have nontrivial solutions in $C^{2+\delta}(\mathbb{R}^n)$.

Theorem 2.1. *The operator L is normally solvable and has a finite dimensional kernel if and only if Condition 1 is satisfied.*

The proof of this theorem is given in [7], [8] in a more general formulation.

Since the limiting operators L_ϕ have constant coefficients, we can formulate Condition 1 in an equivalent form.

Condition 1′. The matrices

$$M(\phi, \xi) = -|\xi|^2 a_\phi + i\sum_{j=1}^{n} \xi_j b_{j\phi} + c_\phi$$

do not have zero eigenvalues for any $\phi \in S$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$.

To obtain the equivalence of these conditions we should apply the generalized Fourier transform to the differential problem. Condition 1′ is more convenient and we use it below. For the degree construction we need not only Conditions 1 and 1′ but also more strong conditions:

Condition 2. For any $\phi \in S$ and $\lambda \geq 0$ the problem $L_\phi u - \lambda u = 0$ does not have nontrivial solutions in $C^{2+\delta}(\mathbb{R}^n)$.

Condition 2′. The matrices

$$M_\lambda(\phi, \xi) = -|\xi|^2 a_\phi + i \sum_{j=1}^{n} \xi_j b_{j\,\phi} + c_\phi - \lambda I_p$$

do not have zero eigenvalues for any $\phi \in S$, $\xi \in \mathbb{R}^n$, and $\lambda \geq 0$. Here I_p is the identity matrix.

Under additional conditions on the operator, Condition 2′ follows from Condition 1′. Indeed, suppose that for some positive λ and $\xi \in \mathbb{R}^n$, $\det M_\lambda(\phi, \xi) = 0$. We find additional conditions on the operators such that there exists $\eta \in \mathbb{R}^n$ for which

$$M_\lambda(\phi, \xi) = M(\phi, \eta). \tag{2.2}$$

Comparing the real and imaginary parts of these matrices, we have

$$(|\eta|^2 - |\xi|^2) a_\phi = \lambda I_p, \quad \sum_{j=1}^{n} (\xi_j - \eta_j) b_{j\,\phi} = 0. \tag{2.3}$$

From the first equality immediately follows that $a_\phi = I_p$ and $|\eta|^2 \neq |\xi|^2$. The second equality is satisfied if the matrices $b_{j\,\phi}$ are linearly dependent. In this case the same vectors ξ and η can be used to satisfy the first equality. We have proved the following proposition.

Proposition 2.2. *If $a_\phi = I_p$ and the matrices $b_{j\,\phi}$ are linearly dependent for all ϕ, then Condition 2′ follows from Condition 1′.*

Corollary 2.3. *Let Condition 1′ be satisfied. If the matrices $b_{j\,\phi}$ are linearly dependent for all ϕ then the index of the operator L is zero.*

Proof. Consider first the case where $a_\phi = I_p$. To prove the corollary it is sufficient to consider the homotopy $L - \tau\lambda$, $\tau \in [0, 1]$. For λ sufficiently large the operator $L - \lambda$ is invertible and its index is zero. Since the homotopy takes place in the class of normally solvable operators with finite dimensional kernel, the index does not change.

It remains to note that the index of the operator $a_\phi^{-1}(x)L$ is the same as the index of the operator L and that multiplication by the matrix a_ϕ^{-1} does not change the linear dependence of the matrices $b_{j\,\phi}$. $\square$

Corollary 2.4. *If Condition 1′ is satisfied and one of the matrices b_j is zero, then the index of the operator L is zero.*

The proof is obvious.

We note that in the case $n > p^2$ the matrices b_{j_ϕ} are linearly dependent and the index of the operator is zero. In particular it is the case for the multidimensional scalar equation $n > 1$, $p = 1$. In the one-dimensional case the index can be different from zero [2]. It can also be equal to zero but Condition $2'$ may not be satisfied. It shows in particular that there exist different homotopy classes of Fredholm operators with the same index.

We conclude this section with the conjecture that if $n > 1$ and Condition $1'$ is satisfied, then the index of the operator L is zero.

3. Proper operators

In this section we show that the restriction of the operator (1.1) on every bounded set is proper. It means that the intersection of the inverse image of any compact set with a ball $\|u\| \le K$, $K > 0$ is compact. For the degree construction we need to consider the operators depending on a parameter. Along with the operator (1.1), we consider also the operator

$$A(u,\tau) = a(x,\tau)\Delta u + \sum_{i=1}^{n} b_i(x,\tau)\frac{\partial u}{\partial x_i} + F(x,u,\tau), \tag{3.1}$$

where $\tau \in [0,1]$ and the following hypothesis are satisfied:

(H1) $a(x,\tau)$, $b_i(x,\tau) \in C^\delta(\mathbb{R}^n)$ for any $\tau \in [0,1]$ and $F'(x,u,\tau) \in C^\delta_\mu(\mathbb{R}^n, |u| \le M)$ for any $\tau \in [0,1]$ and $0 < M < \infty$;

(H2) $\|a(x,\tau) - a(x,\tau_0)\|_\delta \to 0$, $\|b_i(x,\tau) - b_i(x,\tau_0)\|_\delta \to 0$ as $\tau \to \tau_0$ and $\|(F(x,u,\tau) - F(x,u,\tau_0))\mu\|_\delta \to 0$ as $\tau \to \tau_0$ uniformly in u on every bounded set;

(H3) $a(x,\tau)$, $b_i(x,\tau)$ and $c(x,\tau) \equiv F'_u(x,0,\tau)$ have directional limits $a(\phi,\tau)$, $b_i(\phi,\tau)$ and $c(\phi,\tau)$, respectively. The matrix-functions $a(\phi,\tau)$, $b_i(\phi,\tau)$ and $c(\phi,\tau)$ are continuous with respect to ϕ;

(H4) Denote

$$\widetilde{a}(\phi,r,\tau) = a(x,\tau), \quad \widetilde{b}_i(\phi,r,\tau) = b_i(x,\tau),$$
$$\widetilde{F}'_u(\phi,r,u,\tau) = F'_u(x,u,\tau), \quad \phi \in S, r \ge 0.$$

Then

$$\lim_{\substack{r\to\infty\\ \phi\to\phi_0}} \widetilde{a}(\phi,r,\tau) = a(\phi_0,\tau), \quad \lim_{\substack{r\to\infty\\ \phi\to\phi_0}} \widetilde{b}_i(\phi,r,\tau) = b_i(\phi_0,\tau),$$

$$\lim_{\substack{r\to\infty\\ \phi\to\phi_0\\ u\to 0}} \widetilde{F}'_u(\phi,r,u,\tau) = c(\phi_0,\tau).$$

We also introduce the following notations

$$E_1 = C^{2+\delta}_\mu(\mathbb{R}^n), \quad E_1' = E_1 \times [0,1], \quad E_2 = C^\delta_\mu(\mathbb{R}^n),$$

and consider the operator $A(u,\tau)$ as acting from E_1' to E_2.

Theorem 3.1. *Suppose that hypothesis* (H1)–(H4) *are satisfied and the operator*

$$L(\tau)u = a(x,\tau)\Delta u + \sum_{i=1}^{n} b_i(x,\tau)\frac{\partial u}{\partial x_i} + c(x,\tau)u$$

satisfies Condition 1 *for any* $\tau \in [0,1]$. *Then the restriction of the operator* $A(u,\tau)$ *on any bounded set in* E_1' *is proper.*

Proof. Let $B_R = \{(u,\tau) \in E_1',\ \|u\|_{E_1} \le R,\ \tau \in [0,1]\}$, and D be a compact set in E_2. We choose a sequence $\{u_k,\tau_k\} \in G = A^{-1}(D) \cap B_R$. We will show that there exists a converging subsequence of the sequence $\{u_k,\tau_k\}$. Without loss of generality we can assume that $\tau_k \to \tau_0$ as $k \to \infty$.

Since the sequence $\{u_k\}$ is bounded in E_1, then there exists a subsequence $\{u_{k_l}\}$ converging to some limiting function $u_0 \in E_1$ uniformly in $C^2(\mathbb{R}^n)$. We should show that this convergence takes place in E_1. Let

$$A(u_k,\tau_k) = f_k \in D.$$

Since D is compact then without loss of generality we can suppose that $f_k \to f_0$ in E_2. Hence $A(u_0,\tau_0) = f_0$. Put

$$v_k = u_k\mu, \quad v_0 = v_0\mu, \quad w_k = v_k - v_0, \quad g_k = f_k\mu, \quad g_0 = f_0\mu.$$

Then w_k satisfies the following equation

$$\begin{aligned}(3.2)\quad & a(x,\tau_0)\Delta w_k + \sum_{i=1}^{n}(b_i(x,\tau_0) - 2\mu_i a(x,\tau_0))\frac{\partial w_k}{\partial x_i} \\ &+ \bigg(a(x,\tau_0)(2|\mu_\Sigma|^2 - \mu_\Delta) - \sum_{i=1}^{n} b_i(x,\tau_0)\mu_i + B_k(x,\tau_0)\bigg)w_k = h_k - h_0,\end{aligned}$$

where

$$B_k(x,\tau_0) = \int_0^1 F_u'(x,u_0(x) + t(u_k(x) - u_0(x)),\tau_0)\,dt, \quad |\mu_\Sigma|^2 = \sum_{i=1}^{n}\mu_i^2$$

and

$$\begin{aligned}h_k - h_0 = {} & g_k - g_0 + (F(x,u_k,\tau_0) - F(x,u_k,\tau_k))\mu \\ &+ (a(x,\tau_0) - a(x,\tau_k))\bigg(\Delta v_k + (2|\mu_\Sigma|^2 - \mu_\Delta)v_k - \sum_{i=1}^{n} 2\mu_i\frac{\partial v_k}{\partial x_i}\bigg) \\ &+ \sum_{i=1}^{n}(b_i(x,\tau_0) - b_i(x,\tau_k))\bigg(\frac{\partial v_k}{\partial x_i} - \mu_i v_k\bigg).\end{aligned}$$

To prove the theorem we should show that $w_k \to 0$ in $C^{2+\delta}(\mathbb{R}^n)$.

We note first that $h_k \to h_0$ in $C^\delta(\mathbb{R}^n)$. Indeed, we put $h_k - h_0 = g_k - g_0 + G_1 + G_2$, where $G_1 = -(F(x, u_k, \tau_k) - F(x, u_k, \tau_0))\mu$ and G_2 contains all other terms. By construction $g_k \to g_0$ in $C^\delta(\mathbb{R}^n)$. By virtue of the conditions on F, $G_1 \to 0$ in $C^\delta(\mathbb{R}^n)$. All other terms have the form $y_k z_k$ where $y_k \to 0$ and z_k is uniformly bounded in $C^\delta(\mathbb{R}^n)$.

From the inequalities

$$|u_k(x)| \le \frac{K}{\mu(x)}, \quad |u_0(x)| \le \frac{K}{\mu(x)},$$

where K is a positive constant, we obtain the convergence

$$B_k(x, \tau_0) \to c_{\phi_0}, \quad r \to \infty, \quad \phi \to \phi_0.$$

Here $x = (r, \phi)$. By virtue of the convergence $u_n \to u_0$ in $C^2(\mathbb{R}^n)$, $w_k \to 0$ in C^2 on every bounded set.

We show next that $w_k \to 0$ in $C(\mathbb{R})$. If this convergence does not take place, then there exists a sequence $\{x^{(k)}\}$ such that $|w_k(x^{(k)})| \ge \delta > 0$. This sequence cannot be bounded because of the uniform convergence on every bounded set. If we consider the corresponding spherical coordinates $\{r^{(k)}, \phi^{(k)}\}$, then without loss of generality we can assume that $\phi^{(k)} \to \phi_0$ for some $\phi_0 \in S$. The function $\widetilde{w}_k(x) = w_k(x + x^{(k)})$ satisfies the equation

$$\begin{aligned}(3.3)\quad & a(x + x^{(k)}, \tau_0)\Delta\widetilde{w}_k + \sum_{i=1}^{n}\Big(b_i(x + x^{(k)}, \tau_0) - 2\widetilde{\mu}_i a(x + x^{(k)}, \tau_0)\Big)\frac{\partial \widetilde{w}_k}{\partial x_i} \\ & + \Big(a(x + x^{(k)}, \tau_0)(2|\widetilde{\mu}_\Sigma|^2 - \widetilde{\mu}_\Delta) - \sum_{i=1}^{n} b_i(x + x^{(k)}, \tau_0)\widetilde{\mu}_i\Big)\widetilde{w}_k \\ & + B_k(x + x^{(k)}, \tau_0)\widetilde{w}_k = h_k(x + x^{(k)}) - h_0(x + x^{(k)}),\end{aligned}$$

where $\widetilde{\mu}_i(x) = \mu_i(x + x^{(k)})$, $\widetilde{\mu}_\Sigma(x) = \mu_\Sigma(x + x^{(k)})$ and $\widetilde{\mu}_\Delta(x) = \mu_\Delta(x + x^{(k)})$). Passing to the limit in this equation on every bounded set, we obtain that the equation $L_{\phi_0}(\tau_0)u = 0$ has a nonzero solution. This contradicts Condition 1 and proves the uniform convergence $w_k \to 0$.

Since the sequence $\{w_k\}$ is uniformly bounded in $C^{2+\delta}(\mathbb{R}^n)$, then $w_k \to 0$ in $C^2(\mathbb{R}^n)$. To prove that this convergence takes place in $C^{2+\delta}(\mathbb{R}^n)$, we rewrite the equation (3.2) in the form

$$a\Delta w_k = -\sum_{i=1}^{n}((b_i(x, \tau_0) - 2\mu_i a(x, \tau_0)))\frac{\partial w_k}{\partial x_i} + p_k,$$

where

$$p_k = -\Big(a(x, \tau_0)(2|\mu_\Sigma|^2 - \mu_\Delta) - \sum_{i=1}^{n} b_i(x, \tau_0)\mu_i + B_k(x, \tau_0)\Big)w_k + h_k - h_0.$$

The Schauder estimates give $\|w_k\|_{2+\delta} \le C(\|p_k\|_\delta + \|w_k\|_{1+\delta})$. It remains to note that $\|p_k\|_\delta \to 0$ as $k \to \infty$. □

The choice of function spaces plays an important role for properness of the operators. Consider the following example. Let the operator $A(u) = u'' + F(u)$ act from $C^{2+\delta}(\mathbb{R})$ to $C^\delta(\mathbb{R})$ and the function $F(u)$ be sufficiently smooth and satisfy the following conditions:

$$F(0) = 0,\ F(u) < 0,\ 0 < u < u_0,\ F(u) > 0,\ u_0 < u \le 1,\ \int_0^1 F(u)\,du = 0.$$

Then it can be easily shown that there exists a solution $u(x)$ of the problem $A(u) = 0$, $\quad u(\pm\infty) = 0$. Moreover, this solution is invariant up to translation in space, i.e. all functions $u(x+h)$, $-\infty < h < \infty$ are also solutions of this problem. Hence the inverse image of 0 is bounded in the space $C^{2+\delta}(\mathbb{R})$ but not compact and the operator is not proper. If we consider Sobolev spaces instead of the Hölder spaces, the operator is not proper neither. The solution $u(x)$ decreases exponentially at infinity and it is integrable. The family of solutions $u(x+h)$ is not compact. However if we consider the weighted spaces then the norm $\|u(x+h)\|_\mu^{2+\delta}$ goes to infinity as $h \to \pm\infty$. An intersection of this family of solutions with any closed bounded set is compact. It is important to emphasize here that the growth of the weight function at infinity should be slower than exponential. Otherwise solutions of the equation will not belong to the weighted space.

4. Topological degree

We recall the definition of a topological degree. Let E_1 and E_2 be Banach spaces. Suppose we are given a class Φ of operators acting from E_1 to E_2 and a class H of homotopies, i.e. maps

$$A(u,\tau) :\ E_1 \times [0,1] \to E_2, \quad \tau \in [0,1],\ u \in E_1 \tag{4.1}$$

such that for any $\tau \in [0,1]$ fixed, $A(u,\tau) \in \Phi$. Suppose that for any bounded open set $D \subset E_1$ and any operator $A \in \Phi$ such that $A(u) \ne 0$, $u \in \partial D$ (∂D denotes the boundary of the set D) there is an integer $\gamma(A, D)$ satisfying the following conditions:

(i) (Normalization) There exists a linear bounded operator $J : E_1 \to E_2$ with a bounded inverse defined on the whole E_2 such that for any bounded open set $D \subset E_1$, $0 \in D$, $\gamma(J, D) = 1$

(ii) (Additivity) Let D, D_1, $D_2 \in E_1$ be domains in E_1, $D_i \subset D$, $i = 1, 2$, $D_1 \cap D_2 = \emptyset$. Suppose that $A \in \Phi$ and

$$A(u) \ne 0, \quad u \in \overline{D} \setminus (D_1 \cup D_2). \tag{4.2}$$

Then $\gamma(A, D) = \gamma(A, D_1) + \gamma(A, D_2)$.

(iii) (Homotopy invariance) Let $A(u, \tau) \in H$ and

$$A(u,\tau) \neq 0, \quad u \in \partial D, \quad \tau \in [0,1]. \tag{4.3}$$

Then $\gamma(A(\,\cdot\,, 0), D) = \gamma(A(\,\cdot\,, 1), D)$.

The integer $\gamma(A, D)$ is called *topological degree*.

We use here the construction of the degree for Fredholm and proper operators [5]. It concerns operators $\widetilde{A}$ acting from E_1 into itself such that the operator $\widetilde{A} + \lambda I$ is Fredholm for all $\lambda \geq 0$, the operator $\widetilde{A}(u, \tau)$ is proper as acting from $E_1 \times [0, 1]$ to E_1 and has two derivatives withrespect to u and τ. The degree in this form cannot be applied directly to the operators (3.1) acting in different spaces and we need to reduce them to operators acting in the same spaces. We use the approach similar to that in [15].

Let E_0 be an open bounded set in E_1. Let further Φ be the set of all proper operators A acting from E_0 to E_2, which are continuous and have two Frechet derivatives and such that A' satisfies Condition 2. Denote $J_k u = \Delta u - ku$, $k > 0$. We consider J_k as operator acting from E_1 to E_2 and denote Φ_k the set of all operators of the form (1.1) such that the operator $A(u) + \sigma J_k u$ satisfies Condition 1 for all $\sigma \geq 0$.

Lemma 4.1. *For any $k > 0$, $\Phi = \Phi_k$.*

Proof. Condition $2'$ for an operator $A \in \Phi$ equivalent to Condition 2 has the form:

$$\det M(\xi, \phi, \mu) \neq 0, \quad M(\xi,\phi,\mu) = -|\xi|^2 a_\phi + i \sum_{i=1}^{n} \xi_i b_{i\phi} + c_\phi - \mu I_p,$$

for any $\xi \in \mathbb{R}^p$, $\phi \in S$, $\mu \geq 0$. Condition $1'$ for an operator $A + \sigma J_k \in \Phi_k$ has the form

$$\det \widetilde{M}(\xi, \phi, \sigma) \neq 0, \quad \widetilde{M}(\xi,\phi,\sigma) = -|\xi|^2 a_\phi + i \sum_{i=1}^{n} \xi_i b_{i\phi} + c_\phi + \sigma(-|\xi|^2 - k) I_p$$

for any $\xi \in \mathbb{R}^p$, $\phi \in S$, $\sigma \geq 0$.

If for some ξ, ϕ and μ, $\det M(\xi, \phi, \mu) = 0$, then for the same ξ and ϕ we can choose $\sigma \geq 0$ such that $M(\xi, \phi, \mu) = \widetilde{M}(\xi, \phi, \sigma)$. Consequently $\det \widetilde{M}(\xi, \phi, \sigma) = 0$. Inversely, if $\det \widetilde{M}(\xi, \phi, \sigma) = 0$, then we can put $\mu = \sigma(|\xi|^2 + k)$. Hence $\det M(\xi, \phi, \mu) = 0$. □

Consider now the operator $\widetilde{A}(u) = J_k^{-1} A(u) : E_1 \to E_1$. By virtue of the properties of the operator A, $\widetilde{A} + \lambda I$ is Fredholm for all $\lambda \geq 0$ and $\widetilde{A}(u, \tau)$ is proper. Hence the degree $\gamma(\widetilde{A}, D)$ is defined for any domain $D \subset E_1$ such that $\widetilde{A}(u) \neq 0$, $u \in \partial D$. If $\widetilde{A}(u) \neq 0$, $u \in \partial D$, then $A(u) \neq 0$, $u \in \partial D$ and we put by definition $\gamma(A, D) = \gamma(\widetilde{A}, D)$. The normalization operator is J_k for a positive k.

Thus the topological degree is defined (cf. [3], [4], [15]). The uniqueness of the topological degree is proved in [15] under more general assumptions.

5. Application to a reaction-diffusion problem

In this section we consider the problem

$$aw^{''} + cw' + F(w,x) = 0, \quad \lim_{x\to\pm\infty} = w_\pm, \quad w_+ < w_-,. \tag{5.1}$$

Here $w = (w_1, \ldots , w_p)$, $F = (F_1, \ldots , F_p)$, a is a constant diagonal matrix with positive diagonal elements a_i, the inequality between the vectors is understood component-wise. We assume that the vector-valued function F is sufficiently smooth.

Suppose that the following conditions are satisfied:

(C1) If $F_i(u_0, x_0) = 0$ for some $w_+ < u_0 < w_-$, $x_0 \in \mathbb{R}$, then

$$\frac{\partial F_i(u_0, x_0)}{\partial u_j} > 0, \quad j = 1, \ldots , p, \ j \neq i, \quad \frac{\partial F_i(u_0, x_0)}{\partial x} < 0. \tag{5.2}$$

In this case we call the system (5.1) *locally monotone*. This class of systems was introduced in [10] for the case where the nonlinearity F did not depend on x explicitly,

(C2) $F(w_\pm, x) \equiv 0$, $x \in \mathbb{R}$ and there exist the limits

$$F^\pm(u) = \lim_{x\to\pm\infty} F(u,x), \quad F^{\pm^{'}}(u) = \lim_{x\to\pm\infty} F_u^{'}(u,x) \tag{5.3}$$

uniform with respect to u,

(C3) If $F_i^+(u_0) = 0$ ($F_i^-(u_0) = 0$), $w_+ \le u_0 \le w_-$, then there exists such N that

$$\frac{\partial F_i(u,x)}{\partial x} < 0, \quad u \in U(u_0) \cap \{w_+ < u_0 < w_-\}, \ x \ge N \ (x \le -N). \tag{5.4}$$

Here $U(u_0)$ is a neighbourhood of the point u_0,

(C4) For all $u \in \mathbb{R}^p$

$$\frac{\partial F_i^\pm(u)}{\partial u_j} > 0, \quad i, \ j = 1, \ldots , p, \ j \neq i. \tag{5.5}$$

All eigenvalues of the matrices $F^{+^{'}}(w_\pm)$ and $F^{-^{'}}(w_\pm)$ lie in the left-half plane, the functions $F^+(u)$ and $F^-(u)$ have finite numbers of zeros in the interval $w_+ < u < w_-$, the matrices $F^{+^{'}}$ and $F^{-^{'}}$ taken at these zeros are invertible and their principal eigenvalues are positive.

Consider the problems

$$aw^{''} + cw' + F^+(w) = 0, \quad w(\pm\infty) = w_\pm, \tag{5.6}$$

and

$$aw'' + cw' + F^-(w) = 0, \quad w(\pm\infty) = w_\pm. \tag{5.7}$$

We recall that if (C4) is satisfied then there exists unique values $c = c_+$ and $c = c_-$ such that the problems (5.5) and (5.6), respectively, have monotone solutions [10], [16]. For each of these problems the monotone solution is unique up to translation in space and

$$c_\pm = \inf_{\rho\in K} \sup_{x,i} \frac{a_i\rho_i'' + F_i^\pm(\rho)}{-\rho_i'} = \sup_{\rho\in K} \inf_{x,i} \frac{a_i\rho_i'' + F_i^\pm(\rho)}{-\rho_i'},$$

where K is the set of twice continuously differentiable monotically decreasing functions $\rho(x) = (\rho_1(x), \dots, \rho_n(x))$ with the limits $\rho(\pm\infty) = w_\pm$.

The main result of this section is given by the following theorem.

Theorem 5.1. *Suppose that conditions* (C1)–(C4) *are satisfied. If* $c_+ < c < c_-$*, then there exists a monotonically decreasing solution of the problem* (5.1).

The particular case of this theorem where the inequalities (5.2) are satisfied for all $u_0 \in \mathbb{R}^p$ and $x_0 \in \mathbb{R}$ is proved in [13]. In this case the system is called *monotone* and it is the class of systems for which comparison theorems are valid. For locally monotone systemsthey are not applicable and we cannot use them to prove existence of solutions as it is done in [13]. To prove Theorem 5.1 we use the topological degree and the Leray–Schauder method. We will construct a homotopy to reduce the locally monotone system to a monotone system and we will obtain a priori estimates of solutions. Together with the results of [13] it will prove existence of solutions for (5.1).

The remaining part of this section is devoted to the proof of Theorem 5.1.

To define the operators corresponding to the problem (5.1) we introduce a sufficiently smooth function $\psi(x)$ such that $\psi(x) = w_-$ for $x \le -1$ and $\psi(x) = w_+$ for $x \ge 1$. We put

$$A(u) = a(u+\psi)'' + c(u+\psi)' + F(u+\psi, x).$$

We assume that all conditions of Sections 2 and 3 are satisfied and consider this operator as acting from $C_\mu^{2+\delta}$ to C_μ^δ. Then the topological degree can be defined for it.

5.1. Model system. Let $F(u,x)$ satisfy condition (C1). Consider the first component $F_1(u,x)$. For each $u_3, \dots, u_p$ and x fixed, zero line of the function F_1 is a single valued smooth function $u_2 = \phi_1(u_1)$. We have

$$F_1 < 0 \text{ if } u_2 < \phi(u_1), \quad F_1 > 0 \text{ if } u_2 > \phi(u_1).$$

To show the dependence of ϕ on $u_3, \dots, u_p$ and x we write it also as

$$u_2 = \phi_1(u_1, u_3, \dots, u_p, x).$$

By virtue of (5.2),

$$\frac{\partial \phi_1}{\partial u_j} < 0, \quad j = 3, \dots, p, \quad \frac{\partial \phi_1}{\partial x} > 0.$$

We define $\Phi_1(u,x)$ as

$$\Phi_1(u,x) = b_1(u_2 - \phi_1(u_1, u_3, \dots, u_p, x)),\ b_1 > 0.$$

Then

$$\frac{\partial \Phi_1}{\partial u_j} > 0, \quad j = 2, \dots, p, \quad \frac{\partial \Phi_1}{\partial x} < 0.$$

Similarly we define Φ_i for $i = 2, \dots, p$. Consider the problem

$$aw'' + cw' + \Phi(w,x) = 0, \quad w(\pm\infty) = w_\pm.$$

This system is monotone and it has a unique monotonically decreasing solution $w_0(x)$ [13]. Moreover the eigenvalue problem

$$au'' + cu' + \Phi'_w(w_0(x),x)w = \lambda u, \quad u(\pm\infty) = 0$$

has all spectrum in the left half-plane. It allows us to conclude that the index of this solution considered as a stationary point of the corresponding operator equals 1.

5.2. Separation of monotone solutions. Consider the problem

$$aw'' + cw' + F_\tau(w,x) = 0, \quad w(\pm\infty) = w_\pm \tag{5.8}$$

depending on a parameter $\tau \in [0,1]$. We suppose that conditions (C1)–(C4) are satisfied for each $\tau \in [0,1]$ and N in (C3) can be chosen independently of τ. The homotopy considered below will satisfy this condition.

Suppose that there are two sequences of solutions $\{w_k^{(1)}\}$ and $\{w_k^{(2)}\}$, where the first one corresponds to the values $\tau = \tau_k^{(1)}$ of the parameter and consists of vector-valued functions each component of which is monotonically decreasing. The second sequence corresponds to $\tau = \tau_k^{(2)}$ and for each function there is a nonmonotone component. We show that these two sequences are separated in $C_\mu^{2+\delta}(\mathbb{R})$, i.e. there is a positive constant εsuch that

$$\|w_k^{(1)} - w_m^{(2)}\|_{C_\mu^{2+\delta}} \geq \varepsilon, \quad \text{for all } k,\ m.$$

Suppose that it is not so. Then for some subsequences, the norm of the difference converges to 0. Without loss of generality, we can assume that the subsequences coincide with the original sequences,

$$\|w_k^{(1)} - w_k^{(2)}\|_{C_\mu^{2+\delta}} \to 0, \quad k \to \infty. \tag{5.9}$$

We will show below that the monotone solutions are uniformly bounded in the norm $C_\mu^{2+\delta}(\mathbb{R})$. Hence without loss of generality we can assume that the sequence

$\{w_k^{(1)}\}$ converges in $C^1(\mathbb{R})$ to some limiting function $w_0(x)$. Obviously, it is a solution of the problem (5.8) for some τ.

Lemma 5.3. *If* $\|w_k^{(1)} - w_0\|_{C^1} \to 0$, *then* $w_0'(x) < 0,\ x \in \mathbb{R}$.

Proof. Since $w_0(x)$ is a limit of monotone functions, then $w_0'(x) \le 0$. Suppose that for some i and x_0, $w_{0_i}'(x_0) = 0$. Then $w_{0_i}''(x_0) = 0$ and from (5.8) $F_{\tau,i}(w_0(x_0), x_0) = 0$. Differentiate the i-th equation in (5.8) with respect to x:

$$aw_i''' + cw_i'' + \sum_{j=1}^{p} \frac{\partial F_{\tau,i}}{\partial w_j} w_j' + \frac{\partial F_{\tau,i}}{\partial x} = 0.$$

From the definition of the local monotonicity and since $w_j' \le 0$, we obtain $w_i'''(x_0) > 0$. This contradiction proves the lemma. □

We use also the following lemma ([16, p. 209]).

Lemma 5.4. *Let $b(x)$ be a matrix with positive off-diagonal elements, $b_+ = \lim_{x\to\infty} b(x)$, $q > 0$ be a vector such that*

$$b(x)q < 0, \quad x \ge r, \quad b_+ q < 0$$

for some r. If the vector-valued function u satisfies the inequality

$$au'' + cu' + b(x)u \le 0$$

for $x \ge r$ and $u(r) > 0$, then $u(x) \ge 0$ for $x \ge r$.

From the convergence of the monotone solutions to $w_0(x)$ and from (5.9), it follows

$$\|w_k^{(2)} - w_0\|_{C^1} \to 0, \quad k \to \infty. \tag{5.10}$$

Without loss of generality we can assume that $\tau_k \to \tau_0$ for some $\tau_0 \in [0,1]$, that the first component of the vector valued function $w_k^{(2)}(x)$ is not monotone in a neighborhood of a point x_k and $x_k \to \infty$ as $k \to \infty$.

Since $w_0(x) \to w_+$ as $x \to \infty$ and $F_1^+(w_+) = 0$, then there exist $\sigma > 0$ and N such that

$$\frac{\partial F_{\tau,1}}{\partial w_j} > 0, \quad j = 2, \ldots, p, \quad \frac{\partial F_{\tau,1}}{\partial x} < 0,$$

for $w_+ - \sigma e \le w < w_+$, $x \ge N$. Here $e = (1, \ldots, 1)$ and N can be chosen independently of τ for τ sufficiently close to τ_0.

On the other hand, by virtue of convergence (5.10) $w_k^{(2)'}(x_0) < 0$ for any x_0 and k sufficiently large. This contradicts Lemma 5.4. Indeed, let $u(x) = -w_k^{(2)'}(x)$. Then

$$au'' + cu' + F_{\tau_k}'(w_k^{(2)}, x)u = \frac{\partial F_{\tau_k}(w_k^{(2)}, x)}{\partial x} < 0$$

and $u(r) > 0$ for some k and r sufficiently large. Existence of the vector q follows from the fact that the principal eigenvalue of the matrix $F'_{\tau_0}(w_+)$ is negative. Hence $w_k^{(2)'}(x) \le 0$ for $x \ge \max(N, r)$. This contradicts the assumption that the functions $w_k^{(2)}$ are not monotone.

5.3. A priori estimates of monotone solutions. It can be easily verified that monotone solutions of the problem (5.8) are uniformly bounded in $C^{2+\delta}(\mathbb{R})$. A priori estimates of solutions in the space $C_\mu^{2+\delta}(\mathbb{R})$ are reduced to check that the functions $(w - \psi)\mu$ are also uniformly bounded. Each of them is certainly bounded uniformly in x because $w - \psi$ decreases exponentially at infinity. So by the uniform boundedness we mean that

$$|(w - \psi)\mu| \le M$$

with the same M constant for all solutions.

Denote $U(w_\pm)$ neighbourhoods of the points $w_\pm$ where the solutions behave exponentially. More precisely, we can choose positive constants K and ε such that

$$|w(x) - w_+| \le K \exp^{-\varepsilon x}, \quad |w(x) - w_-| \le K \exp^{\varepsilon x}$$

if $w(x) \in U(w_\pm)$. Let x^+ and x^- be such that $w(x^+) \in \partial U(w_+)$, $w(x^-) \in \partial U(w_-)$. It remains to estimate $|x^+|$ and $|x^-|$ uniformly for all solutions (see [16] for more details).

Suppose that there is a sequence of solutions $\{w_k\}$ such that $w_k(x_k^+) \in \partial U(w_+)$ and $x_k^+ \to -\infty$. Then $x_k^- \to -\infty$. The functions $u_k(x) = w_k(x + x_k)$ satisfy the equation

$$au_k'' + cu_k' + F(u_k, x + x_k^+) = 0.$$

Passing to the limit as $k \to \infty$, we obtain a solution $u^+(x)$ of the equation

$$au'' + cu' + F^-(u) = 0. \tag{5.11}$$

It is easy to verify that $u^+(x) \to w_+$ as $x \to \infty$. If $u^+(-\infty) = w_-$, we obtain a contradiction since the problem

$$au'' + cu' + F^-(u) = 0, \quad u(\pm\infty) = w_\pm$$

does not have monotone solutions with $c \ne c_-$.

If $u^+(-\infty) \ne w_-$, then it is an unstable intermediate zero of the function $F^-(u)$. Hence $c < 0$ (see Lemma 5.5 below). We consider then the functions $v_k(x) = w_k(x + x_k^-)$ and proceeding as above, we obtain a solution $v^-(x)$ of the equation (5.11) such that $v^-(-\infty) = w_-$, $v^-(\infty) \ne w_+$. Hence $c > 0$. This contradiction shows that x_k^+ cannot converge to $-\infty$.

We used here the following lemma [16].

Lemma 5.5. *If $u(x)$ is a monotone solution of* (5.11) *such that $u(-\infty) = w_-$ and $u(\infty) \neq w_+$, then $c > 0$. If $u(\infty) = w_+$ and $u(-\infty) \neq w_-$, then $c < 0$.*

For what follows we need also a generalization of this result for the case where F depends explicitly on x.

Lemma 5.6. *Let $u(x)$ be a monotone solution of the equation*

$$au'' + cu' + F(u, x) = 0 \tag{5.12}$$

such that $u(-\infty) = w_-$, $u(\infty) \neq w_+$. Then $c > 0$. If $u(\infty) = w_+$ and $u(-\infty) \neq w_-$, then $c < 0$.

Proof. Suppose that $u(-\infty) = w_-$ and $u(\infty) = u_0 \neq w_+$. Then $F^+(u_0) = 0$ and the principal eigenvalue of the matrix $F^{+'}(u_0)$ is positive. Then there exists a vector $p > 0$ such that $pF^{+'}(u_0) > 0$. We have for x sufficiently large

$$(p, F(u, x)) = (p, F(u, x) - F(u_0, x)) + (p, F(u_0, x)) > (p, B(x)(u - u_0)) > 0.$$

Here $B(x) = \int_0^1 F_u'(tu + (1-t)u_0, x)\, dt$. Multiplying (5.12) by p and integrating from x_0 to ∞, we obtain

$$-(p, au'(x_0)) + c(p, u_0 - u(x_0)) + \int_{x_0}^{\infty} (p, F(u, x))\, dx = 0.$$

Since $u(x)$ is monotonically decreasing, then $c > 0$.

The second case can be considered similarly. □

Suppose now that the sequence $\{x_k^+\}$ remains bounded and $x_k^- \to -\infty$. Passing to the limit as above, we obtain that there exists a solution $u(x)$ of the equation (5.11) such that $u(-\infty) = w_-$, $u(\infty) \neq w_+$. From Lemma 5.6 it follows that $c > 0$. On the other hand there exists a solution $v(x)$ of the equation (5.12) such that $v(\infty) = w_+$, $v(-\infty) \neq w_-$. Hence $c < 0$. We obtain a contradiction.

Similarly we consider all other cases and show that both sequences $\{x_k^+\}$ and $\{x_k^-\}$ remain bounded. This gives a priori estimates of solutions.

5.4. Homotopy. We prove Theorem 5.1 by the Leray–Schauder method. For this we need to construct a homotopy of the system (5.1) to a model system such that the corresponding operator $A_\tau(u) : E_1' \to E_2$ satisfies the following properties:

1. There exists a domain $D \subset E_1$ such that the operator $A_1(\tau)$ corresponding to the model system has a nonzero degree $\gamma(A_1, D)$,
2. There are a priori estimates of solutions during the homotopy,

$$A_\tau(u) \neq 0, \quad u \in \partial D,\ \tau \in [0, 1].$$

Then the properties of the topological degree allow us to conclude that there is a solution of the equation $A_0(u) = 0$ in D.

This general scheme should be modified in the case under consideration because we obtain a priori estimates only for monotone solutions. On the other hand we prove also that monotone and nonmonotone solutions are separated in the function space. Hence we can construct a domain D_τ, $\tau \in [0,1]$ in $C^{2+\delta}_\mu(\mathbb{R})$ depending on the parameter and such that it contains all solutions $u_M = w_M - \psi$ of the equation $A_\tau(u) = 0$ and does not contain solutions $u_N = w_N - \psi$. Here w_M denotes monotone solutions of the problem (5.8) and w_N nonmonotone solutions.

Consider a ball $B_R(u_M)$ with the center u_M and the radius R. If R is sufficiently small, then for each τ fixed, the union of all these balls,

$$D_\tau = \bigcup_{u_M} B_R(u_M)$$

does not contain the functions u_N. Moreover, since the operator $A(u)$ is proper and all solutions u_M are uniformly bounded, then the set $\{u_M\}$ is compact. Hence, we can consider only a union of a finite number of balls.

Consider the problem

$$aw'' + cw' + \tau\Phi^+ + (1-\tau)F^+ = 0, \quad w(\pm\infty) = w_\pm, \tag{5.13}$$

where the function Φ is defined in Section 5.1. For $\tau \in [0,1]$ it is monotone and the nonlinearity $G^+_\tau = \tau\Phi^+ + (1-\tau)F^+$ has the same zeros as F^+. Hence there exists a unique monotone solution of (5.13) with $c = c^+(\tau)$.

Similarly a monotone solution of the problem

$$aw'' + cw' + \tau\Phi^- + (1-\tau)F^- = 0, \quad w(\pm\infty) = w_\pm \tag{5.14}$$

exists for $c = c^-(\tau)$. Let us first define F_τ as

$$F_\tau = (1-\tau)F + \tau\Phi.$$

Then the system (5.5) is locally monotone. To apply the results of Sections 5.2 and 5.3 we should verify that

$$c^-(\tau) > c^+(\tau), \quad \tau \in [0,1]. \tag{5.15}$$

By assumption $c^-(0) > c^+(0)$. By construction of the function Φ, $\Phi^-(w) > \Phi^+(w)$ for $w_+ < w < w_-$. Hence $c^-(1) > c^+(1)$. However (5.15) may be not satisfied for some $\tau \in (0,1)$. Without loss of generality we can assume that $c^-(\tau) > 0$, $\tau \in [0,1]$. Otherwise we could modify the nonlinearity in such a way that this condition holds (see [14]).

To obtain the required inequality we modify the homotopy F_τ to the form $F_\tau = s_\tau(x)[(1-\tau)F + \tau\Phi]$, where $s_\tau(x)$ is a positive twice continuously differentiable with respect to x and τ function such that $s_\tau(x) = s^+(\tau)$ for $x \geq 1$ and

$s_\tau(x) = s^-(\tau)$ for $x \le -1$ for some values $s^+(\tau)$ and $s^-(\tau)$. It is easy to verify that the system (5.5) remains locally monotone and the systems (5.13) and (5.14) are monotone. The values $c^\pm(\tau)$ of the velocity are replaced by $\sqrt{s^\pm(\tau)}c^\pm(\tau)$ in this case. We can choose $s^\pm(\tau)$ such that

$$\sqrt{s^-(\tau)}c^-(\tau) > \sqrt{s^+(\tau)}c^+(\tau), \quad \tau \in [0,1]$$

and $s^\pm(0) = s^\pm(1) = 1$.

It remains to note that (C3) is satisfied for N independent of τ since F_τ and F have the same zeros, $F_\tau = s_\tau[(1-\tau)F + \tau\Phi]$ and $s_\tau^{'}(x) \equiv 0$ for $|x| \ge 1$.

The model system with $F_1(w,x)$ is monotone and it has a monotone solution [13]. As it was pointed out above the index of this solution equals 1. From the global stability of the solution follows its uniqueness [13]. Thus $\gamma(A_1, D_1) = 1$.

From a priori estimates of monotone solutions and by construction of the domain D_τ we have

$$A_\tau(u) \ne 0, \quad u \in \partial D_\tau.$$

Hence $\gamma(A_0, D_0) = \gamma(A_1, D_1) = 1$. This proves existence of monotone solutions of the problem (5.1).

6. Nonexistence of the degree

We consider the scalar equation (5.1) ($p = 1$) and put $c = 0$, $w_+ = 0$, $w_- = 1$. Suppose that

$$\int_0^1 F^-(w)\,dw > 0, \qquad \int_0^1 F^+(w)\,dw < 0.$$

Then there exists a monotonically decreasing solution $w(x)$ of this problem [13]. The principal eigenvalue of the problem

$$v^{''} + F'(w(x),x)v = \lambda v, \quad v(\pm\infty) = 0$$

is negative. Indeed, $-w'(x)$ is a positive solution of the inequality

$$v^{''} + F'(w(x),x)v \le 0.$$

Hence all eigenvalues lie in the left-half plane [13]. Thus $w(x)$ is a unique monotonically decreasing solution of the problem

$$w^{''} + F(w,x) = 0, \quad w(\infty) = 0, \quad w(-\infty) = 1. \tag{6.1}$$

It is easy to construct a continuous deformation $F_\tau(w,x)$, $\tau \in [0,1]$ such that $F_0(w,x) = F(w,x)$, $\partial F_1/\partial x < 0$, and both integrals

$$\int_0^1 F_1^\pm(w)\,dw$$

are positive. During this deformation the monotone solution of the problem

$$w'' + F_\tau(w,x) = 0, \quad w(-\infty) = 1, \quad w(\infty) = 0 \tag{6.2}$$

disappears [13].

Consider the operator $A(u)$ defined above as acting from $C_0^{2+\delta}$ to C_0^δ. Here the subscript 0 denotes the subspace of functions converging to 0 at infinity. If the functions $F(w,x)$ and ψ'' are Hölder continuous, then the operator A is bounded and continuous. Denote u_M and u_N the functions such that $w = u_M + \psi$ and $w = u_N + \psi$ are monotone and nonmonotone, respectively. As we discussed above, monotone and nonmonotone solutions of the equation $A(u) = 0$ are separated in the function space. It means that $\|u_M - u_N\| \geq \varepsilon$ for some positive ε and for any monotone and nonmonotone solutions. As in the previous section we can construct a domain D_τ depending on the parameter τ, containing all monotone solutions and which does not contain nonmonotone solutions. A priori estimates of monotone solutions in $C_0^{2+\delta}$ are obvious. Hence the degree $\gamma(A_\tau, D_\tau)$ does not depend on τ. It remains to note that $\gamma(A_0, D_0) = 1$ because the linearized operator $A_0'(w)$ has all eigenvalues in the left-half plane, $\gamma(A_1, D_1) = 0$ because there are no monotone solutions of the equation $A_1(u) = 0$. This contradiction shows that the degree cannot be constructed with this choice of spaces.

Denote τ_c the value of the parameter τ for which

$$\int_0^1 F^+{}_{\tau_c}(w)\,dw = 0. \tag{6.3}$$

Consider the problem

$$w'' + F^+{}_{\tau_c}(w) = 0, \quad w(-\infty) = 1, \quad w(\infty) = 0. \tag{6.4}$$

It has a monotone solution $w_c(x)$. Let $w_\tau(x)$ be a solution of (5.1) for $\tau < \tau_c$. Then

$$w_\tau(x) \sim w_c(x - h(\tau)), \quad \tau \to \tau_c, \tag{6.5}$$

and $h(\tau) \to \infty$ as $\tau \to \tau_c$. The asymptotic (6.5) shows how the solution disappears being bounded in the norm $C_0^{2+\delta}$. It occurs because the limiting problem (6.4) has a solution. If we impose additional conditions such that the limiting problem does not have solutions, the degree can be defined (cf. [3]).

The situation is different if we consider the weighted spaces. The norm of the solution $w_\tau(x) - \psi(x)$ goes to infinity in the space $C_\mu^{2+\delta}$ as $\tau \to \tau_c$. Hence the condition $A(u,\tau) \neq 0$, $u \in \partial D_\tau$ cannot be satisfied (D_τ is uniformly bounded) and disappearance of the solution does not contradict the homotopy invariance. The degree can be defined.

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Cristelle Barillon
School of Mathematical Sciences
Tel Aviv University, Ramat Aviv
69978 Tel Aviv, ISRAEL

E-mail address: cristell@math.tau.ac.il

Vitaly A. Volpert
Analyse Numérique, U.M.R. 5585 CNRS
Université Claude Bernard Lyon I
43, bd. du 11 Novembre 1918
69622 Villeurbanne Cedex, FRANCE

E-mail address: volpert@lan.uiv-lyon1.fr