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ON COHOMOLOGY OF THE INVARIANT PART OF AN ISOLATING BLOCK

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Dedicated to the memory of Juliusz P. Schauder

Abstract. In this paper we review some old and new results on cohomology of the maximal invariant set inside of an isolating block B. In particular, we prove the following one: If $u \cup cv$ is nonzero for some $u \in \overline{H}^*(B)$ and $v \in \overline{H}^*(B, B^-)$ then the restriction of u to $\overline{H}^*(S)$ is nontrivial.

Let ϕ be a flow on a topological space X. For a subset B of X we define its *invariant part* (with respect to ϕ) as

$$\operatorname{Inv} B := \operatorname{Inv}_\phi B := \{x \in B : \phi_t(x) \in B \text{ for all } t \in \mathbb{R}\}.$$

B is called an *isolating block* provided it is compact, $\operatorname{Inv} B$ is contained in the interior of B and both the sets

$$\begin{aligned} B^+ &:= \{x \in B : \exists\{\varepsilon_n\},\ 0 < \varepsilon_n \to \infty,\ \phi_{\varepsilon_n}(x) \notin B\}, \\ B^- &:= \{x \in B : \exists\{\varepsilon_n\},\ 0 < \varepsilon_n \to \infty,\ \phi_{-\varepsilon_n}(x) \notin B\} \end{aligned}$$

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are compact. The compactness condition imposed on $B^{\pm}$ is equivalent to the continuity of the maps

$$\sigma_B^+ : \ B \ni x \to \sup\{t \geq 0 : \phi(x, [-t, 0]) \subset B\} \in [0, \infty],$$
$$\sigma_B^- : \ B \ni x \to \sup\{t \geq 0 : \phi(x, [0, t]) \subset B\} \in [0, \infty],$$

see the proof of the Ważewski Theorem in [2, p. 25]. The notion of isolating block was introduced in [3], where smooth flows was considered. In the topological setting its definition was stated in [1]. Our definition is slightly less restrictive then the one in [1], but it does not affect the results presented below.

In the sequel we put $S := \operatorname{Inv} B$. Let $\overline{H}^*$ denote the Alexander–Spanier cohomology functor having coefficients in some fixed abelian group. We consider the following problem:

PROBLEM. Determine properties of $\overline{H}^*(S)$ by a finite number of data on topology of B and $B^{\pm}$.

Almost the same question was posed in [5, Problem 2]; it concerned sets arising in filtrations of vector-fields on manifolds. Actually, those sets are also isolating blocks in the above sense.

Below we formulate some results on the Problem (reversing of the time-variable leads to the corresponding results in which B^- is replaced by B^+). The simplest situation arises if one of the sets $B^{\pm}$ is empty. Using the tautness of Alexander–Spanier cohomology and continuity of the map σ_B^- it is easy to prove:

THEOREM 1. *If $B^- = \emptyset$ then $\overline{H}^*(S) = \overline{H}^*(B)$.*

As simple examples show, in general $\overline{H}^*(S)$ cannot be completly calculated by the data on B and $B^{\pm}$ only, and one can try to get "lower bound" estimates. A version of the classical Ważewski Retract Theorem delivers such an estimate on $\overline{H}^0(S)$:

THEOREM 2. *If B^- is not a strong deformation retract of B then $S \neq \emptyset$, i.e. $\overline{H}^0(S) \neq 0$.*

COROLLARY 1. *If $\overline{H}^*(B, B^-) \neq 0$ then $S \neq \emptyset$.*

For the block B define

$$A := \{x \in B : \sigma_B^+(x) + \sigma_B^-(x) = \infty\} \quad \text{and} \quad A^{\pm} := A \cap B^{\pm}.$$

The next result comes from [1].

THEOREM 3. *There exists a long exact sequence*

$$\cdots \to \overline{H}^q(B, B^-) \to \overline{H}^q(S) \to \overline{H}^q(A^-) \to \overline{H}^{q+1}(B, B^-) \to \cdots .$$

As one can easily check, the above result is more general then Corollary 1. Nevertheless, it has limited application to our problem since usually the topology of the set A^- is not a priori known.

In order to state other results on the problem we define the notion of cohomological category (see [4]). Assume that $\overline{H}^*$ has real coefficients. Let $q \geq 1$ and $k \geq 0$ be fixed integers. Define

(a) $c^{q,k}(\emptyset) = 0$.
(b) $c^{q,k}(X) = 1$ if and only if $X \neq \emptyset$ and $\dim \overline{H}^q(X) \leq k$.
(c) $c^{q,k}(X) \leq n$ if and only if there exists a closed cover $\{A_1, \dots, A_n\}$ of X such that $\operatorname{rank} e_i^* \leq k$ for $i = 1, \dots, n$, where $e_i^* : \overline{H}^*(X) \to \overline{H}^*(A_i)$ is induced by the inclusion.
(d) $c^{q,k}(X) = \infty$ if and only if $c^{q,k}(X) \neq n$ for every integer n.

The following two results were proved in [4].

Theorem 4. *If B is an* ANR *and $c^{q,k}(B/B^-) \geq 3$ then $\dim \overline{H}^q(S) > k$.*

Theorem 5. *If B is an* ANR *and B/B^- has the homotopy type of a compact oriented $2q$-dimensional manifold then*

$$\dim \overline{H}^q(S) \geq \frac{1}{2} \dim \overline{H}^q(B/B^-).$$

Now we assume that $\overline{H}^*$ has coefficients in a ring-with-unit. The last result presented here was stated in [6].

Theorem 6. *If for $u \in \overline{H}^*(B)$ there exists $v \in \overline{H}^*(B, B^-)$ such that $u \cup v \neq 0$ then the restriction $u|_S \in \overline{H}^*(S)$ is nonzero.*

In particular, if $v \in \overline{H}^*(B, B^-)$ is nonzero then $1 \cup v = v \neq 0$ (where $1 \in \overline{H}^0(B)$ is the unit element), hence $1|_S \neq 0$ which means $S \neq 0$. Thus Theorem 6 is a generalization of Corollary 1.

Proof of Theorem 6. The diagram

$$\begin{array}{ccccc}
\overline{H}^*(B) & \otimes & \overline{H}^*(B, B^-) & \xrightarrow{\cup} & \overline{H}^*(B, B^-) \\
\downarrow & & \downarrow & & \downarrow \\
\overline{H}^*(A) & \otimes & \overline{H}^*(A, A^-) & \xrightarrow{\cup} & \overline{H}^*(A, A^-)
\end{array}$$

(in which the vertical arrows are generated by the inclusions) commutes and $(A, A^-) \hookrightarrow (B, B^-)$ induces an isomorphism by [1, Lemma 4.3], hence

$$u|_A \cup v|_{(A,A^-)} = (u \cup v)|_{(A,A^-)} \neq 0.$$

Thus $u|_A \neq 0$. Since $S \hookrightarrow A$ induces an isomorphism by [1, Proposition 4.6], one gets

$$u|_S = (u|_A)|_S \neq 0. \qquad \square$$

Example. Let K^r denote the r-dimensional unit ball and let S^{r-1} be its boundary. Assume that a block B is homeomorphic to $(K^m \times K^n) \times S^1$ such that B^- is transformed onto $(K^m \times S^{n-1}) \times S^1$. If u is a generator of $\overline{H}^1(B)$ and v is a generator of $\overline{H}^n(B, B^-)$ then $u \cup v \neq 0$. It follows by Theorem 6 that $u|_S \neq 0$, hence $\overline{H}^1(S) \neq 0$. That result cannot be obtained directly from Theorem 3 because no information on A^- is provided, and from Theorem 4 because $c^{q,k}(B/B^-) \leq 2$ for each q and k.

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