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SHARKOVSKIĬ THEOREM FOR MULTIDIMENSIONAL PERTURBATIONS OF ONE-DIMENSIONAL MAPS II

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Abstract. We present a multidimensional generalization of the Sharkovskiĭ Theorem concerning the appearance of periodic points for the self-maps on the real line.

Introduction

Let $f : X \to X$ be a map. We say that $x \in X$ is a periodic point of period n if $f^n(x) = x$ and $f^k(x) \neq x$ for $k = 1, \ldots , n - 1$.

We begin with recalling the Sharkovskiĭ Theorem.

Theorem 1. *Let the ordering of positive integers be as follows:*

$$3 \lhd 5 \lhd 7 \lhd \ldots \lhd 2 \cdot 3 \lhd 2 \cdot 5 \lhd 2 \cdot 7 \lhd \ldots \lhd 2^2 \cdot 3 \lhd 2^2 \cdot 5 \lhd \ldots \lhd 2^3 \cdot 3 \lhd 2^3 \cdot 5 \lhd \ldots \lhd 2^3 \lhd 2^2 \lhd 2 \lhd 1.$$

Let $f : I \to \mathbb{R}$ be a continuous map of an interval into the real line. If $n \lhd k$ and f has a periodic point of period n then f also has a periodic point of period k.

The ordering described in Theorem 1 is called *the Sharkovskiĭ ordering.* We reserve the symbol "$\lhd$" for this order.

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Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous map. Let V be a real Banach space and let the continuous decomposition $V=\mathbb{R}\oplus W$ be given. According to this decomposition we will represent elements $v\in V$ as pairs $v=(x,w)$, where $x\in\mathbb{R}$ and $w\in W$. Let $F:[0,1]\times V\to V$ be a continuous and compact map. We will use the notation F_λ for the partial map with the fixed λ, so $F_\lambda(v):=F(\lambda,v)$ for $v\in V$. Suppose that $F_0(x,w)=(f(x),0)$. We say that the maps F_λ are multidimensional perturbations of f. Let us recall that a continuous map is called *compact*, if and only if it maps bounded sets into relatively compact sets.

The main theorem proved in this paper is stated below

THEOREM 2. *Let f, F be as above. Suppose f has a point of period n. For every $r>0$ there exists $\lambda_0>0$ such that, for $\lambda\le\lambda_0$, if $n\lhd m$, $m\neq n$ then F_λ has a periodic point of period m in the set $\mathbb{R}\oplus B_W(r)$, where $B_W(r)$ is an open ball in W of radius r.*

The above theorem was proved in [9] with the additional assumption that F is uniformly continuous.

In this paper we present only modifications to the arguments from [9] required for the proof of Theorem 2, so without referring to [9] this note is rather unreadable.

Sections 1–5 correspond to analogous sections in [9]. In all these sections but 3 even the header of the section is preserved.

The idea of the modifications can be explained as follows. In a study of periodic points it is enough to consider only points which are in the image of the map, which in our case is a compact set.

Another modification is a simpler proof of the existence of a nested sequence of topological horseshoes for one-dimensional maps (see Section 3).

Section 6 contains some remarks concerning the relevance of the presented results in the context of ordinary differential equations.

1. Topological theorems

This section is a modification of analogous section in [9]. We will use the notations used there. The changes begin with a new definition of *covering relations.*

Let us fix an $r>0$. We define N, $L(N)$, $R(N)$, $S_L(N)$, $S_R(N)$ as in [9].

DEFINITION 1. Let $f:(-\infty,\infty)\times\overline{B_W(r)}\to V$ be a continuous and compact map and $A\subset V$ such that

$$\overline{f(N)}\cap N\subset A,\quad A\cap L(N_i)\neq\emptyset,\quad A\cap R(N_i)\neq\emptyset.$$

We say that N_i *f-covers horizontally N_j with respect to A* if and only if

$$f(A\cap N)\subset(-\infty,\infty)\times B_W(r),\tag{1}$$

and either

$$f(L(N_i)\cap A)\subset S_L(j) \quad\text{and}\quad f(R(N_i)\cap A)\subset S_R(j), \tag{2}$$

or

$$f(L(N_i)\cap A)\subset S_R(j) \quad\text{and}\quad f(R(N_i)\cap A)\subset S_L(j). \tag{3}$$

We will use the following graphical notation for this relation $N_i \overset{f,A}{\Longrightarrow} N_j$. In the case $N \subset A$ we will often omit the set A (this coincides with the notation from [9]).

Let us fix an $n>0$. Let $\alpha=(\alpha_0,\dots,\alpha_{n-1})\in\{0,1,\dots,K-1\}^n$. Let

$$Z=\bigcup_{i=0}^{K-1} Z_k \subset V.$$

For an indexed family of the continuous maps

$$f_i : (-\infty,\infty)\times\overline{B_W(r)}\to V, \quad \text{for } i=1,\dots,n,$$

we define

$$Z_\alpha := \{v\in Z_{\alpha_0} \mid f_i\circ\ldots\circ f_1(v)\in Z_{\alpha_i}, \text{ for } i=1,\dots,n-1\},$$

and similarly, for an indexed family of homotopies

$$F_i : [0,1]\times(-\infty,\infty)\times\overline{B_W(r)}\to V, \quad \text{for } i=1,\dots,n,$$

and $F_{i,\lambda}$ we set

$$Z^\lambda_\alpha := \{v\in Z_{\alpha_0} \mid F_{i,\lambda}\circ\ldots\circ F_{1,\lambda}(v)\in Z_{\alpha_i}, \text{ for } i=1,\dots,n-1\}.$$

The following theorem is a generalization of theorems about topological horseshoes by Mischaikow and Mrozek [4, Theorem 2.3] and Zgliczyński [8].

Theorem 3. *Let $\alpha=(\alpha_0,\dots,\alpha_{n-1})\in\{0,\dots,K-1\}^n$. Let the maps*

$$F_i : [0,1]\times(-\infty,\infty)\times\overline{B_W(r)}\to V, \quad \textit{for } i=1,\dots,n$$

be continuous and compact. Let

$$\operatorname{Im} F=\bigcup_i \overline{F_i([0,1]\times N)}.$$

Suppose that we have the following relations, for every $\lambda\in[0,1]$,

$$N_{\alpha_{i-1}} \overset{F_{i,\lambda},\operatorname{Im} F}{\Longrightarrow} N_{\alpha_i}, \quad \textit{for } i=1,\dots,n-1,$$
$$N_{\alpha_{n-1}} \overset{F_{n,\lambda},\operatorname{Im} F}{\Longrightarrow} N_{\alpha_0},$$

then the fixed point index $I(F_{n,\lambda} \circ \ldots \circ F_{1,\lambda}, \operatorname{int} N_\alpha^\lambda)$ *is well defined and constant* (*i.e. does not depend on* λ).

PROOF. For $\delta > 0$ define

$$L(N_i,\delta) := [a_{2i}, a_{2i}+\delta] \times \overline{B_W(r)}, \tag{4}$$
$$R(N_i,\delta) := [a_{2i+1}-\delta, a_{2i+1}] \times \overline{B_W(r)}, \tag{5}$$
$$V(N_i,\delta) := L(N_i,\delta) \cup R(N_i,\delta), \tag{6}$$
$$H(N_i,\delta) := [a_{2i}, a_{2i+1}] \times (\overline{B_W(r)} \setminus B_W(r-\delta)), \tag{7}$$
$$H(N,\delta) := H(N_0,\delta) \cup H(N_1,\delta) \cup \ldots \cup H(N_{K-1},\delta). \tag{8}$$

For $\delta < a_{2i+1} - a_{2i}$ the sets $L(N_i,\delta)$, $R(N_i,\delta)$, $H(N_i,\delta)$ are contained in N_i and are equal to the δ-thickened left vertical, right vertical and the union of horizontal edges in N_i, respectively.

Let $Z \subset N$. We introduce the notation

$$\operatorname{Fix}_\lambda(Z) := \{x \in Z \mid F_{n,\lambda} \circ \ldots \circ F_{1,\lambda}(x) = x\}. \tag{9}$$

It follows from the compactness of F_i that $\operatorname{Im} F$ is a compact set. It is easy to see that, for $\lambda \in [0,1]$,

$$\operatorname{Fix}_\lambda(N_\alpha^\lambda) \subset \operatorname{Im} F. \tag{10}$$

Obviously, $\operatorname{Fix}_\lambda(N_\alpha^\lambda)$ is a compact set.

From the compactness of $\operatorname{Im} F$ and assumptions concerning covering relations we easily conclude that there exists $\delta > 0$, such that for $i = 1, \ldots, n$ and $\lambda \in [0,1]$ the following conditions hold

$$F_{i,\lambda}(V(N_{\alpha_{i-1}}, 2\delta) \cap \operatorname{Im} F) \cap N_{\alpha_{i \bmod n}} = \emptyset, \tag{11}$$
$$H(N_{\alpha_i}, 2\delta) \cap F_{i,\lambda}(N_{\alpha_{i-1}} \cap \operatorname{Im} F) = \emptyset. \tag{12}$$

We define

$$C_i := (a_{2i}+\delta, a_{2i+1}-\delta) \times B_W(r-\delta), \tag{13}$$
$$D_i := (a_{2i}+2\delta, a_{2i+1}-2\delta) \times B_W(r-2\delta). \tag{14}$$

We have

$$B(D_i,\delta) \subset C_i, \qquad B(C_i,\delta) \subset \operatorname{Int} N_i. \tag{15}$$

Obviously the sets D_α^λ and C_α^λ are both open and the set N_α^λ is closed for $\lambda \in [0,1]$.

We will show now that, for $\lambda \in [0,1]$,

$$F_{n,\lambda} \circ \ldots \circ F_{1,\lambda}(x) \neq x, \quad \text{provided } x \in N_\alpha^\lambda \setminus D_\alpha^\lambda. \tag{16}$$

Suppose that (16) does not hold. Then there exists an $x \in N_\alpha^\lambda \setminus D_\alpha^\lambda$ and $0 \le i_0 \le n-1$ such that

$$F_{n,\lambda} \circ \ldots \circ F_{1,\lambda}(x) = x, \tag{17}$$

$$F_{i_0,\lambda} \circ \ldots \circ F_{1,\lambda}(x) \in (N_{\alpha_{i_0}} \setminus D_{\alpha_{i_0}}). \tag{18}$$

It follows from (12) and (14) that

$$F_{i_0,\lambda} \circ \ldots \circ F_{1,\lambda}(x) \in V(N_{\alpha_{i_0}}, 2\delta). \tag{19}$$

It follows from this and from (11) that

$$F_{i_0+1,\lambda} \circ \ldots \circ F_{1,\lambda}(x). \notin N_{\alpha_{(i_0+1) \bmod n}}. \tag{20}$$

So, if $i_0 < n-1$ then $x \notin N_\alpha^\lambda$, and if $i_0 = n-1$ then it follows from (20) that x can not be a fixed point for the map $F_{n,\lambda} \circ \ldots \circ F_{1,\lambda}$. In both cases we get a contradiction, so (16) holds.

From (16) it follows immediately that

$$\mathrm{Fix}_\lambda(D_\alpha^\lambda) = \mathrm{Fix}_\lambda(C_\alpha^\lambda) = \mathrm{Fix}_\lambda(\mathrm{int} N_\alpha^\lambda) = \mathrm{Fix}_\lambda(N_\alpha^\lambda). \tag{21}$$

Let $\lambda_0 \in [0,1]$. It follows from the uniform continuity of continuous maps on compact sets that there exists an interval Λ open in $[0,1]$, $\lambda_0 \in \overline{\Lambda}$, such that for every $\lambda_1, \lambda_2 \in \overline{\Lambda}$, $i = 1, \ldots, n$ and $x \in \mathrm{Im}\, F$ holds

$$|F_{i,\lambda_1} \circ \ldots \circ F_{1,\lambda_1}(x) - F_{i,\lambda_2} \circ \ldots \circ F_{1,\lambda_2}(x)| \le \delta. \tag{22}$$

We will show now the following inclusions for $\lambda \in \overline{\Lambda}$

$$\mathrm{Im}\, F \cap D_\alpha^\lambda \subset \mathrm{Im}\, F \cap C_\alpha^{\lambda_0} \subset \mathrm{Im}\, F \cap \mathrm{int}(N_\alpha^\lambda). \tag{23}$$

Let $x \in \mathrm{Im}\, F \cap D_\alpha^\lambda$. Then

$$F_{i,\lambda} \circ \ldots \circ F_{1,\lambda}(x) \in D_{\alpha_i}, \quad \text{for } i = 1, \ldots, n-1.$$

However, it follows from (15), (22) that

$$F_{i,\lambda_0} \circ \ldots \circ F_{1,\lambda_0}(x) \in C_{\alpha_i}, \quad \text{for } i = 0, \ldots, n-1.$$

Thus $x \in \mathrm{Im}\, F \cap C_\alpha^{\lambda_0}$. The proof of the second inclusion is analogous. Because $\mathrm{Fix}_\lambda(C_\alpha^{\lambda_0}) \subset \mathrm{Im} F$, we obtain from (21) and (23) for $\lambda \in \overline{\Lambda}$

$$\mathrm{Fix}_\lambda(C_\alpha^{\lambda_0}) = \mathrm{Fix}_\lambda(N_\alpha^\lambda). \tag{24}$$

Since $\mathrm{Fix}_\lambda(N_\alpha^{\lambda_0})$ is a compact set then it follows from (21) and (24) that the fixed point indices of the map $F_{n,\lambda} \circ \ldots \circ F_{1,\lambda}$ relatively to the sets $\mathrm{int}\, N_\alpha^\lambda$, $C_\alpha^{\lambda_0}$, D_α^λ are well defined and from the excision property of the fixed point index [9] we conclude

$$\mathrm{I}(F_{n,\lambda} \circ \ldots \circ F_{1,\lambda}, D_\alpha^\lambda) = \mathrm{I}(F_{n,\lambda} \circ \ldots \circ F_{1,\lambda}, C_\alpha^{\lambda_0}) = \mathrm{I}(F_{n,\lambda} \circ \ldots \circ F_{1,\lambda}, \mathrm{int} N_\alpha^\lambda), \tag{25}$$

for all $\lambda \in \overline{\Lambda}$. Substituting $\lambda := \lambda_0$ we derive

$$\mathrm{I}(F_{n,\lambda_0} \circ \ldots \circ F_{1,\lambda_0}, D_\alpha^{\lambda_0}) = \mathrm{I}(F_{n,\lambda_0} \circ \ldots \circ F_{1,\lambda_0}, C_\alpha^{\lambda_0}). \tag{26}$$

From (21) and (24) it follows that

$$\mathrm{Fix}_{\overline{\Lambda}}(C_\alpha^{\lambda_0}) := \bigcup_{\lambda \in \overline{\Lambda}} \mathrm{Fix}_\lambda(C_\alpha^{\lambda_0}) \subset C_\alpha^{\lambda_0}, \tag{27}$$

$$\mathrm{Fix}_{\overline{\Lambda}}(C_\alpha^{\lambda_0}) = \bigcup_{\lambda \in \overline{\Lambda}} \mathrm{Fix}_\lambda(N_\alpha^{\lambda}). \tag{28}$$

From the above condition and the compactness of the maps $F_{i,\lambda}$ we see that $\mathrm{Fix}_{\overline{\Lambda}}(C_\alpha^{\lambda_0})$ is compact.

Now, from the homotopy property of the fixed point index, we obtain

$$\mathrm{I}(F_{n,\lambda} \circ \ldots \circ F_{1,\lambda}, C_\alpha^{\lambda_0}) = \mathrm{I}(F_{n,\lambda_0} \circ \ldots \circ F_{1,\lambda_0}, C_\alpha^{\lambda_0}) \quad \text{for all } \lambda \in \overline{\Lambda}. \tag{29}$$

From (25), (26) and (29) we conclude

$$\mathrm{I}(F_{n,\lambda} \circ \ldots \circ F_{1,\lambda}, D_\alpha^{\lambda}) = \mathrm{I}(F_{n,\lambda_0} \circ \ldots \circ F_{1,\lambda_0}, D_\alpha^{\lambda_0}) \quad \text{for all } \lambda \in \overline{\Lambda}. \tag{30}$$

From the connectedness of $[0,1]$, (30) and (25) we get

$$\mathrm{I}(F_{n,\lambda} \circ \ldots \circ F_{1,\lambda}, \mathrm{Int}\, N_\alpha^{\lambda}) = \mathrm{I}(F_{n,0} \circ \ldots \circ F_{1,0}, \mathrm{Int}\, N_\alpha^{0}) \quad \text{for all } \lambda \in [0,1].$$

This finishes the proof. □

The following theorem calculates the fixed point index of the composition $F_{n,\lambda} \circ \ldots \circ F_{1,\lambda}$ on the set $\mathrm{int}\, N_\alpha^\lambda$ for multidimensional perturbations of one-dimensional maps.

THEOREM 4. *Let $\alpha = (\alpha_0, \ldots, \alpha_{n-1}) \in \{0, \ldots, K-1\}^n$. Let $F_i : [0,1] \times N \to V$ for $i = 1, \ldots, n$ be continuous and compact. Suppose that there exist the one-dimensional maps $f_i : \mathbb{R} \to \mathbb{R}$ such that*

$$F_{i,0}(x,y) = (f_i(x), 0), \quad \text{for } i = 1, \ldots, n.$$

Let

$$\mathrm{Im}\, F = \bigcup_i \overline{F_i([0,1] \times N)}.$$

Suppose that, for every $\lambda \in [0,1]$, we have the following relations

$$N_{\alpha_{i-1}} \overset{F_{i,\lambda}, \mathrm{Im}\, F}{\Longrightarrow} N_{\alpha_i}, \quad \text{for } i = 1, \ldots, n-1,$$
$$N_{\alpha_{n-1}} \overset{F_{n,\lambda}, \mathrm{Im}\, F}{\Longrightarrow} N_{\alpha_0},$$

then $I(F_{n,\lambda} \circ \ldots \circ F_{1,\lambda}, \mathrm{int}\, N_\alpha^\lambda) = \pm 1$.

PROOF. It was shown in [9, Theorem 4] that $I(F_{n,0} \circ \ldots \circ F_{1,0}, \mathrm{int}\, N_\alpha^0) = \pm 1$. The assertion of the theorem follows now from Theorem 3. □

Now we are going to define notions of *a topological horseshoe* and *a topological prehorseshoe*.

Let $a_0 < a_1$, $a_2 < a_3$, $(a_0, a_1) \cap (a_2, a_3) = \emptyset$ and $r > 0$, $N_0 = [a_0, a_1] \times \overline{B_W(r)}$, $N_1 = [a_2, a_3] \times \overline{B_W(r)}$ and $N = N_0 \cup N_1$.

Definition 2. A continuous and compact map $f : N \to V$ will be called *a topological horseshoe* if and only if there exists a compact homotopy $F : [0,1] \times N \to V$, such that $F_1 = f$ and F_0 is one-dimensional, such that for $A := \overline{F([0,1] \times N)}$ and $\lambda \in [0,1]$ the following covering relations hold

$$N_i \overset{F_\lambda, A}{\Longrightarrow} N_j, \quad \text{for } i, j = 0, 1.$$

Definition 3. A continuous and compact map $f : N \to V$ will be called *a topological prehorseshoe* if and only if there exists a compact homotopy $F : [0,1] \times N \to V$, such that $F_1 = f$ and F_0 is one-dimensional, such that for $A := \overline{F([0,1] \times N)}$ and $\lambda \in [0,1]$ the following covering relations hold

$$N_0 \overset{F_\lambda, A}{\Longrightarrow} N_0, \quad N_0 \overset{F_\lambda, A}{\Longrightarrow} N_1, \quad N_1 \overset{F_\lambda, A}{\Longrightarrow} N_0.$$

With these definitions the theorems from [9] concerning the existence of symbolic dynamics for topological (pre)horseshoes are also true.

2. Nested sequences of topological horseshoes

Definition 4. Let $f : (-\infty, \infty) \times \overline{B_W(r)} \to V$ be continuous and compact. Let $l \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$, $0 < l < k$. Let I_i^s for $i = 0, 1$ and $s = l, \ldots, k$ be closed intervals such that $I_i^s \supset I_i^{s+1}$ and $\operatorname{int} I_0^l \cap \operatorname{int} I_1^l = \emptyset$. Let $N_i^s := I_i^s \times \overline{B_W(r)}$. We say that f has *an* (l,k)*-nested sequence of topological horseshoes* (*of vertical size* r), if and only if there exists a continuous and compact homotopy map $F : [0,1] \times (-\infty, \infty) \times \overline{B_W(r)} \to V$ such that $F_1 = f$ and F_0 is one-dimensional and for $A := \overline{F([0,1], N_0^l \cup N_1^l)}$ and $\lambda \in [0,1]$ holds

$$N_i^s \overset{F_\lambda^s, A}{\Longrightarrow} N_j^l, \quad \text{for } i, j = 0, 1 \text{ and } s = l, \ldots, k.$$

Definition 5. Let $f : (-\infty, \infty) \times \overline{B_W(r)} \to V$ be continuous and compact. Let $l \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$ $1 < l < k$. Let I_i^s for $i = 0, 1$ and $s = l-1, \ldots, k$ be closed intervals such that $I_i^s \supset I_i^{s+1}$ and $\operatorname{int} I_0^l \cap \operatorname{int} I_1^l = \emptyset$. Let $N_i^s := I_i^s \times \overline{B_W(r)}$. We say that f has *an* (l,k)*-nested sequence of topological horseshoes with a prehorseshoe* (*of vertical size* r), if and only if there exists a continuous and compact homotopy map $F : [0,1] \times (-\infty, \infty) \times \overline{B_W(r)} \to V$ such that $F_1 = f$ and F_0 is one-dimensional and for $A := \overline{F([0,1], N_0^{l-1} \cup N_1^{l-1})}$ and $\lambda \in [0,1]$ holds

$$N_i^s \overset{F_\lambda^s, A}{\Longrightarrow} N_j^{l-1}, \qquad \text{for } i, j = 0, 1 \text{ and } s = l, \ldots, k,$$

$$N_0^{l-1} \overset{F_\lambda^{l-1},A}{\Longrightarrow} N_j^{l-1}, \quad \text{for } i,j = 0,1,$$
$$N_1^{l-1} \overset{F_\lambda^{l-1},A}{\Longrightarrow} N_0^{l-1}.$$

With these definitions the results concerning existence of all but a finite number of periods for nested sequences of topological horseshoes from [9] are true.

3. Nested sequences of topological horseshoes in dimension one

In this section we present a modification to the argument from [9] concerning the existence of the nested sequences of the topological horseshoes for one-dimensional maps which have a point of odd period greater than 1.

In [9] the case of period 3 was treated as follows: from the Sharkovskiĭ theorem we know that there exist a point of period 5 and then we considered case by case all possible permutations, which can be induced by the orbit of period 5 . This was the content of the proof of Theorem 23 in [9].

Here we consider period 3 separately. Due to this for period 5 we are left with only one case — described by the Stefan diagram [7] and treated in Theorem 25 in [9].

Lemma 5. *Let $I \subset \mathbb{R}$ be a closed segment and $f : I \to \mathbb{R}$ be continuous. Suppose that f has a periodic point of period 3, then f has an $(4,\infty)$-nested sequence of topological horseshoes.*

Proof. Let $x_0 < x_1 < x_2$ be an orbit of period 3 for f. Without any loss of generality we may assume that $I = [x_0, x_2]$.

Without any loss of generality we can also assume that

$$f(x_0) = x_1, \quad f(x_1) = x_2, \quad f(x_2) = x_0, \tag{31}$$

because the case of $f(x_0) = x_2$ can be obtained from this via symmetry $x \mapsto -x$. Since $I \subset f([x_1, x_2])$ we can define

$$y_1 := \inf\{y \mid y > x_1,\ f(y) = x_1\}. \tag{32}$$

It follows immediately from the Darboux theorem that

$$x_1 < y_1 < x_2, \quad f(y_1) = x_1, \quad [x_1, x_2] \subset f([x_1, y_1]), \quad I \subset f^2([x_1, y_1]).$$

Observe that $y_1 \in (x_1, x_2) \subset f((x_0, x_1))$, hence we can define

$$y_2 = \sup\{y \mid y < x_1,\ f(y_2) = y_1\}. \tag{33}$$

We have

$$x_0 < y_2 < x_1 < y_1 < x_2. \tag{34}$$

We set

$$I_0 := [x_1, y_1], \quad I_1 := [y_2, x_1]. \tag{35}$$

We have $[x_1, x_2] \subset f(I_0)$ and $I \subset f^2(I_0)$. For I_1 we obtain

$$[y_1, x_2] \subset f(I_1), \quad [x_0, x_1] \subset f^2(I_1), \quad [x_1, x_2] \subset f^3(I_1), \quad I \subset f^4(I_1).$$

We can now easily construct inductively the sets I_j^s for $s \geq 4$ and $j = 0, 1$ with the following properties

$$\begin{aligned} I_j^4 &\subset I_j, \quad j = 0,1, \\ I_j^{s+1} &\subset I_j^s, \quad j = 0,1, \\ f^s(I_j^s) &= I, \quad j = 0,1. \end{aligned}$$

This gives us an $(4, \infty)$-nested sequence of topological horseshoes. □

We summarize Theorems 22 and 25 from [9] and the above lemma into the following statement.

THEOREM 6. *Let $I \subset \mathbb{R}$ be a closed segment. Let $f : I \to \mathbb{R}$ be continuous. Suppose f has a periodic point of odd period n, $n > 1$. Then f has a $(n+1, \infty)$-nested sequence of topological horseshoes with a prehorseshoe.*

4. Existence of infinitely many periodic points for perturbations

Let W, $V = \mathbb{R} \oplus W$ be Banach spaces. Suppose that we have a continuous and compact map $F : [0,1] \times V \to V$ such that $F_0(x, w) = (f(x), 0)$. Contrary to the analogous section in [9] we do not require for F to be uniformly continuous.

THEOREM 7. *Suppose f has a periodic point of odd period n, $n > 1$. $k > 0$. There exists $p \leq \max\{n, 5\}$, such that for every $r > 0$ there exists $\lambda_0 > 0$ such that for $\lambda < \lambda_0$ F_λ has a $(p+1, p+k)$-nested sequence of topological horseshoes with a prehorseshoe of vertical size r.*

PROOF. Let p be the smallest odd period for f greater than 3. It follows from Theorem 6 that f as a one-dimensional map has a $(p+1, \infty)$-nested sequence of topological horseshoes with a prehorseshoe.

Let the segments I_i^s, where $s = p, p+1, \ldots$ and $i = 0, 1$ be as in Definition 4. Let us fix an $r > 0$. We define $N_i^s := I_i^s \times \overline{B_W(r)}$, for $i = 0, 1$ and $s = p, p+1, \ldots$. We set $N^s := N_0^s \cup N_1^s$. Let

$$A = \overline{F\left([0,1], \bigcup N^p\right)}.$$

Observe that from the compactness of F it follows that the set A is compact. From a uniform continuity of F on the compact set $[0,1]\times A$ for sufficiently small λ we have

$$\overline{F^s_\lambda(N^p\cap A)}\subset(-\infty,\infty)\times B_W(r),\quad s=p,\dots,p+k. \tag{36}$$

The remaining conditions of the definition of the covering relations are obtained in the same way for sufficiently small λ. Hence F_λ is a $(p+1,p+k)$-nested sequence of topological horseshoes with prehorseshoe for λ sufficiently small λ.□

Proceeding further as in [9] we obtain the following

THEOREM 8. *Suppose f has a periodic point of odd period n, $n>1$. There exists an integer $M(n)$ such that, for every $r>0$, there exists $\lambda_0>0$ such that, for $\lambda\le\lambda_0$ and $m>M(n)$, F_λ has a periodic point of period m in the set $\mathbb{R}\oplus B_W(r)$.*

5. Continuation of individual periodic orbits

In this section we assume that W and $V=\mathbb{R}\oplus W$ are Banach spaces. $F:[0,1]\times V\to V$ is a continuous and compact map, such that $F_0(x,w)=(f(x),0)$. Contrary to the analogous section in [9] we do not assume a uniform continuity of F.

Suppose that f has a nontrivial periodic point x, so we can define segments I, I_i and $D(f,x)$ as in [9].

The aim of this section is to show that many periodic orbits, which exist for f by the Sharkovskiĭ theorem, continue for small λ. The only modification in comparison to [9] is in the proof of the following theorem

THEOREM 9. *Let $p\in\mathbb{N}$, $p>0$. Suppose that there exist $i_0,i_1,\dots,i_{p-1},i_p$ where $i_0=i_p$ such that*

$$D_{i_p i_{p-1}}\dots D_{i_1 i_0}=-1.$$

Then for every $r>0$ there exists $\lambda_0>0$ such that for $\lambda<\lambda_0$ there exists a periodic point z_λ such that

$$F^l_\lambda(z_\lambda)\in\operatorname{Int} I_{i_l}\times B_W(r),\quad \textit{for } l=0,\dots,p-1, \tag{37}$$

$$F^p_\lambda(z_\lambda)=z_\lambda. \tag{38}$$

Before we prove the above theorem we need one technical lemma, which was proved in [9].

LEMMA 10. *Suppose that assumptions of Theorem 9 are satisfied. Then there exists a family of segments* $J_l = [d_l, u_l]$ *for* $l = 0, \dots, p-1$ *such that*

$$J_l \subset \operatorname{Int} I_{i_l}, \tag{39}$$

$$f(I_{l-1}) = I_l, \tag{40}$$

$$f^p(d_0) > d_0, \quad f^p(u_0) < u_0. \tag{41}$$

PROOF OF THEOREM 9. Let us fix $r > 0$. Let

$$\operatorname{Im} F = \overline{F([0,1] \times I \times \overline{B_W(r)})}. \tag{42}$$

It follows from a uniform continuity of F on $[0,1] \times \operatorname{Im} F$ that for sufficiently small λ we have

$$\overline{F^s_\lambda((I \times \overline{B_W(r)}) \cap \operatorname{Im} F)} \subset (-\infty, \infty) \times B_W(r), \quad \text{for } s = 1, \dots, p. \tag{43}$$

Let $J_l = [d_l, u_l]$, for $l = 0, \dots, p-1$ be a family of segments constructed in Lemma 10. We show now that

$$\mathrm{I}(F^p_0, \operatorname{Int} J_0 \times B_W(r)) = 1. \tag{44}$$

To this end we use the homotopy property of the fixed point index (see [9], [1], [3]). We define the homotopy $H : [0,1] \times V \to V$ by

$$H(\lambda, (x,y)) = (1-\lambda)(f^p(x), 0) + \lambda((d_0 + u_0)/2, 0). \tag{45}$$

Let us remark that it follows from (41) for $\lambda \in [0,1]$ and $y \in W$ that

$$x(H_\lambda(d_0, y)) > d_0, \qquad x(H_\lambda(u_0, y)) < u_0, \tag{46}$$

$$\operatorname{Fix}(H, J_0 \times \overline{B_W(r)}) \cap \partial(J_0 \times \overline{B_W(r)}) = \emptyset. \tag{47}$$

Therefore, by the homotopy property of the fixed point index and by the formula for an index of affine maps [9], we obtain (44).

Let $\lambda_0 > 0$ be such that for $\lambda < \lambda_0$ holds

$$x(F^p_\lambda(d_0, y)) > d_0, \quad x(F^p_\lambda(u_0, y)) < u_0, \tag{48}$$

for $|y| \le r$, $(d_0, y) \in \operatorname{Im} F$, $(u_0, y) \in \operatorname{Im} F$. The existence of such λ_0 follows from (41) and a uniform continuity of F^p on the set $[0,1] \times \{(J_0 \times \overline{B_W(r)}) \cap \operatorname{Im} F\}$.

From (48) and (43) we get

$$\operatorname{Fix}(F^p_\lambda, J_0 \times \overline{B_W(r)}) \cap \partial(J_0 \times \overline{B_W(r)}) = \emptyset, \tag{49}$$

for $\lambda < \lambda_0$. Therefore, we can apply again the homotopy property to F_λ. We obtain

$$\mathrm{I}(F^p_\lambda, \operatorname{int} J_{i_0} \times B_W(r)) = 1. \tag{50}$$

Now to obtain (37) we find $\lambda_1 \leq \lambda_0$ such that for $\lambda < \lambda_1$ holds

$$F^l_\lambda(\{J_l \times \overline{B_W(r)}\} \cap \operatorname{Im} F) \subset \operatorname{Int} I_{l+1} \times B_W(r), \quad \text{for } l = 1, \dots, p-1. \tag{51}$$

Again the existence of λ_1 follows from (39), (40) and the uniform continuity of F on compacts. □

Proceeding further as in [9] we obtain the following theorem.

Theorem 11. *Suppose f has a point of odd period n, $n \geq 3$. Then for every $M \in \mathbb{N}$ and every $r > 0$ there exists λ_0 such that for $\lambda \leq \lambda_0$ holds*

(∗) *if $m \leq M$ and $n \lhd m$, $m \neq n$ then F_λ has a periodic point of period m in $\mathbb{R} \oplus B_W(r)$.*

Proof of Theorem 2. Observe that Theorem 2 for n odd follows immediately from the above theorem and Theorem 8. The proof of Theorem 2 for even periods is exactly the same as in [9] with the similar modifications as these presented in this section. □

6. Conclusions and outlook

Theorem 2 presented here is an example of the following prototype theorem:

Theorem 12. *Let $I \subset \mathbb{R}$ be a closed segment and $f : I \to \mathbb{R}$ be a continuous map. Suppose that f has some interesting dynamical property* **A**. *Then there exists $\delta = \delta(f, A)$, $n = n(f)$ such that every continuous and compact map $P : I \oplus W \to \mathbb{R} \oplus W$, satisfying*

$$|P^i(x,y) - (f^i(x), 0)| \leq \delta, \quad (x,y) \in Z,\ i = 1, \dots, n, \tag{52}$$

where Z is a suitable compact set containing the set

$$\overline{P(I \oplus \overline{B}_W(r))} \cap I \oplus \overline{B}_W(r)),$$

has some interesting property **A'** (*similar to property* **A**).

To see that Theorem 2 is indeed an example of realization of the above prototype theorem observe that λ_0 from its conclusion is given by the following condition

$$|F^i_\lambda(x,y) - F^i_0(x,y)| \leq \delta(n, F_{0|I}), \quad (x,y) \in Z,\ i = 1, \dots, M(n),\ \lambda \leq \lambda_0, \tag{53}$$

where n is the smallest (in the Sharkovskiĭ ordering) period for F_0, I is a closed interval spanned by the periodic orbit of period n, $M(n)$ is obtained in Theorem 8 and

$$Z = \overline{F([0, \lambda_0] \times I \oplus \overline{B}_W(r))} \cap I \oplus \overline{B}_W(r)).$$

Given the map P, we define homotopy F as follows

$$F_\lambda(x,y) = \lambda P(x,y) + (1-\lambda)(f(x),0). \tag{54}$$

In applications the map P can be given as a Poincaré map for an ordinary or partial differential equation. If the map P possesses an nearly one-dimensional attractor, then we can define *a one-dimensional model map f* and we can treat P as *a multidimensional perturbation of one-dimensional map f* through the homotopy (54).

In fact a direct application of Theorem 2 in this case is usually impossible, because the bounds (53) are very hard to verify even with computer assistance. However, the methods used in the proof of Theorem 2: topological horseshoes and a continuation of individual orbits can be applied under much weaker conditions. For an example of such approach to the studies of strange attractors for Rössler and Lorenz equations the reader is referred to papers [10] and [2], respectively. In those papers the existence of symbolic dynamics was proved with computer assistance, we used computer to rigorously check assumptions of our theorems concerning topological horseshoes.

We think that the methods developed in [9] and the present paper can be applied to prove *prototype theorem* also for other dynamical properties of topological origin like: topological entropy, other forcing relations between periodic orbits, kneading sequences.

For development concerning topological horseshoes in the case of more than one expanding direction the reader is referred to other papers by author [11]–[13].

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