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EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR BOUNDARY VALUE PROBLEMS OF 2D ORDER ODE

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Abstract. We study the existence and multiplicity of positive solutions of the boundary value problem

$$u'' + sf(t,u) = 0, \quad u(0) = u(1) = 0,$$

where s is a positive parameter and f is a non-negative function. We indicate examples with more than two solutions.

1. Introduction

In this paper, we study positive solutions of the Dirichlet problem

$$u'' + sf(t,u) = 0, \quad u(0) = u(1) = 0, \tag{1}$$

for positive values of s and in case f is non-negative. With such assumptions, non-trivial solutions are positive for $t \in (0,1)$. Early results on positive solutions can be found in the survey paper by Wong [17].

We consider assumptions so that the number of solutions of problem (1) depends on the value of the parameter s. Typically, for large values of s there is no solution and the number of solutions increases as s becomes small. This problem has been widely studied. The model example

$$u'' + sh(t)e^u = 0, \quad u(0) = u(1) = 0, \tag{2}$$

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was studied by Choi [2]. Korman and Ouyang [11] have considered nonlinearities with polynomial growth and in case of convex nonlinearities they obtain an exact count of the number of solutions. Systems with an exponential nonlinearity are considered in Ki Sik Ha and Yong-Hoon Lee [8]. Autonomous equations with asymptotically linear nonlinearities can be found in Mironescu and Radulescu [14].

It is known that natural assumptions for the Dirichlet problem allow f to be singular at both end points 0 and 1. This remark goes back to Rosenblatt [15] in 1933. A more recent study can be found in Habets and Zanolin [7]. Here, we use the framework in [7], i.e. f is supposed to satisfy some Carathéodory conditions and is bounded on compact sets by a function $h(t)$ such that $t(1-t)h(t) \in L^1(0,1)$. In this case solutions are in $\mathcal{C}([0,1]) \cap W^{2,1}_{\rm loc}((0,1))$. This is clear from the example

$$u'' + \frac{s}{t} = 0, \quad u(0) = u(1) = 0, \tag{3}$$

whose solution is $u(t) = st\log(1/t)$. Problems (1) with singular nonlinearities have been studied by Lomtatidze [13], Wong [16], Ki Sik Ha and Yong-Hoon Lee [8].

Positive solutions of (1) were obtained using various methods such as upper and lower solutions [11], [13], [8], bifurcation theory [11], fixed point theory [1], variational methods [12]. We will base our analysis on the method of upper and lower solutions except for the last sections where we use a shooting argument together with a detailed analysis of variational equations. Most of our results can be extended to the derivative dependent case

$$u'' + sf(t,u,u') = 0, \quad u(0) = u(1) = 0.$$

This implies obtaining an a-priori bound on the derivative. One way is to impose some Nagumo condition $|f(t,u,u')| \le \psi(t)\phi(|u'|)$, where $\psi \in L^p(0,1)$ and $\phi \in \mathcal{C}(\mathbb{R})$ satisfy appropriate conditions. This approach rules out singular problems such as (3). An alternative way uses bounding functions (see [9], [10]) but is quite elaborate. For these reasons, we do not consider here derivative dependent nonlinearities.

The paper is organized as follows. In Section 2, we recall the theoretical backgrounds from the theory of lower and upper solution we use. These can be found in [4] and [5]. The third section deals with the existence of one solution. Several cases are considered which ensure the existence of at least one positive solution if $s > 0$ is smaller than some s_0 and zero solution for $s > s_0$. In the next section, we study cases where there are at least two positive solutions for $s < s_0$, one if $s = s_0$ and no solution if $s > s_0$. An exact count of the number of solutions is worked out in Sections 5 for non-autonomous problems but under strict convexity assumptions. The last section considers the model example (2) and investigates cases where there are more than two solutions.

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2. Preliminary results

Let us first introduce the notation. We shall write $\mathbb{R}^+ = [0,\infty)$, and "for a.e." instead of "for almost every". We consider the set of functions

$$\begin{aligned} \mathcal{A} &:= \{u \in L^1_{\mathrm{loc}}(0,1) \mid s(1-s)u(s) \in L^1(0,1)\}, \\ W^{k,p}(I) &:= \{u \in \mathcal{C}^{k-1}(I) \mid u^{(k)} \in L^p(I)\}, \text{ where } I \text{ is an interval}, \\ W^{2,\mathcal{A}}(0,1) &:= \{u \in W^{1,1}(0,1) \mid u'' \in \mathcal{A}\}, \end{aligned}$$

and the norms

$$||u||_{\mathcal{A}} := \int_0^1 t(1-t)|u(t)|\,ds, \qquad \|u\|_\infty := \sup_{t\in[0,1]} |u(t)|.$$

To study the nonlinear boundary value problem

$$u'' + f(t,u) = 0, \quad u(0) = u(1) = 0, \tag{4}$$

it is necessary to investigate first the corresponding linear problem. The following proposition is easy to verify.

Proposition 2.1. *There exists $K > 0$ such that, if $h \in \mathcal{A}$, the problem*

$$u'' + h(t) = 0, \quad u(0) = 0,\ u(1) = 0, \tag{5}$$

has a solution $u \in W^{2,\mathcal{A}}(0,1)$ with

$$u(t) = \int_0^1 G(t,s)h(s)\,ds,$$

where $G(t,s)$ is the Green function corresponding to (5). *Further, one has*

$$||u||_\infty \le K||h||_{\mathcal{A}}$$

and if $h \ge 0$, then $u \ge 0$.

Our next definition concerns the regularity conditions we require on the nonlinearity $f(t,u)$. A function $f(t,u)$ defined on $[0,1]\times\mathbb{R}^+$ is said to satisfy $\mathcal{A}$-*Carathéodory conditions* (resp. to satisfy L^∞-*Carathéodory conditions*) if

(a) for almost every $t \in [0,1]$, $f(t,\cdot)$ is continuous,
(b) for any $u \in \mathbb{R}^+$, the function $f(\cdot,u)$ is measurable,
(c) for all $r > 0$, exists $h_r \in \mathcal{A}$ (respectively, exists $h_r \in L^\infty$), for all $u \in [0,r]$ and for a.e. $t \in [0,1]$, $|f(t,u)| \le h_r(t)$.

The basic concept of lower and upper solutions we use is defined as follows. A function $\alpha \in \mathcal{C}([0,1])$ is a *lower solution* of (4) if $\alpha(0) \le 0$, $\alpha(1) \le 0$ and for any $t_0 \in (0,1)$, one of the following is satisfied: either $D_-\alpha(t_0) < D^+\alpha(t_0)$, or there exists an open interval $I_0 \subset (0,1)$ such that $t_0 \in I_0$, $\alpha \in W^{2,1}(I_0)$ and, for almost every $t \in I_0$,

$$\alpha''(t) + f(t, \alpha(t)) \ge 0.$$

In the same way, a function $\beta \in \mathcal{C}([0,1])$ is an *upper solution* of (4) if $\beta(0) \ge 0$, $\beta(1) \ge 0$ and for any $t_0 \in (0,1)$, one of the following is satisfied: either $D^-\beta(t_0) > D_+\beta(t_0)$, or there exists an open interval $I_0 \subset (0,1)$ such that $t_0 \in I_0$, $\beta \in W^{2,1}(I_0)$ and, for almost every $t \in I_0$,

$$\beta''(t) + f(t, \beta(t)) \le 0.$$

We can state now the following existence result, which can be found in [7] for continuous f and in [3] for the $\mathcal{A}$-Carathéodory case.

Proposition 2.2. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfy $\mathcal{A}$-Carathéodory conditions. Assume that α and β are lower and upper solutions of* (4) *such that, for any $t \in [0,1]$, $\alpha(t) \le \beta(t)$. Then the problem* (4) *has at least one solution $u \in W^{2,\mathcal{A}}(0,1)$ such that, for all $t \in [0,1]$,*

$$\alpha(t) \le u(t) \le \beta(t).$$

The next tool we need concerns the relation between lower and upper solutions and degree theory. Let us recall first that the problem (4) can be written $u = Tu$, where $T : \mathcal{C}_0([0,1]) \to \mathcal{C}_0([0,1])$ is defined by

$$Tu(t) := \int_0^1 G(t,s) f(s, u(s))\, ds, \tag{6}$$

and $G(t,s)$ is the Green function corresponding to (5).

Next, we must introduce strict lower and upper solutions. The definitions we propose are not the most general ones but they suffice for our purpose. A function $\alpha \in \mathcal{C}([0,1])$ is a *strict lower solution* of (4) if $\alpha(0) < 0$, $\alpha(1) < 0$ and for any $t_0 \in (0,1)$, one of the following is satisfied:

(a) $D_-\alpha(t_0) < D^+\alpha(t_0)$,

(b) there exists an interval $I_0 \subset [0,1]$ and $\varepsilon > 0$ such that $t_0 \in \operatorname{int} I_0$, $\alpha \in W^{2,1}(I_0)$ and for almost every $t \in I_0$, for all $u \in [\alpha(t), \alpha(t) + \varepsilon]$, we have

$$\alpha''(t) + f(t, u) \ge 0.$$

In the same way, a function $\beta \in \mathcal{C}([0,1])$ is a *strict upper solution* of (4) if $\beta(0) > 0$, $\beta(1) > 0$ and for any $t_0 \in (0,1)$, one of the following is satisfied:

(a) $D^-\beta(t_0) > D_+\beta(t_0)$,

(b) there exists an interval $I_0 \subset [0,1]$ and $\varepsilon > 0$ such that $t_0 \in \operatorname{int} I_0$, $\beta \in W^{2,1}(I_0)$ and for almost every $t \in I_0$, for all $u \in [\beta(t) - \varepsilon, \beta(t)]$, we have

$$\beta''(t) + f(t,u) \le 0.$$

Now we can state the relation between lower and upper solutions and degree theory (see [4] or [5]).

Proposition 2.3. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a $\mathcal{A}$-Carathéodory function and assume α and β are strict $W^{2,1}$-lower and upper solutions of the problem* (4) *such that $\alpha < \beta$. Let*

$$\Omega = \{u \in \mathcal{C}_0([0,1]) \mid \textit{ for all } t \in [0,1], \alpha(t) < u(t) < \beta(t)\},$$

and $T : \mathcal{C}_0([0,1]) \to \mathcal{C}_0([0,1])$ be the operator defined by (6)*. Then, we have*

$$\deg(I - T, \Omega) = 1.$$

3. Existence of one solution

In this section, we consider the problem

$$u'' + sf(t,u) = 0, \quad u(0) = u(1) = 0. \tag{7}$$

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ satisfy $\mathcal{A}$-Carathéodory conditions and assume $f(t,0) \not\equiv 0$. Then, there exists $s_0 \in (0,\infty) \cup \{\infty\}$ such that*

(a) *for any $s \in (0, s_0)$,* (7) *has at least one positive solution,*
(b) *for any $s > s_0$,* (7) *has no solution.*

Proof. *Claim 1.* For any $s > 0$ small enough, there is a positive solution u_s of (7).

Define $h \in \mathcal{A}$ such that

$$\text{for a.e. } t \in [0,1] \text{ and all } u \in [0,1], \quad |f(t,u)| \le h(t).$$

Define now β_1 to be the solution of

$$\beta_1'' + h(t) = 0, \quad \beta_1(0) = \beta_1(1) = 0.$$

From Proposition 2.1, we know that

$$\beta_1(t) \ge 0 \quad \text{and} \quad ||\beta_1||_\infty \le K||h||_{\mathcal{A}}.$$

Next, we choose $\widehat{s} > 0$ small enough so that $\widehat{s}K||h||_{\mathcal{A}} \le 1$ and define, for $s \in (0, \widehat{s}]$, $\beta := s\beta_1$. We compute then

$$\beta'' + sf(t,\beta) = s(f(t,\beta) - h(t)) \le 0.$$

Hence, $\alpha = 0$ and $\beta \geq \alpha$ are lower and upper solutions for (7) and the claim follows from Proposition 2.2.

Claim 2. Define $s_0 = \sup\{s \mid (7) \text{ has a solution }\} \in (0,\infty) \cup \{\infty\}$. Then for any $\overline{s} \in (0, s_0)$ problem (7) has a positive solution.

Let us notice first that solutions of (7) for $s > 0$ are positive so that the existence of solutions is equivalent to the existence of positive solutions. From Claim 1, it follows that $s_0 > 0$. Let us fix $\overline{s} \in (0, s_0)$ and $s_1 \in [\overline{s}, s_0]$ such that (7), with $s = s_1$, has a solution u_1, which is positive. Notice that $\alpha = 0$ and $\beta = u_1$ are lower and upper solutions for (7), with $s = \overline{s}$, and the claim follows from Proposition 2.2. □

Remarks. (a) If in the proof of Claim 1 we let s go to zero, we have that $||\beta||_\infty$ and hence $||u_s||_\infty$ goes to zero. This shows there is, in the space (u, s), a "branch" of solutions that goes out of the origin.

Further, if we assume $f \in \mathcal{C}^1$, we can apply the implicit function theorem to the equation

$$\Phi(s, u) = u'' + sf(t, u) = 0,$$

and prove that this "branch" is actually a curve parametrized by s.

(b) If $s_1 < s_2$, it follows from the proof of Claim 2 that there exist corresponding solutions u_1 and u_2 which are ordered: $u_1 \leq u_2$. It can also be proved that for any $s \in (0, s_0)$ there exists a minimal positive solution u_s which is an increasing function of s, i.e. $s_1 \geq s_2$ implies that, for all $t \in [0, 1]$, $u_{s_1}(t) \geq u_{s_2}(t)$.

(c) If we consider the example (3) it is clear that solutions do not have in general bounded derivatives.

That same example (3) has a positive solution for any $s > 0$. Hence without additional assumptions we might have $s_0 = \infty$. The next result gives conditions so that $s_0 \in \mathbb{R}^+$.

Theorem 3.2. *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}^+$ satisfy $\mathcal{A}$-Carathéodory conditions. Assume $f(t, 0) \not\equiv 0$ and there exists $k \in L^\infty(0, 1)$ such that $k \geq 0$, $k(t) \not\equiv 0$ and*

$$\text{for a.e. } t \in [0, 1] \text{ and all } u \geq 0,\ \ f(t, u) \geq k(t)u.$$

Then, there exists $s_0 \in \mathbb{R}^+$ such that

(a) *for any $s \in (0, s_0)$, (7) has at least one positive solution,*
(b) *for any $s > s_0$, (7) has no solution.*

Proof. From Theorem 3.1, it is enough to prove that for $s > 0$ large enough, there is no solution of (7). Consider the eigenvalue problem

$$u'' + \lambda k(t)u = 0, \quad u(0) = u(1) = 0,$$

and let $\lambda_1 > 0$ and φ_1 be its first eigenvalue and eigenfunction. Assume there exists a solution u of (7) with $s > \lambda_1$. We compute

$$0 = \int_0^1 \frac{d}{dt}(u'(t)\varphi_1(t) - u(t)\varphi_1'(t))\,dt \le \int_0^1 (\lambda_1 - s)k(t)u(t)\varphi_1(t)\,dt < 0,$$

which is impossible. $\square$

Remarks. (a) Notice there is no point in assuming that $k \in \mathcal{A}$, since we can replace $k(t)$ by $\min(k(t), 1)$, which is in $L^\infty(0,1)$.

(b) It follows from the proof that $s_0 \le \lambda_1$. Hence, in case k is constant, $\lambda_1 = \pi^2/k$ and there is no solution if $s \in (\pi^2/k, \infty)$.

To better understand the role of the lower bound on $f(t,u)$, we can write the following modification of Theorem 3.2.

Proposition 3.3. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ satisfy $\mathcal{A}$-Carathéodory conditions. Assume $f(t,0) \not\equiv 0$ and there exists a continuous function $g : \mathbb{R} \to \mathbb{R}^+$ such that*

(i) $\sup_{u_0>0} \int_0^{u_0} \frac{du}{\sqrt{G(u_0) - G(u)}} < \infty$, *where* $G(u) := \int_0^u g(v)\,dv$,
(ii) $\liminf_{u\to 0} g(u)/u > 0$,
(iii) *for a.e.* $t \in [0,1]$ *and all* $u \ge 0$, $f(t,u) \ge g(u)$.

Then, there exists $s_0 \in \mathbb{R}^+$ such that

(a) *for any $s \in (0, s_0)$, (7) has at least one positive solution,*
(b) *for any $s > s_0$, (7) has no solution.*

Proof. Assumption (ii) is such that for a small enough and s large enough, $\alpha(t) = a\sin\pi t$ is a lower solution of

$$u'' + sg(u) = 0, \quad u(0) = u(1) = 0. \tag{8}$$

Any solution $u(t)$ of (7) is an upper solution for (8). Hence if $s_0 = \infty$, we deduce from Proposition 2.2 that the problem (8) has solutions for any value of s. But this is impossible since assumption (i) implies that for s large enough, (8) has no solution. $\square$

We can obtain more precise results for linearly bounded nonlinearities.

Proposition 3.4. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ satisfy L^∞-Carathéodory conditions. Assume $f(t,0) \not\equiv 0$ and for some $a > 0$, $b > 0$ and all $u \ge 0$ $f(t,u) \le a + bu$. Then, for $s \in (0, \pi^2/b)$, (7) has a positive solution.*

Proof. Take

$$\beta(t) = B(\cos\sqrt{sb}(t - 1/2) - \cos\sqrt{sb}/2) \ge 0,$$

and compute for B large enough

$$\begin{aligned}\beta''(t)+ &sf(t,\beta(t)) \leq \\ &\leq -sbB\cos\sqrt{sb}(t-1/2)+s[a+bB(\cos\sqrt{sb}(t-1/2)-\cos\sqrt{sb}/2)] \\ &= s[a-bB\cos\sqrt{sb}/2]<0.\end{aligned}$$

The proof follows now from Proposition 2.2 with $\alpha(t)=0$. □

The situation is somewhat different if f is also asymptotically linear near the origin as follows from the following result.

Proposition 3.5. *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}^+$ satisfy $\mathcal{A}$-Carathéodory conditions. Assume that for some $c\in[0,\infty)$, $d\in[0,\infty)\cup\{\infty\}$, for a.e. $t\in[0,1]$ and all $u>0$, $c\leq f(t,u)/u\leq d$. Then, for $s\in(0,\pi^2/d)\cup(\pi^2/c,\infty)$, (7) has no positive solution.*

Proof. Assume $u(t)$ is a positive solution of (7) and let $v(t)=\sin\pi t$. If $sc>\pi^2$, we have the contradiction

$$0=\int_0^1\frac{d}{dt}(u'(t)v(t)-u(t)v'(t))\,dt=\int_0^1\left(\pi^2-s\frac{f(t,u(t))}{u(t)}\right)u(t)v(t)\,dt<0.$$

Similarly, if $sd<\pi^2$, we obtain

$$0=\int_0^1\frac{d}{dt}(u'(t)v(t)-u(t)v'(t))\,dt=\int_0^1\left(\pi^2-s\frac{f(t,u(t))}{u(t)}\right)u(t)v(t)\,dt>0. \quad \square$$

Notice that the nonexistence of positive solutions for $s\in(\pi^2/c,\infty)$ is essentially the remark (b) after Theorem 3.2.

Examples. Consider first the problem

$$u''+s(u+h(t)e^{-u})_+=0,\quad u(0)=u(1)=0,$$

where $h(t)\in(0,1]$ and $u_+=\max(u,0)$. We have here, with the notations of Propositions 3.4 and 3.5, $a=1$, $b=1$, $c=1$, $d=\infty$ so that these propositions imply there exists a solution if $s\in(0,\pi^2)$ and no solution if $s\in(\pi^2,\infty)$ i.e. $s_0=\pi^2$.

Consider next the problem

$$u''+s\max(1,u-1)=0,\quad u(0)=u(1)=0.$$

Here $a=1$, $b=1$, $c=1/2$, $d=\infty$ so that we can deduce from the same propositions there is a solution if $s\in[0,\pi^2)$ and there is no solution if $s\in(2\pi^2,\infty)$, i.e. $s_0\in[\pi^2,2\pi^2]$.

Another example is

$$u''+s(u-1)_+=0,\quad u(0)=u(1)=0.$$

Here $c=0$, $d=1$ and there is no solution if $s\in[0,\pi^2)$.

4. Existence of multiple solutions

The type of problem we consider, can have unique solutions as it is clear from the example

$$u'' + s(u+1)_+ = 0, \quad u(0) = u(1) = 0,$$

whose solution

$$u(t) = \frac{\cos(\sqrt{s}(t-1/2))}{\cos(\sqrt{s}/2)} - 1,$$

is unique and defined for $s \in (0, \pi^2)$. To obtain more solutions, some structural assumptions on the nonlinearity f are necessary.

Theorem 4.1. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ satisfy $\mathcal{A}$-Carathéodory conditions and assume that given $r > 0$ and $\eta > 1$, there exists some $\varepsilon > 0$ such that*

(9) *for all u_1, $u \in [0, r]$ and for a.e. $t \in [0,1]$,*

$$u_1 \le u \le u_1 + \varepsilon \Rightarrow f(t,u) \le \eta f(t, u_1).$$

Suppose $f(t,0) \not\equiv 0$ and further that for some $k \in L^\infty(0,1)$ such that $k \ge 0$, $k(t) \not\equiv 0$, and $a > 0$, we have

$$\text{for a.e. } t \in [0,1], \text{ and all } u \ge 0, \quad f(t,u) \ge k(t) u (1 + u^a).$$

Then, there exists $s_0 \in \mathbb{R}^+$ such that

(a) *for any $s \in (0, s_0)$, (7) has at least two positive solutions,*
(b) *for $s = s_0$, (7) has at least one positive solution,*
(c) *for any $s > s_0$, (7) has no solution.*

Proof. Define s_0 from Theorem 3.2, fix $\overline{s} \in (0, s_0)$ and choose $s_1 \in (\overline{s}, s_0)$. Let u_1 be a solution of (7) with $s = s_1$ and $\varepsilon \in (0,1]$ be small enough so that for all $u, v \in [0, ||u_1||_\infty + 1]$ and for a.e. $t \in [0,1]$

$$v \le u \le v + \varepsilon \Rightarrow f(t,u) \le \frac{s_1}{\overline{s}} f(t,v).$$

Claim 1. If T is defined by (6) and $\Omega = \{u \in \mathcal{C}_0 \mid \text{for all } t \in [0,1],\ -1 < u(t) < u_1(t) + \varepsilon\}$, we have

$$\deg(I - \overline{s}T, \Omega) = 1.$$

Notice that $\alpha = -1$ and $\beta = u_1 + \varepsilon$ are strict lower and upper solutions of (7) with $s = \overline{s}$. The claim follows then from Proposition 2.3.

Claim 2. There exists $R_0 > 0$ such that for all positive solutions u of

$$u'' + \overline{s} k(t) u^{1+a} = 0, \quad u(0) = u(1) = 0, \tag{10}$$

we have $||u||_\infty \le R_0$.

Assume, by contradiction, there exists a sequence $(u_n)_n$ of solutions of (10) such that $\|u_n\|_\infty = A_n \geq n$. Notice that $u_n(t) \geq A_n\varphi(t)$ with $\varphi(t) = \min(t, 1-t)$. Hence, for n large enough, we have the contradiction

$$\begin{aligned} 0 &= -\int_0^1 (u_n''(t) + \pi^2 u_n(t)) \sin \pi t\, dt \\ &= \int_0^1 (\overline{s}k(t)u_n^{1+a}(t) - \pi^2 u_n(t)) \sin \pi t\, dt \\ &\geq \bigg[\overline{s}\bigg(\int_0^1 k(t)\varphi^{1+a}(t) \sin \pi t\, dt\bigg) A_n^\alpha - \pi^2\bigg] A_n > 0. \end{aligned}$$

Claim 3. There exists $R > 0$ such that solutions of (7) with $s \geq \overline{s}$ satisfy $||u||_\infty \leq R$.

Assume the claim is wrong. Hence, there exist $(s_n)_n \subset [\overline{s}, s_0]$ and solutions u_n of (7), with $s = s_n$, such that $||u_n||_\infty \to \infty$, as $n \to \infty$.

Let $\mu > 0$ and φ be the first eigenvalue and eigenfunction of the problem

$$\varphi'' + \mu k(t)\varphi = 0, \quad \varphi(1/4) = \varphi(3/4) = 0.$$

Let us extend φ on $[0,1]$ so that $\varphi(t) = 0$ on $[0, 1/4] \cup [3/4, 1]$ and define $\alpha_0 \in W^{2,\infty}(0,1)$ to be such that its graph is the concave envelop of the graph of φ on $[0,1]$. Hence there exist $1/4 < t_0 < t_1 < 3/4$ such that α_0 is linear on $[0, t_0]$ and on $[t_1, 1]$, and $\alpha_0(t) = \varphi(t)$ on $[t_0, t_1]$. Consider now $\alpha = A\alpha_0$. It is easy to see that if A is large enough

$$\alpha''(t) + \overline{s}k(t)\alpha^{1+a}(t) = Ak(t)\varphi(t)(\overline{s}A^a\varphi^a(t) - \mu) > 0, \quad \text{on } [t_0, t_1].$$

Hence, α is a lower solution for (10). Choose now u_n such that $u_n \geq \alpha$ and notice that u_n is an upper solution for (10). If $\|\alpha\|_\infty > R_0$, we get from Proposition 2.2 a solution of (10) with maximum larger than R_0, which contradicts Claim 2.

Claim 4. Problem (7), with $s = \overline{s}$, has at least two solutions.

It follows from Claim 1, there is a solution in Ω. From Claim 3, Theorem 3.2 and the properties of the degree, we deduce that $\deg(I - \overline{s}T, B(0,R)) = \deg(I - (s_0+1)T, B(0,R)) = 0$. Hence, we get $\deg(I - \overline{s}T, B(0,R) \setminus \Omega) = -1$, which proves the existence of a second solution in $B(0,R) \setminus \Omega$.

Claim 5. For $s = s_0$, (7) has at least one solution.

Consider a sequence $(s_n)_n \subset [0, s_0]$ such that $\lim_{n\to\infty} s_n = s_0$ and corresponding solutions u_n of (7) with $s = s_n$. From Claim 3 and the $\mathcal{A}$-Carathéodory conditions on $f(t,u)$, there exists $h \in \mathcal{A}$ such that

$$\begin{aligned} |u_n'(t)| &= s_n\bigg[-\int_0^t rf(r, u_n(r))\, dr + \int_t^1 (1-r)f(r, u_n(r))\, dr\bigg] \\ &\leq s_n\bigg[\int_0^t rh(r)\, dr + \int_t^1 (1-r)h(r)\, dr\bigg] \in L^1(0,1). \end{aligned}$$

Hence, from Arzelà–Ascoli Theorem, there exists a subsequence $(u_{n_i})_i$ which converges in $\mathcal{C}([0,1])$ to some function v_0. From the closedness of the derivative, $v_0 \in W^{2,1}_{\mathrm{loc}}((0,1))$ and satisfies (7), with $s = s_0$. □

Remarks. (a) Notice that condition (9) is satisfied if $f(t,u)$ is continuous and $f(t,u) > 0$.

(b) If $f \in \mathcal{C}^1$, we can arrange the two solutions u_s and v_s such that $\|u_s\|_\infty \to 0$ and $\|v_s\|_\infty \to \infty$ as $s \to 0$. If the sequence $(v_s)_s$ is bounded as $s \to 0$, then for some subsequence $v_s \to v$ and v satisfies $v''(t) = 0$, $v(0) = v(1) = 0$. Hence $v = 0$, but in a neighbourhood of the origin there exists a single branch of solutions.

(c) We can arrange the two solutions so that $u_s(t) < v_s(t)$ for $t \in (0,1)$. If not, we can prove the existence of solutions between $\alpha(t) = 0$ and $\beta(t) = \min(u_s(t), v_s(t))$ (see [5]).

5. Convex nonlinearities

The main result of this section presents conditions on the nonlinearity $f(t,u)$ so that Theorem 4.1 gives an exact count of the number of solutions of (7). A model example of such nonlinearities is $f(t,u) = h(t)e^u$.

Theorem 5.1. *Assume $f \in \mathcal{C}^1([0,1] \times \mathbb{R}, \mathbb{R}^+)$ is such that*

(i) *$f(t,u) = f(1-t,u)$ and $f(\,\cdot\,,u)$ is nondecreasing on $[0,1/2]$,*
(ii) *$f(t,\,\cdot\,)$ is a strictly convex function,*
(iii) *there exist $k > 0$ and $a > 0$ so that for all $t \in [0,1]$ and $u \geq 0$, we have $f(t,u) \geq ku(1+u^a)$.*

Then, there exists $s_0 \in \mathbb{R}^+$ such that

(a) *for any $s \in (0,s_0)$, (7) has exactly two positive solutions u_1 and u_2 and for any $t \in (0,1)$, $u_1(t) < u_2(t)$,*
(b) *for $s = s_0$, (7) has exactly one positive solution,*
(c) *for any $s > s_0$, (7) has no solution.*

Moreover, these solutions are symmetric, i.e. $u_i(t) = u_i(1-t)$.

The proof of this theorem uses several auxiliary results. The first one concerns ordered solutions.

Proposition 5.2. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ satisfy $\mathcal{A}$-Carathéodory conditions. Assume solutions of the Cauchy problem are unique and that for a.e. $t \in [0,1]$, function $f(t,\,\cdot\,)$ is strictly convex. Then, if u_1, u_2 and u_3 are solutions of (7), they are not ordered.*

Proof. Assume that for all $t \in (0,1)$, $u_1(t) \geq u_2(t) \geq u_3(t)$. From the uniqueness of solutions of the Cauchy problem, we can assume the inequalities are strict. Let $v_1(t) := u_1(t) - u_3(t) > 0$ and $v_2(t) := u_2(t) - u_3(t) \in (0, v_1)$. Notice that we can write

$$f(t, u_1(t)) - f(t, u_3(t)) = a_1(t)(u_1(t) - u_3(t)),$$

and

$$f(t, u_2(t)) - f(t, u_3(t)) = a_2(t)(u_2(t) - u_3(t)),$$

where, using the strict convexity of f, we have $a_1(t) > a_2(t)$. Hence, we obtain the contradiction

$$0 = v_1' v_2 - v_1 v_2' |_0^1 = s \int_0^1 (a_2 - a_1) v_1 v_2 \, dt < 0.$$

Notice that $(a_2 - a_1) v_1 v_2 \in L^1(0,1)$ since $(a_2 - a_1) \in \mathcal{A}$ and $v_i \in H_0^1(0,1)$, i.e. $|v_i(t)| \le K\sqrt{t(1-t)}$. □

Our next result concerns the nonexistence of nonsymmetric solutions.

Proposition 5.3. *Let $f \in \mathcal{C}^1([0,1] \times \mathbb{R}, \mathbb{R}^+)$ be such that $f(t,u) = f(1-t, u)$ and $f(\,\cdot\,, u)$ is nondecreasing on $[0,1/2]$. Then positive solutions of (7) are symmetric, i.e. $u(t) = u(1-t)$.*

This proposition can be found in Gidas, Ni and Nirenberg [6] for PDE. The proof in the ODE case is identical to the one in [6] and will be omitted.

Proposition 5.4. *Assume $f \in \mathcal{C}^1([0,1] \times \mathbb{R}, \mathbb{R}^+)$ is such that*

(i) *$f(t,u) = f(1-t,u)$ and $f(\,\cdot\,,u)$ is nondecreasing on $[0,1/2]$,*
(ii) *$f(t,\,\cdot\,)$ is a strictly convex function.*

Then problem (7) has at most two solutions.

Proof. Define $u(t;a)$ to be the solution of

$$u'' + sf(t,u) = 0, \quad u(0) = 0, \ u'(0) = a. \tag{11}$$

Let $0 \le a_1 < a_2 < a_3$ be such that $u(t; a_i)$ are three solutions of (7). We know from Proposition 5.3 that these solutions are symmetric.

Define now the function $z(t;a) = du(t;a)/da$ which satisfies the Cauchy problem

$$z'' + s\frac{\partial f(t, u(t;a))}{\partial u} z = 0, \quad z(0) = 0, \ z'(0) = 1. \tag{12}$$

Let $r_1(a)$ be the first positive zero of $z(t;a)$. Notice that $r_1(a)$ is a continuous function of a.

Claim 1. $r_1(a_i) \ge 1/2$.

Assume $r := r_1(a_i) < 1/2$ and notice that

$$\frac{d}{dt}(z'(t;a_i)u'(t;a_i) + sf(t,u(t;a_i))z(t;a_i)) = s\frac{\partial f(t,u(t;a_i))}{\partial t} z(t;a_i). \tag{13}$$

The following contradiction follows then

$$\begin{aligned}0 &\le s\int_0^r \frac{\partial f(t,u(t;a_i))}{\partial t} z(t;a_i)\,dt \\ &= (z'(t;a)u'(t;a) + sf(t,u(t;a))z(t;a))|_0^r = z'(r;a)u'(r;a) - a < 0.\end{aligned}$$

Claim 2. $r_1(a)$ is a decreasing function.

Let $b < c$ and assume $r_1(b) \le r_1(c)$. Define $m \in [b,c)$ to be such that $r := r_1(m) = \min_{[b,c]} r_1(a)$. Let us fix $t \in (0,r)$. We have for all $a \in [m,c]$, $z(t;a) > 0$ and therefore $u(t;c) > u(t;m)$. This implies

$$\frac{\partial f}{\partial u}(t,u(t;c)) > \frac{\partial f}{\partial u}(t,u(t;m)).$$

Now, we obtain

$$\begin{aligned}z(r;c)z'(r;m) &= (z(t;c)z'(t;m) - z'(t;c)z(t;m))\Big|_0^r \\ &= s\int_0^r \bigg(\frac{\partial f}{\partial u}(t,u(t;c)) - \frac{\partial f}{\partial u}(t,u(t;m))\bigg) z(t;c)z(t;m)\,dt > 0.\end{aligned}$$

Hence, $z(r;c) < 0$ and $r_1(c) < r \le r_1(b)$, which contradicts our assumption.

Conclusion. From Claims 1 and 2, we know that for $a \le a_3$, $r_1(a) \ge r_1(a_3) \ge 1/2$. Hence for any $t \in [0,1/2]$, $u(t;a)$ is an increasing function in a. In particular, $u(t;a_1) \le u(t;a_2) \le u(t;a_3)$ for $t \in [0,1/2]$. As these solutions are symmetric, they are ordered on $[0,1]$ which contradicts Proposition 5.2. □

Our next step is to prove that, if problem (7) has two solutions for some $s_1 > 0$, it has at least one solution in some neighbourhood of s_1. To this end, we will use the following lemma.

Lemma 5.5. *Let $f \in \mathcal{C}^1([0,1]\times\mathbb{R},\mathbb{R}^+)$, $f(t,0) \not\equiv 0$ and $s_1 > 0$. Assume there exists a function $\overline{u}(t)$ which is positive on $[0,1]$ and such that*

$$\overline{u}'' + s_1 f(t,\overline{u}) = 0. \tag{14}$$

Then for s near enough s_1, problem (7) has a positive solution.

Proof. Let $\beta(t)$ be the solution of

$$\beta'' + sf(t,\beta) = 0, \quad \beta(0) = \overline{u}(0),\ \beta'(0) = \overline{u}'(0).$$

For s near enough s_1, $\beta(t)$ is positive. Hence it is an upper solution for (7) and the claim follows from Proposition 2.2 with $\alpha = 0$. □

Lemma 5.6. *Assume $f \in \mathcal{C}^1([0,1]\times\mathbb{R},\mathbb{R}^+)$ is such that*

(i) *$f(t,u) = f(1-t,u)$ and $f(\cdot\,,u)$ is nondecreasing on $[0,1/2]$,*
(ii) *$f(t,\cdot\,)$ is a strictly convex function and*
(iii) *$f(t,0) \not\equiv 0$.*

Assume s_1 is such that there exist two solutions of problem

$$u'' + s_1 f(t, u) = 0, \quad u(0) = u(1) = 0. \tag{15}$$

Then for s near enough s_1, the problem (7) *has at least a positive solution.*

Proof. Define $u(t; a)$ to be the solution of (11) with $s = s_1$. Let $a_1 < a_2$ be such that $u(t; a_i)$ are the given solutions of (15). From Proposition 5.3 these solutions are symmetric. The function $z(t; a) = du(t; a)/da$ satisfies the Cauchy problem (12) with $s = s_1$. Define next $r_i(a)$ to be the i-th positive zero of $z(t; a)$ and notice that $r_i(a)$ are continuous functions of a. It follows from Claims 1 and 2 in the proof of Proposition 5.4 that $r_1(a)$ is a decreasing function such that $r_1(a_i) \geq 1/2$.

Claim 1. $r_2(a_i) > 1$.

Let us assume the two first positive zeros, $r_1 := r_1(a_i)$ and $r_2 := r_2(a_i)$, of $z(t; a_i)$ are in $[1/2, 1]$. We deduce then from (13) the contradiction

$$\begin{aligned} 0 &\leq s_1 \int_{r_2}^{r_1} \frac{\partial f(t, u(t; a_i))}{\partial t} z(t; a_i)\, dt \\ &= (z'(t; a_i) u'(t; a_i) + s_1 f(t, u(t; a_i)) z(t; a_i))|_{r_1}^{r_2} \\ &= z'(r_2; a_i) u'(r_2; a_i) - z'(r_1; a_i) u'(r_1; a_i) < 0. \end{aligned}$$

Claim 2. $z(1, a_1) \geq 0$.

This follows from the fact that a_1 corresponds to the solution $u(t; a)$ for the smallest value of a, i.e. $u(1; a) < 0$ if $a < a_1$ and $u(1; a_1) = 0$.

Conclusion. If $z(1, a_1) > 0$, then by Claim 1, $z(t, a_1) > 0$ for all $t \in (0, 1]$. It follows that, for $a > a_1$ and $a - a_1$ small enough, $u(t, a) > 0$ on $(0, 1]$. Hence, there exists a function $\overline{u}(t)$ which is positive on $[0, 1]$ and satisfies (14) which implies the affirmation of the Lemma 5.5 holds true.

If $z(1, a_1) = 0$, i.e. $r_1(a_1) = 1$, and since $r_1(a)$ is a decreasing function, we have $r_1(a_2) < 1$ and by Claim 1 $z(1, a_2) < 0$. This implies that for, $a < a_2$ and $a - a_2$ small enough, $u(t, a) > 0$ on $(0, 1]$. Once again, there exists a function $\overline{u}(t)$ which is positive on $[0, 1]$ and satisfies (14) which proves the lemma. □

Proof of Theorem 5.1. Define s_0 from Theorem 4.1. From Proposition 5.4 there is exactly two solutions for $s \in (0, s_0)$. From Lemma 5.6 we deduce that there is exactly one solution for $s = s_0$. □

6. A model example

The model problem

$$u'' + sh(t)e^u = 0, \quad u(0) = u(1) = 0, \tag{16}$$

can exhibit a variety of multiplicity results. We know (see Theorem 5.1) that there is at most two solutions, if h is nondecreasing on $[0,1/2]$, positive and symmetric (i.e. $h(t)=h(1-t)$). Further, these solutions are symmetric.

The situation is different if h is decreasing on $[0,1/2]$. If we choose

$$h(t)=1 \text{ if } t\in[0,t_0]\cup[1-t_0,1], \quad h(t)=0 \text{ if } t\in(t_0,1-t_0), \tag{17}$$

and $t_0\in(0,1/2)$, Proposition 6.1 hereunder shows that, here also, (16) has at most two symmetric solutions. However, for small values of s, Proposition 6.2 establishes the existence of four solutions, two of them being symmetric, the others being non-symmetric. Figure 1 provides such a bifurcation diagram. It represents the initial slope of solutions as a function of s. Symmetric solutions correspond to bold face lines.

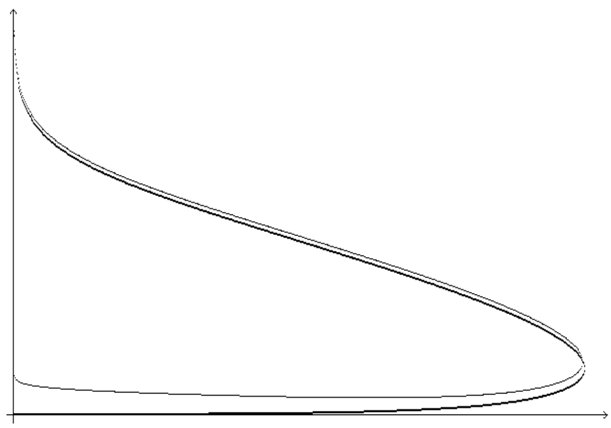

Figure 1

There is some computational evidence that the geometry of the set of solutions can be much more complicate. For example, for small values of the parameter s, problem (16) with

$$h(t)=1 \quad \text{if } t\in[0,0.1]\cup[0.9,1], \qquad h(t)=0.001 \quad \text{if } t\in(0.1,0.9)$$

exhibits numerically six solutions, two symmetric ones and four non-symmetric ones. Further, in some interval of the parameter s, we compute eight solutions, four of them being symmetric. This is represented in Figure 2, where we used the same conventions as for Figure 1.

Our first result concerns symmetric solutions of (16).

Proposition 6.1. *There exists $s_0\in\mathbb{R}^+$ such that for any $s\in(0,s_0)$ problem* (16)*, with h defined in* (17)*, has exactly two positive symmetric solutions $u_1(\,\cdot\,;s)$ and $u_2(\,\cdot\,;s)$ which are such that:*

(a) *for any $t\in(0,1)$, $u_1(t)<u_2(t)$,*

(b) $\lim_{s\to 0} \|u_1(\,\cdot\,; s)\|_\infty = 0$,
(c) $\lim_{s\to 0} \|u_2(\,\cdot\,; s)\|_\infty = \lim_{s\to 0} u_2'(0; s) = \infty$.

Proof. Consider the auxiliary problem

$$v'' + se^v = 0, \quad v(0) = v(2t_0) = 0, \tag{18}$$

From Theorem 5.1, there exists s_0 such that for $s < s_0$ (18) has exactly two solutions, which are symmetric and ordered. Notice next there is a bijection between symmetric solutions u and v of (16) and (18) given by $u(t) = v(t)$ if $t \in [0, t_0]$, $u(t) = v(t_0)$ if $t \in (t_0, 1 - t_0)$, $u(t) = v(t - 1 + 2t_0)$ if $t \in [1 - t_0, 1]$.

Claim (b) follows from Remark (a) after Theorem 3.1.

Claim (c) follows from Remark (b) after Theorem 4.1 and the fact that $u'(0; s) \geq u(t_0; s)/t_0 = \|u(\,\cdot\,; s)\|_\infty/t_0$. □

Remark. Notice that $u_1'(0; s) < u_2'(0; s)$ since $u_1(t; s) < u_2(t; s)$.

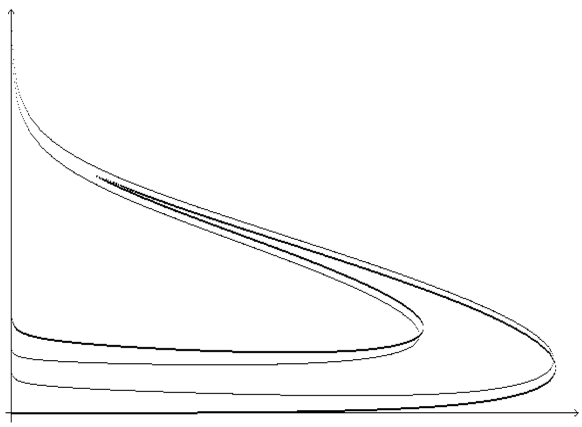

Figure 2

The main result of this section completes Proposition 6.1.

Proposition 6.2. *There exists $s_1 \in \mathbb{R}^+$ such that for any $s \in (0, s_1)$ problem* (16)*, with h defined in* (17)*, has at least four positive solutions.*

Proof. We shall prove this result using a shooting method. Consider the problem

$$u'' + sh(t)e^u = 0, \quad u(0) = 0,\ u'(0) = a,$$

and write its solution $u(t; a, s)$. Solutions of (16) are the functions $u(t; a, s)$ such that

$$u(1; a, s) = 0. \tag{19}$$

Hence, we have to prove that for small values of s, (18) has at least four positive solutions $a_i(s)$.

Consider first the solutions u_1 and u_2 given in Proposition 6.1 and define $a_1(s) = u_1'(0,s)$ and $a_2(s) = u_2'(0,s)$. These are solutions of (18).

To obtain one additional solution, we shall use the following claim, which will be proved later.

Claim. For small values of s,

$$\frac{\partial}{\partial a} u(1; a_1, s) > 0 \quad \text{and} \quad \frac{\partial}{\partial a} u(1; a_2, s) > 0.$$

It follows then by an intermediate value theorem that there exists $a_3 \in (a_1, a_2)$ such that $u(1; a_3, s) = 0$.

From Proposition 6.1, the corresponding solution $u_3(t; s) = u(t; a_3, s)$ cannot be symmetric. Hence, we have a fourth solution $u_4(t; s) = u_3(1-t; s)$. $\square$

In order to prove the above claim, notice that

$$z(t; a, s) = \frac{\partial}{\partial a} u(t; a, s)$$

is the solution of the Cauchy problem

$$z'' + sh(t)e^{u(t;a,s)} z = 0, \quad z(0) = 0, \; z'(0) = 1.$$

Lemma 6.3. *For small values of s,*

$$z(1; a_1, s) = \frac{\partial}{\partial a} u(1; a_1, s) > 0.$$

Proof. Assume by contradiction $z(\,\cdot\,; a_1, s)$ has a first positive zero $r_1 \in (0,1]$. As $\lim\limits_{s\to 0} \|u_1(\,\cdot\,; s)\|_\infty = 0$, we have, for s small enough,

$$sh(t)e^{u(t;a_1,s)} \le se^{\|u_1\|_\infty} < \left(\frac{\pi}{r_1}\right)^2.$$

Let $v(t) = \sin(\pi t/r_1)$. We obtain then the contradiction

$$0 = (zv' - z'v)|_0^{r_1} = \int_0^{r_1} (sh(t)e^{u(t;a_1,s)} - (\pi/r_1)^2) z(t)v(t)\, dt < 0. \qquad \square$$

Lemma 6.4. *For small values of s,*

$$z(1; a_2, s) = \frac{\partial}{\partial a} u(1; a_2, s) > 0.$$

Proof. Let us write $z(t) := z(t; a_2, s)$ and $u(t) := u(t; a_2, s)$.

Claim 1. $z(t) > 0$ for $t \in (0, t_0]$.

Assume $z(\,\cdot\,)$ has a first positive zero $r_1 \in (0, t_0]$. Since

$$\frac{d}{dt}(u'(t)z'(t) + se^{u(t)} z(t)) = 0,$$

we have

$$0 = (u'(t)z'(t) + se^{u(t)}z(t))|_0^{r_1} = u'(r_1)z'(r_1) - a_2.$$

This leads to the contradiction $0 \geq u'(r_1)z'(r_1) = a_2 > 0$.

Claim 2. $\lim_{s\to 0} z(t_0/2) = t_0/2$.

Let us first prove that

$$se^{u(t_0/2)} = se^{u(t_0/2;a_2,s)} \leq k^2 := \left(\frac{2\pi}{t_0}\right)^2.$$

If not, notice that $se^{u(t)} \geq k^2$ on $[t_0/2, t_0]$, let $v(t) = \sin k(t - t_0/2)$ and compute

$$0 > (zv' - z'v)|_{t_0/2}^{t_0} = \int_{t_0/2}^{t_0} zv(se^u - k^2)\,dt \geq 0,$$

which is a contradiction. Next we compute

$$\frac{a_2^2 - u'^2(t_0/2)}{2} = \int_{t_0/2}^{0} u''u'\,dt = s\int_0^{t_0/2} e^u u'\,dt = s(e^{u(t_0/2)} - 1) \leq k^2,$$

and

$$s\int_0^{t_0/2} e^u\,dt = -\int_0^{t_0/2} u''\,dt = |a_2 - u'(t_0/2)| \leq \frac{2k^2}{a_2 + u'(t_0/2)} \leq \frac{2k^2}{a_2}.$$

As a_2 goes to infinity, this proves

$$\lim_{s\to 0} s\int_0^{t_0/2} e^u\,dt = 0. \tag{20}$$

We can write now

$$\begin{aligned} z\left(\frac{t_0}{2}\right) &= z(0) + z'(0)\frac{t_0}{2} + \int_0^{t_0/2}\left(\frac{t_0}{2} - \tau\right)z''(\tau)\,d\tau \\ &= \frac{t_0}{2} - s\int_0^{t_0/2}\left(\frac{t_0}{2} - \tau\right)e^{u(\tau)}z(\tau)\,d\tau. \end{aligned}$$

As further $z(t)$ is concave on $[0, t_0]$, we have $z(t) \leq t \leq 1$ and we deduce that

$$\left|z\left(\frac{t_0}{2}\right) - \frac{t_0}{2}\right| \leq s\int_0^{t_0/2} e^{u(\tau)}\,d\tau,$$

and the claim follows from (20).

Claim 3. $\lim_{s\to 0} z(t_0) = 0$.

As

$$\frac{d}{dt}\left(\frac{u'^2}{2} + se^u\right) = 0 \quad\text{and}\quad \frac{d}{dt}(u'z' + se^u z) = 0, \quad\text{on } [0, t_0]$$

integrating on this interval, we obtain

$$\frac{a_2^2}{2} + s = se^{u(t_0)} \quad\text{and}\quad a_2 = se^{u(t_0)}z(t_0).$$

As further $a_2 \to \infty$ as $s \to 0$, we have

$$\lim_{s\to 0} z(t_0) = \lim_{s\to 0} \frac{\alpha_2}{\alpha_2^2/2 + s} = 0.$$

Claim 4. For s small enough, $z(t)$ has a first zero $r_1 \in (t_0, 1/2)$.

Notice that on $[0, t_0]$, function $z(t)$ is concave. Hence, it follows from Claims 2 and 3 that, for s small enough,

$$z'(t_0) \le \frac{z(t_0) - z(t_0/2)}{t_0/2} \le -\frac{1}{2}.$$

As $z(t)$ is linear on $[t_0, 1-t_0]$, the claim follows then from Claim 3.

Claim 5. $z(t)$ has a second zero $r_2 \in (1-t_0, 1)$.

As $z(t)$ is linear on $[t_0, 1-t_0]$ it has no second zero in this interval. Assume $z(t) < 0$ on $(1-t_0, 1)$ and define the function $\widehat{z}(t) = z(1-t)$ which is such that

$$\widehat{z}'' + sh(t)e^{u(t)}\widehat{z} = 0, \quad \widehat{z}(1-r_1) = \widehat{z}(1) = 0.$$

One obtains then the contradiction

$$0 = (z\widehat{z}' - z'\widehat{z})|_{1-r_1}^{1} = z(1)\widehat{z}'(1) - z(1-r_1)\widehat{z}'(1-r_1) > 0.$$

Claim 6. $z(t) > 0$ for $t \in (r_2, 1]$.

Assume $z(t)$ has a third zero $r_3 \in (r_2, 1]$. As in Claim 5 we have the contradiction

$$0 = (z\widehat{z}' - z'\widehat{z})|_{r_2}^{r_3} = -z'(r_3)\widehat{z}(r_3) + z'(r_2)\widehat{z}(r_2) > 0. \qquad \square$$

Remark. Consider problem (16) with $h(t) = 1$ if $t \in [0, t_0] \cup [1-t_0, 1]$ and $h(t) = k > 0$ if $t \in (t_0, 1-t_0)$. Going back to the proof of Proposition 6.2, the existence of four solutions, for $k = 0$, was a consequence of the fact that, in this case, $u(1; a, s)$ has two zeros a_1 and a_2 with positive slope $\partial u(1; a_i, s)/\partial a$, i.e. for $\varepsilon > 0$ small enough $u(1; a_1 - \varepsilon, s) < 0 < u(1; a_1 + \varepsilon, s)$ and $u(1; a_2 - \varepsilon, s) < 0 < u(1; a_2 + \varepsilon, s)$. This situation still holds true for small perturbations of h and therefore, given s small, we have the existence of at least three positive solutions for k small enough. This is the situation illustrated in Figure 2. It shows that Theorem 5.1 does not hold without assuming that $f(\,\cdot\,, u)$ is nondecreasing on $[0, 1/2]$.

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