EXISTENCE OF MANY SIGN-CHANGING NONRADIAL SOLUTIONS FOR SEMILINEAR ELLIPTIC PROBLEMS ON THIN ANNULI

ALFONSO CASTRO — MARCEL B. FINAN

Abstract. We study the existence of many nonradial sign-changing solutions of a superlinear Dirichlet boundary value problem in an annulus in $\mathbb{R}^N$. We use Nehari-type variational method and group invariance techniques to prove that the critical points of an action functional on some spaces of invariant functions in $H^{1,2}_0(\Omega_\epsilon)$, where $\Omega_\epsilon$ is an annulus in $\mathbb{R}^N$ of width $\epsilon$, are weak solutions (which in our case are also classical solutions) to our problem. Our result generalizes an earlier result of Castro et al. (See [4])

1. Introduction

In this article we discuss the existence of many sign-changing nonradial solutions of semilinear elliptic equations on an annulus in $\mathbb{R}^N$, $N \geq 2$:

$$\Omega_\epsilon := \{ x \in \mathbb{R}^N : 1 - \epsilon < |x| < \epsilon \},$$

where $\epsilon > 0$.

We consider the Dirichlet boundary value problem

\begin{equation}
\begin{cases}
\Delta u + f(u) = 0 & \text{in } \Omega_\epsilon, \\
u = 0 & \text{on } \partial \Omega_\epsilon,
\end{cases}
\end{equation}

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where the non-linearity \( f \) is of class \( C^1(\mathbb{R}) \) and satisfies the following assumptions:

(A1) \( f(0) = 0 \) and \( f'(0) < \lambda_1 \), where \( \lambda_1 \) is the smallest eigenvalue of \(-\Delta\) with zero Dirichlet boundary condition in \( \Omega_\varepsilon \).

(A2) \( f'(u) > f(u)/u \) for all \( u \neq 0 \).

(A3) (Superlinearity)
\[
\lim_{|u| \to \infty} \frac{f(u)}{u} = \infty.
\]

(A4) (Subcritical growth) There exist constants \( p \in (1, (N + 2)/(N - 2)) \) and \( C > 0 \) such that
\[
|f'(u)| \leq C(|u|^{p-1} + 1) \quad \text{for all} \quad u \in \mathbb{R}.
\]

(A5) There exist constants \( m \in (0, 1) \) and \( \rho \) such that
\[
uf(u) \geq \frac{2}{m} F(u) > 0,
\]
where \( |u| > \rho \) and \( F(u) = \int_0^u f(s) \, ds \).

If \( N = 2 \), then \( p \in (1, \infty) \). A typical nonlinearity is the function \( f(t) = t^3 \), although our results are not restricted to an odd nonlinearity.

We note that the condition \( f'(0) < \lambda_1 \) is necessary for the existence of sign-changing solutions (see [2]).

In [11], Wang proved that, over a smooth bounded domain, problem (1.1) has a positive solution, a negative solution, and a third solution with no information about its sign. In [2], Castro et al. proved the existence of a third solution that changes sign exactly once. Later in [4], they established the existence of a nonradial sign-changing solution when the underlying domain is an annulus in \( \mathbb{R}^N \). Furthermore, if the annulus is two dimensional they proved that (1.1) has many sign-changing nonradial solutions. The purpose of this paper is to extend their result to higher dimensions.

Our main result is the following

**Theorem 1.1.** Assume \( f \) satisfies (A1)-(A5). Then for each positive integer \( k \) there exists \( \varepsilon_1(k) > 0 \) such that if \( 0 < \varepsilon < \varepsilon_1(k) \) then (1.1) has \( k \) sign-changing nonradial solutions.

In our context, by a solution to (1.1) we mean a function \( u \in H^{1,2}_0(\Omega_\varepsilon) \) that satisfies

\[
\int_{\Omega_\varepsilon} (\nabla u \cdot \nabla v - vf(u)) \, dx = 0,
\]
for all \( v \in C_0^\infty(\Omega_\varepsilon) \), where \( H^{1,2}_0(\Omega_\varepsilon) \) is the Sobolev space with inner product \( \langle u, v \rangle = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, dx \) (see [1]). Note that (1.2) is obtained by multiplying the
equation in (1.1) by \( v \) and integrating by parts. So classical solutions of (1.1) (that is, the ones which are in \( C^2(\Omega_\varepsilon) \cap C(\overline{\Omega_\varepsilon}) \)) are also weak solutions. By the assumptions on \( f \) and the regularity theory for elliptic boundary value problems (see [7]), a weak solution of (1.1) is also a classical solution.

The left-hand side of (1.2) is just the Fréchet derivative of the functional

\[
J(u) = \int_{\Omega_\varepsilon} \left\{ \frac{1}{2} |\nabla u|^2 - F(u) \right\} \, dx
\]

defined on \( H^{1,2}_0(\Omega_\varepsilon) \). Note that \( J \in C^2(H^{1,2}_0(\Omega_\varepsilon), \mathbb{R}) \) (see [10]). Moreover, \( u \) is a solution to (1.1) if and only if \( u \) is a critical point of \( J \).

Instead of looking for sign-changing critical points of the functional \( J \) on \( H^{1,2}_0(\Omega_\varepsilon) \), we look for them on a subset of a submanifold of invariant functions in \( H^{1,2}_0(\Omega_\varepsilon) \).

Our main tools for proving existence and multiplicity results consist of an idea in [8] and [9] and critical point theory, i.e., we consider the functional \( J \) defined above and the functional

\[
\gamma(u) = \int_{\Omega_\varepsilon} (|\nabla u|^2 - uf(u)) \, dx.
\]

For a positive integer \( k \), we define

\[
H(\varepsilon, k) := \text{Fix}(G(k))
\]

\[
= \{ v \in H^{1,2}_0(\Omega_\varepsilon) : v(gx, Ty) = v(x, y), \text{ for all } (g, T) \in G(k) \}
\]

\[
= \{ v \in H^{1,2}_0(\Omega_\varepsilon) : v(x, y) = u(x, |y|), \text{ for some } u \text{ which satisfies } u(gx, |y|) = u(x, |y|) \text{ for all } g \in G_k \},
\]

where \( G(k) = G_k \times O(N - 2) \), \( O(j) \) denotes the group of \( j \times j \) orthogonal matrices, and

\[
G_k := \left\{ g \in O(2) : \begin{align*}
g(x_1, x_2) &= \left( x_1 \cos \frac{2\pi l}{k} + x_2 \sin \frac{2\pi l}{k}, -x_1 \sin \frac{2\pi l}{k} + x_2 \cos \frac{2\pi l}{k} \right), \\ (x_1, x_2) &\in \mathbb{R}^2, \ l \in \mathbb{Z} \end{align*} \right\}.
\]

Note that \( H(\varepsilon, k) \) can be regarded as the class of functions that are periodic of period \( 2\pi/k \) in the \( \theta \) variable, where \( (r, \theta) \) are the polar coordinate of \( x = (x_1, x_2) \), and that depend on \( |y| \), where \( y = (x_3, \ldots, x_N) \).

Also, we consider the Nehari manifold

\[
S(\varepsilon, k) = \{ v \in H(\varepsilon, k) \setminus \{0\} : \gamma(v) = 0 \}.
\]
Of particular interest is the subset of $S(\varepsilon, k)$ given by
\[ S^1(\varepsilon, k) = \{ v \in S(\varepsilon, k) : v_+, v_- \in S(\varepsilon, k) \}, \]
where $v_+(x) = \max \{ v(x), 0 \}$ and $v_-(x) = \min \{ v(x), 0 \}$ are the positive and negative parts of $v$ respectively.

Similarly, we define
\[ H(\varepsilon, \infty) := \{ v \in H^1_0(\Omega_\varepsilon) : v(gx, T y) = v(x, y), \]
\[ \text{for all } (g, T) \in O(2) \times O(N - 2) \} \]
the manifold
\[ S(\varepsilon, \infty) = \{ v \in H(\varepsilon, \infty) \setminus \{ 0 \} : \gamma(v) = 0 \}, \]
and the set
\[ S^1(\varepsilon, \infty) = \{ v \in S(\varepsilon, \infty) : v_+, v_- \in S(\varepsilon, \infty) \}. \]

Note that if $u \in H(\varepsilon, \infty)$ then $u$ is $\theta$-independent.

We consider the following numbers associated with the above sets
\[ j_k^\varepsilon = \inf \limits_{v \in S^1(\varepsilon, k)} J(v), \quad j_\infty^\varepsilon = \inf \limits_{v \in S^1(\varepsilon, \infty)} J(v). \]
We will obtain many sign-changing nonradial solutions to (1.1) by establishing the following properties:

(i) $j_k^\varepsilon$ is achieved by some $u_{\varepsilon, k} \in S^1(\varepsilon, k)$ and $u_{\varepsilon, k}$ is a critical point of $J$ on $H(\varepsilon, k)$.

(ii) $u_{\varepsilon, k}$ is a critical point of $J$ on $H^1_0(\Omega_\varepsilon)$.

(iii) $j_k^\varepsilon < j_\infty^\varepsilon$ for $k \geq 1$ and $0 < \varepsilon < \varepsilon_1(k)$.

(iv) $j_k^\varepsilon < j_n^\varepsilon$ whenever $j_n^\varepsilon < j_\infty^\varepsilon$.

Note that assertion (ii) is related to the symmetric criticality principle: if $u_{\varepsilon, k}$ is a critical point of $J$ on $H(\varepsilon, k)$, then $u_{\varepsilon, k}$ is a critical point of $J$ on $H^1_0(\Omega_\varepsilon)$ (see [12]).

The paper is organized as follows: in Section 2, we discuss assertions (i), (iii), and (iv). In Section 3, we prove Theorem 1.1.

2. Existence results

Assertion (i) of the previous paragraph is a direct consequence of the following theorem

**Theorem 2.1.** For each positive integer $k = 1, 2, \ldots$ and $\varepsilon > 0$ there exists a minimizer $u_{\varepsilon, k}$ of $j_k^\varepsilon$ which changes sign. Moreover, $u_{\varepsilon, k}$ is a critical point of $J$ on $H(\varepsilon, k)$.

**Proof.** This follows from a recent result of Castro, Cossio, and Neuberger [2]. □
As for assertion (iii) we have

**Theorem 2.2.** For a positive integer \( k \), there exists \( \varepsilon_1(k) > 0 \) such that if \( 0 < \varepsilon < \varepsilon_1(k) \) then \( j_k^c < j_\infty^c \). Thus, \( u_{\varepsilon,k} \) is \( \theta \)-dependent.

**Proof.** A proof of this theorem can be found in [6]. □

The following lemma, which establishes assertion (iv), shows that if \( k \) divides \( n \) and \( j_k^\pm < j_\infty^\pm \) then \( j_k^\pm < j_k^\pm \).

**Lemma 2.3.** Let \( f \) satisfies (A1)–(A5). For \( n = 2, 3, \ldots, k = 1, 2, \ldots \), if \( j_k^\pm n < j_k^\pm \) then \( j_k^\pm n < j_k^\pm \).

**Proof.** Fix \( k \) and \( n \). For \( \varepsilon > 0 \), Theorem 2.1 guarantees the existence of a sign-changing minimizer \( u \) of \( J \) on \( S^1(\varepsilon, kn) \). According to Theorem 2.1 and assertion (ii), \( u \) is a solution to (1.1). Furthermore, invoking Theorem 2.2 with \( 0 < \varepsilon < \varepsilon_1^k \), we know that \( u \) is \( \theta \)-dependent. Now, by the regularity theory of elliptic equations we know that \( u \) is a \( C^2 \) function. Let \( x = (r, \theta) \) be the polar coordinate of \( x \in \mathbb{R}^2 \) and write \( u(r, \theta, |y|) \). Then

\[
\int_{\Omega_\varepsilon} |\nabla u|^2 \, dx \, dy = \int_{(r,|y|)}^{2\pi} (u^2 + \frac{1}{r^2} u^2_\theta + |\nabla_y u|^2) \, r \, dr \, d\theta \, dy
\]

and

\[
\int_{\Omega_\varepsilon} F(u) \, dx \, dy = \int_{(r,|y|)}^{2\pi} F(u) \, r \, dr \, d\theta \, dy.
\]

Define the function

\[
v(r, \theta, |y|) = u(r, \theta/\theta, |y|), \quad 0 \leq \theta \leq 2\pi.
\]

Since \( u \) is \( \theta \)-dependent and changes sign so does \( v \). Also,

\[
v_{\pm}(r, \theta + 2\pi/k, |y|) = v_{\pm}(r, \theta, |y|).
\]

It follows that \( v_{\pm} \in H(\varepsilon, k) \).

An easy calculation yields the following equalities

\[
\int_{\Omega_\varepsilon} |\nabla v_{\pm}|^2 \, dx \, dy = \int_{(r,|y|)}^{2\pi} \left( (u_{\pm})_r^2 r \, dr \, d\theta \, dy + \frac{1}{r^2} (u_{\pm})_{\theta}^2 r \, dr \, d\theta \, dy \right)
\]

and

\[
\int_{\Omega_\varepsilon} F(v_{\pm}) \, dx \, dy = \int_{(r,|y|)}^{2\pi} F(u_{\pm}(r, \theta, |y|)) r \, dr \, d\theta \, dy.
\]

Since \( u \) does not belong to \( S^1(\varepsilon, \infty) \) we have

\[
\int_{(r,|y|)}^{2\pi} (u_{\pm})_{\theta}^2 r \, dr \, d\theta \, dy > 0.
\]
This implies that $\gamma(v_{\pm}) < 0$. That is
\[
\int_{\Omega} |\nabla v_{\pm}|^2 \, dx \, dy < \int_{\Omega} v_{\pm} f(v_{\pm}) \, dx \, dy.
\] (2.1)

Now, by Lemma 2.2 of [2] we can find $0 < \alpha < 1$ and $0 < \beta < 1$ such that $\alpha v_+ \in S(\varepsilon, k)$ and $\beta v_- \in S(\varepsilon, k)$. Let $w = \alpha v_+ + \beta v_- \in S^1(\varepsilon, k)$. Using the fact that $P_\varepsilon(\lambda) = \lambda v f(\lambda v)/2 - F(\lambda v)$ is monotonically increasing for $\lambda > 0$ and the definition of $j_k^\varepsilon$ we have
\[
j_k^\varepsilon \leq P_\varepsilon^+(\alpha) + P_\varepsilon^-(\beta) < P_\varepsilon^+(1) + P_\varepsilon^-(1) = J(u) = j_{kn}^\varepsilon.
\]
Putting together all the arguments above we conclude a proof of the lemma. □

3. Proof of Theorem 1.1

Let $k \geq 1$ be an integer. According to Theorem 2.2 there exists $\varepsilon_1(2^k)$ such that if $0 < \varepsilon < \varepsilon_1(2^k)$ then $j_{2^k}^\varepsilon < j_{\infty}^\varepsilon$. Applying Lemma 2.3 to obtain
\[
j_{2^i}^\varepsilon < j_{2^{i+1}}^\varepsilon < \ldots < j_{2^k}^\varepsilon < j_{\infty}^\varepsilon.
\] (3.1)

According to Theorem 2.1 there exists $u_i \in S^1(\varepsilon, 2^i)$, $i = 1, \ldots, k$, such that $j_{2^i}^\varepsilon = J(u_i)$. Moreover, $u_i$ is a solution of (1.1). Also, according to Theorem 2.2, $u_i$ is $\theta$-dependent. Finally, by (3.1), $\{u_i\}_{i=1}^k$ are distinct. The proof of Theorem 1.1 is now complete. □

References


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ALFONSO CASTRO
Division of Mathematics and Statistics
University of Texas at San Antonio
San Antonio, Texas 78249, USA
E-mail address: castro@math.utsa.edu

MARCEL B. FINAN
Department of Mathematics
University of Texas at Austin
Austin, Texas 78712, USA
E-mail address: mbfinan@math.utexas.edu