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CONCENTRATION OF SOLUTIONS FOR A NONLINEAR ELLIPTIC PROBLEM WITH NEARLY CRITICAL EXPONENT

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Abstract. We construct solutions of the Dirichlet problem (1.1)–(1.3) concentrating at strict local maximum point of the coefficient Q either at the boundary or in the interior of Ω. We also prove the existence of solutions concentrating at an interior strict local minimum point of Q.

1. Introduction

The main purpose of this paper is to investigate concentration phenomena for the following Dirichlet problem:

$$-\Delta u = Q(y)u^{2^*-\varepsilon-1} \quad y \in \Omega, \tag{1.1}$$

$$u > 0 \quad y \in \Omega, \tag{1.2}$$

$$u = 0 \quad y \in \partial\Omega, \tag{1.3}$$

where ε is a small positive number, $2^* = 2N/(N-2)$, $N \geq 3$, Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$ and $Q(y)$ is a smooth positive function in $\overline{\Omega}$.

During the last several years, concentration phenomena for elliptic problems involving critical or subcritical exponent have been subject of an extensive research. The main problems are to examine the effect of the topology of the

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domain [3]–[5], [7]–[9], [15], [16], [21], [34], the shape of the domain [1], [2], [6], [10], [18], [19], [22], [24], [26]–[31], [35]–[38] and the shape of the graphs of coefficients [11]–[14], [17], [23], [25], [32], [33] on the number of the solutions. As far as the authors know, there are no results on the existence of solutions for (1.1)–(1.3) concentrating at the boundary or at the interior minimum point of Q.

Problem (1.1)–(1.3) always has a least energy solution. It is easy to check that the least energy solution concentrates at a point $x_\circ$ which is a maximum point of $Q(y)$ in $\overline{\Omega}$. It is worth pointing out that $x_\circ$ is not necessary a critical point of Q if $x_\circ$ is on the boundary.

The aim of this paper is to construct solutions for (1.1)–(1.3) concentrating at various points of Ω. We are mainly interested in constructing solutions concentrating at a strict local maximum point of Q either at the boundary or in the interior of Ω. We shall also construct solutions concentrating at an interior strict local minimum point of Q.

Before we state our main results, we introduce some notation.

Let

$$U_{x,\lambda}(y) = [N(N-2)]^{(N-2)/4}\frac{\lambda^{(N-2)/2}}{(1+\lambda^2|y-x|^2)^{(N-2)/2}}.$$

It is well known that $U_{x,\lambda}$ satisfies

$$\Delta U_{x,\lambda} = U_{x,\lambda}^{2^*-1}, \quad y \in \mathbb{R}^N.$$

Let P denote the projection from $H^1(\Omega)$ into $H^1_\circ(\Omega)$; that is, if $w \in H^1(\Omega)$, then Pw is a unique solution of the following Dirichlet problem

$$\begin{cases} \Delta u = \Delta w & y \in \Omega, \\ u = 0 & y \in \partial\Omega. \end{cases}$$

Let

$$\langle u, v\rangle = \int_\Omega DuDv, \qquad u, v \in \mathrm{H}^1_\circ(\Omega), \tag{1.4}$$

$$\|u\| = \left(\int_\Omega |Du|^2\right)^{1/2}, \quad u \in \mathrm{H}^1_\circ(\Omega), \tag{1.5}$$

$$\begin{aligned} E_{x,\lambda} = \bigg\{v : v \in \mathrm{H}^1_\circ(\Omega),\ \langle v, PU_{x,\lambda}\rangle = \left\langle v, \frac{PU_{x,\lambda}}{\partial\lambda}\right\rangle \\ = \left\langle v, \frac{PU_{x,\lambda}}{\partial x_j}\right\rangle = 0,\ j = 1,\ldots,N\bigg\}. \end{aligned} \tag{1.6}$$

Now we state the main results of this paper.

Theorem 1.1. *Let $x_\circ \in \partial\Omega$ be a strict local maximum point of $Q(x)$ satisfying*

$$Q(x) \le Q(x_\circ) - a|x-x_\circ|^{2+\alpha} \quad \textit{for all } x \in B_\delta(x_\circ)\cap\overline{\Omega}, \tag{1.7}$$

where $\delta > 0$, $a > 0$ *and* $\alpha > 0$ *if* $N \le 4$, $\alpha \in [0, 4/(N-4))$ *if* $N \ge 5$. *Then there is an* $\varepsilon_\circ > 0$, *such that for each* $\varepsilon \in (0, \varepsilon_\circ]$, (1.1)–(1.3) *has a solution of the form*

$$u_\varepsilon = \alpha_\varepsilon P U_{x_\varepsilon, \lambda_\varepsilon} + v_\varepsilon, \tag{1.8}$$

where $v_\varepsilon \in E_{x_\varepsilon, \lambda_\varepsilon}$, *and as* $\varepsilon \to 0$,

$$\alpha_\varepsilon \to Q(x_\circ)^{-1/(2^*-2)}, \tag{1.9}$$

$$\|v_\varepsilon\| \to 0, \tag{1.10}$$

$$x_\varepsilon \to x_\circ, \tag{1.11}$$

$$\lambda_\varepsilon \to \infty, \tag{1.12}$$

$$\lambda_\varepsilon d(x_\varepsilon, \partial\Omega) \to \infty. \tag{1.13}$$

THEOREM 1.2. *Let* $x_\circ \in \Omega$ *be a strict local maximum point of* $Q(x)$. *Then there is an* $\varepsilon_\circ > 0$, *such that for each* $\varepsilon \in (0, \varepsilon_\circ]$, (1.1)–(1.3) *has a solution of the form* (1.8) *satisfying* (1.9)–(1.12).

It is easy to prove that if (1.1)–(1.2) has a solution of the form (1.8) with $x_\varepsilon \to x_\circ \in \Omega$ as $\varepsilon \to 0$, then $x_\circ$ is a critical point of $Q(y)$. Thus a natural question arises whether Theorems 1.1 and 1.2 hold if $x_\circ$ is a strict minimum point. We will give an example in the next section, showing that, in general, it is impossible to construct a solution concentrating at a strict minimum point of $Q(x)$ on the boundary. We will also prove in the next section that for a strict local minimum point $x_\circ \in \Omega$ with $\Delta Q(x_\circ) > 0$, (1.1)–(1.3) has no solution of the form (1.8) satisfying (1.1)–(1.3) if $N \ge 5$. Our next theorem shows that if $Q(x)$ is flat enough around $x_\circ$, (1.1)–(1.3) has a solution concentrating at this point.

THEOREM 1 .3. *Let* $x_0 \in \Omega$ *be a strict local minimum point of* $Q(x)$ *satisfying*

$$|D^l Q(x)| \le C|x - x_\circ|^{L-l}, \quad l = 1, \dots, N-2, \text{ for all } x \in B_\delta(x_\circ), \tag{1.14}$$

$$|Q(x) - Q(x_\circ)| \ge c_0 |x - x_\circ|^L, \tag{1.15}$$

where $L > N-2$ *is a constant. Then there is an* $\varepsilon_\circ > 0$, *such that for each* $\varepsilon \in (0, \varepsilon_\circ]$, (1.1)–(1.3) *has a solution of the form* (1.8) *satisfying* (1.9)–(1.11) *and* $\varepsilon \lambda_\varepsilon^{N-2} \to c_1 > 0$.

In the case $N = 3$ or 4, we can get a better result.

THEOREM 1.4. *Suppose that* $x_\circ \in \Omega$ *is a strict local minimum point of* $Q(x)$. *If one of the following conditions is satisfied:*

(i) $N = 3$;

(ii) $N = 4$ *and*

$$K_3 H(x_\circ, x_\circ) - K_1 \Delta Q(x_\circ) > 0, \tag{1.16}$$

where K_1 *and* K_2 *are the constants from Lemma* A1 (*see Appendix* A), *and* $H(x,y)$ *denotes the regular part of the Green function for* Ω,

then the conclusion of Theorem 1.3 *holds.*

Remark 1.5. We show in Theorem 2.3 that (1.16) is nearly necessary.

Let $K : H^1(\Omega) - \{0\} \to \mathbb{R}$ be a functional defined by

$$K(u) = \frac{\int_\Omega |Du|^2}{(\int_\Omega Q(y)|u|^{2^*-\varepsilon})^{2/(2^*-\varepsilon)}}. \tag{1.17}$$

Following Bahri [3] (see also Rey [34]), in order to prove Theorems 1.1–1.4, we only need to find a critical point of the form $PU_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon$ for $K(u)$, with $\|v_\varepsilon\| \to 0$. Let

$$J(x,\lambda,v) = K(PU_{x,\lambda} + v), \tag{1.18}$$

for all $(x,\lambda,v) \in M = \{x \in \Omega,\ \lambda \geq \lambda_\circ,\ v \in E_{x,\lambda}\}$, where $\lambda_\circ$ is a large positive constant. It is well known that if $\|v\|$ is small enough, $PU_{x,\lambda} + v$ is a critical point of $K(u)$ if and only if $(x,\lambda,v) \in M$ is a critical point of $J(x,\lambda,v)$ on M, see for example [3], [4]. [34]. On the other hand, $(x,\lambda,v) \in M$ is a critical point of $J(x,\lambda,v)$ on M if and only if there are $A \in \mathbb{R}$, $B \in \mathbb{R}$ and $G_j \in \mathbb{R}$, such that

$$\frac{\partial J}{\partial x_i} = B\left\langle \frac{\partial^2 PU_{x,\lambda}}{\partial\lambda\partial x_i}, v\right\rangle + \sum_{j=1}^k G_j \left\langle \frac{\partial^2 PU_{x,\lambda}}{\partial x_j \partial x_i}, v\right\rangle, \quad i = 1,\dots,N, \tag{1.19}$$

$$\frac{\partial J}{\partial \lambda} = B\left\langle \frac{\partial^2 PU_{x,\lambda}}{\partial\lambda^2}, v\right\rangle + \sum_{j=1}^k G_j \left\langle \frac{\partial^2 PU_{x,\lambda}}{\partial x_j \partial \lambda}, v\right\rangle, \tag{1.20}$$

$$\frac{\partial J}{\partial v} = APU_{x,\lambda} + B\frac{\partial PU_{x,\lambda}}{\partial\lambda} + \sum_{j=1}^k G_j \frac{\partial PU_{x,\lambda}}{\partial x_j}. \tag{1.21}$$

Moreover, it is easy to prove that a critical point of the form $PU_{x,\lambda} + v$ for $K(u)$ is positive if $\|v\|$ is small enough, see for example [34].

The proof of the main results of this paper is based on the comparison of energy functionals. This method is very effective when we deal with problems characterised by degeneracy, see [10], [13], [19], [32], [38].

The paper is organized as follows. In Section 2 we present proofs of Theorems 1.1 and 1.2. Since the proof of Theorem 1.2 is similar to that of Theorem 1.1 and even is slightly simpler, we only point out some necessary changes in the proof. Section 3 is devoted to the proofs of Theorems 1.3 and 1.4. Some technical estimates needed in the proofs of our main results are given in the appendices.

2. Proof of Theorems 1.1 and 1.2

First of all, we establish a lower bound for the functional $J(x,\lambda,v)$.

Lemma 2.1. *There is a $\rho'>0$, such that for all $(x,\lambda,v)\in M$ with $\|v\|$ small,*

$$J(x,\lambda,v)\geq J(x,\lambda,0)+\rho'\|v\|^2+O\biggl(\frac{1}{\lambda^2}+\varepsilon^2\ln^2\lambda+\frac{1}{(\lambda d)^{N-2+2\theta}}\biggr),$$

where $d=d(x,\partial\Omega)$, $\theta>0$.

Proof. As in [3] (see also [34]) we expand $J(x,\lambda,v)$ in a neighbourhood of a $v=0$:

$$J(x,\lambda,v)=J(x,\lambda,0)+\langle f_\varepsilon,v\rangle+\frac{1}{2}\langle Q_\varepsilon v,v\rangle+O(\|v\|^{2+\theta_1}), \tag{2.1}$$

where θ_1 is a positive constant, $\langle f_\varepsilon,v\rangle$ is a continuous linear form on $E_{x,\lambda}$ equipped with a scalar product from $H^1_\circ(\Omega)$ given by

$$\langle f_\varepsilon,v\rangle=\frac{2\int_\Omega|DPU_{x,\lambda}|^2}{(\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon})^{2/(2^*-\varepsilon)+1}}\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon-1}v, \tag{2.2}$$

and $\langle Q_\varepsilon,v\rangle$ is a quadratic form on $E_{x,\lambda}\times E_{x,\lambda}$ satisfying

$$\begin{aligned}\langle Q_\varepsilon v,v\rangle=&\frac{2}{(\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon})^{2/(2^*-\varepsilon)}}\\&\cdot\biggl[\int_\Omega|Dv|^2-(2^*-\varepsilon-1)\frac{\int_\Omega|DPU_{x,\lambda}|^2}{\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon}}\\&\cdot\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon-2}v^2\\&+(2^*-\varepsilon+2)\frac{\int_\Omega|DPU_{x,\lambda}|^2}{(\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon})^2}\\&\cdot\biggl(\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon-1}v\biggr)^2\biggr].\end{aligned} \tag{2.3}$$

It follows from Lemma A.2 that

$$\langle f_\varepsilon,v\rangle=O\biggl(\frac{1}{\lambda}+\varepsilon\ln\lambda+\frac{1}{(\lambda d)^{\theta+(N-2)/2}}\biggr)\|v\|, \tag{2.4}$$

Moreover, according to Appendix D in [34] and Lemma A.2, we have

$$\langle Q_\varepsilon v,v\rangle\geq\rho\|v\|^2. \tag{2.5}$$

It is easy to see that Lemma 2.1 follows from (2.1)–(2.5). □

To proceed further we introduce some notations. For two constants $\beta\in(0,1/2)$ and $L>\beta>0$ we define a set

$$D_\varepsilon=\{x:x\in\Omega\cap\overline{B_{\varepsilon^\beta}(x_0)},\ d(x,\partial\Omega)\geq\varepsilon^L\}. \tag{2.6}$$

Let $H(y,x)$ be the regular part of the Green function for Ω. For constants $0<c_\circ<c_1$ we set

$$\lambda^{\varepsilon}_{c_i}(x)=c_i\bigg(\frac{H(x,x)}{\varepsilon}\bigg)^{1/(N-2)}\qquad i=1,2$$

and we define the following set

$$M_{\varepsilon,\delta}=\{(x,\lambda,v):x\in D_\varepsilon,\ \lambda\in[\lambda^{\varepsilon}_{c_\circ}(x),\lambda^{\varepsilon}_{c_1}(x)],\ v\in E_{x,\lambda},\ \|v\|\le\delta\}. \tag{2.7}$$

Constants β, L and c_i will be determined later. We now consider the following minimization problem:

$$\inf\{J(x,\lambda,v):(x,\lambda,v)\in M_{\varepsilon,\delta}\}. \tag{2.8}$$

It is obvious that for each fixed $\varepsilon>0$ problem (2.8) has a minimizer $(x_\varepsilon,\lambda_\varepsilon,v_\varepsilon)$. In order to prove that $(x_\varepsilon,\lambda_\varepsilon,v_\varepsilon)$ is a critical point of $J(x,\lambda,v)$, we only need to prove that $(x_\varepsilon,\lambda_\varepsilon,v_\varepsilon)$ is an interior point of $M_{\varepsilon,\delta}$.

Proof of Theorem 1.1. We prove that if $\varepsilon>0$ is small enough, the minimizer $(x_\varepsilon,\lambda_\varepsilon,v_\varepsilon)$ of (2.8) is an interior point of $M_{\varepsilon,\delta}$. First we show that if $c_\circ$ and c_1 are suitably chosen, then

$$\lambda_\varepsilon\in(\lambda^{\varepsilon}_{\circ}(x_\varepsilon),\lambda^{\varepsilon}_{1}(x_\varepsilon)). \tag{2.9}$$

Since $(x_\varepsilon,\lambda_\varepsilon,v_\varepsilon)$ is a minimum point of (2.8), we have

$$J(x_\varepsilon,\lambda_\varepsilon,v_\varepsilon)\le J(x_\varepsilon,\lambda,0) \tag{2.10}$$

for all $\lambda\in[\lambda^{\varepsilon}_{\circ}(x_\varepsilon),\lambda^{\varepsilon}_{1}(x_\varepsilon)]$.

In view of Lemma 2.1, we get

$$J(x_\varepsilon,\lambda_\varepsilon,0)+\rho\|v_\varepsilon\|^2+O\bigg(\frac{1}{\lambda_\varepsilon^2}+\varepsilon^2\ln^2\lambda_\varepsilon+\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{N-2+2\theta}}\bigg)\le J(x_\varepsilon,\lambda,0). \tag{2.11}$$

It follows from Lemma A.1 that

$$\begin{aligned}&\frac{K_3H(x_\varepsilon,x_\varepsilon)}{\lambda_\varepsilon^{N-2}}+K_2\varepsilon\ln\lambda_\varepsilon\\&\quad+O\bigg(\frac{1}{\lambda_\varepsilon^2}+\frac{\varepsilon}{\lambda_\varepsilon}+\varepsilon^2\ln^2\lambda_\varepsilon+\frac{\varepsilon}{(\lambda_\varepsilon d_\varepsilon)^{N-2}}+\frac{1}{(\lambda_\varepsilon d_\varepsilon)^{N-2+2\theta}}\bigg)\\&\le\frac{K_3H(x_\varepsilon,x_\varepsilon)}{\lambda_\varepsilon^{N-2}}+K_2\varepsilon\ln\lambda\\&\quad+O\bigg(\frac{1}{\lambda^2}+\frac{\varepsilon}{\lambda}+\varepsilon^2\ln^2\lambda+\frac{\varepsilon}{(\lambda d_\varepsilon)^{N-2}}+\frac{1}{(\lambda d_\varepsilon)^{N-2+2\theta}}\bigg).\end{aligned} \tag{2.12}$$

Since $x_\varepsilon\in D_\varepsilon$, we get $\varepsilon^L\le d_\varepsilon\le\varepsilon^\beta$. If we choose β satisfying

$$\beta>\max\bigg\{\frac12-\frac{1}{N-2},0\bigg\}, \tag{2.13}$$

then there exists a $\gamma > 0$, such that

$$(2.14) \qquad \frac{1}{\lambda^2} \le C\left(\frac{\varepsilon}{H(x_\varepsilon, x_\varepsilon)}\right)^{2/(N-2)} \le C\left(d_\varepsilon^{N-2}\varepsilon\right)^{2/(N-2)} \le C\varepsilon^{2\beta+2/(N-2)} = O(\varepsilon^{1+\gamma}),$$

$$(2.15) \qquad \frac{1}{\lambda d_\varepsilon} \le C\left(\frac{\varepsilon}{H(x_\varepsilon, x_\varepsilon)}\right)^{1/(N-2)} \frac{1}{d_\varepsilon} \le C d_\varepsilon \varepsilon^{1/(N-2)} \frac{1}{d_\varepsilon} = C\varepsilon^{1/(N-2)}.$$

Consequently, we have

$$(2.16) \qquad \frac{1}{(\lambda d_\varepsilon})^{N-2+2\theta} = O(\varepsilon^{1+\gamma}), \quad \frac{\varepsilon}{(\lambda d_\varepsilon)^{N-2}} = O(\varepsilon^2), \quad \frac{\varepsilon}{\lambda} = O(\varepsilon^{1+\gamma}).$$

On the other hand, we have

$$(2.17) \qquad \varepsilon^2 \ln \lambda = O\left(\varepsilon^2 \ln \frac{1}{\varepsilon^{1/(N-2)} d_\varepsilon}\right) = O\left(\varepsilon^2 \ln \frac{1}{\varepsilon^{L+1/(N-2)}}\right) = O(\varepsilon^{1+\gamma}).$$

Combining (2.14), (2.16) and (2.17), we see

$$(2.18) \qquad O\left(\frac{1}{\lambda^2} + \frac{\varepsilon}{\lambda} + \varepsilon^2 \ln^2 \lambda + \frac{\varepsilon}{(\lambda d_\varepsilon)^{N-2}} + \frac{1}{(\lambda d_\varepsilon)^{N-2+2\theta}}\right) = O(\varepsilon^{1+\gamma}).$$

Inserting (2.18) into (2.12), we obtain

$$(2.19) \qquad \frac{K_3 H(x_\varepsilon, x_\varepsilon)}{\lambda_\varepsilon^{N-2}} + K_2 \varepsilon \ln \lambda_\varepsilon \le \frac{K_3 H(x_\varepsilon, x_\varepsilon)}{\lambda^{N-2}} + K_2 \varepsilon \ln \lambda + O(\varepsilon^{1+\gamma}).$$

Let

$$(2.20) \qquad \lambda_\varepsilon = t_\varepsilon \left(\frac{H(x_\varepsilon, x_\varepsilon)}{\varepsilon}\right)^{1/(N-2)}, \qquad \lambda = t\left(\frac{H(x_\varepsilon, x_\varepsilon)}{\varepsilon}\right)^{1/(N-2)}.$$

We then have from (2.19)

$$(2.21) \qquad \frac{K_3}{t_\varepsilon^{N-2}} + K_2 \ln t_\varepsilon \le \frac{K_3}{t^{N-2}} + K_2 \ln t + O(\varepsilon^\gamma).$$

Since $K_3/t^{N-2} + K_2 \ln t$, $t > 0$, attains its global minimum at

$$t^* = \left[\frac{(N-2)K_3}{K_2}\right]^{1/(N-2)},$$

we conclude from (2.21) that as $\varepsilon \to 0$,

$$t_\varepsilon \to \left[\frac{(N-2)K_3}{K_2}\right]^{1/(N-2)}.$$

If we choose

$$c_\circ = \frac{1}{2}\left[\frac{(N-2)K_3}{K_2}\right]^{1/(N-2)}, \qquad c_1 = \frac{3}{2}\left[\frac{(N-2)K_3}{K_2}\right]^{1/(N-2)},$$

then, for $\varepsilon > 0$ small, we have

$$\lambda_\varepsilon \in (\lambda^\varepsilon_{c_\circ}(x_\varepsilon), \lambda^\varepsilon_{c_1}(x_\varepsilon)). \tag{2.22}$$

Next we prove that $\|v_\varepsilon\| < \delta$ and x_ε is an interior point of D_ε. Let n be the inward unit normal of $\partial\Omega$ at $x_\circ$. Let

$$z_\varepsilon = x_\circ + \varepsilon n \tag{2.23}$$

and fix $\lambda^*_\varepsilon \in (\lambda^\varepsilon_\circ(z_\varepsilon), \lambda^\varepsilon_1(z_\varepsilon))$. Since $d(z_\varepsilon, \partial\Omega) = \varepsilon$, we have

$$\lambda^*_\varepsilon \sim \left(\frac{1}{\varepsilon\varepsilon^{N-2}}\right)^{1/(N-2)} = \varepsilon^{-(N-1)/(N-2)}.$$

Thus it follows from Lemma A.1 that

$$\begin{aligned} K(PU_{z_\varepsilon,\lambda^*_\varepsilon}) &= \frac{A^{1-2/(2^*-\varepsilon)}}{Q(z_\varepsilon)^{2/(2^*-\varepsilon)}}(1 + O(\varepsilon\ln(1/\varepsilon))) \\ &= \frac{A^{1-2/(2^*-\varepsilon)}}{Q(x_\circ)^{2/(2^*-\varepsilon)}}(1 + O(\varepsilon))(1 + O(\varepsilon\ln(1/\varepsilon))) \\ &= \frac{A^{1-2/(2^*-\varepsilon)}}{Q(x_\circ)^{2/(2^*-\varepsilon)}}(1 + O(\varepsilon\ln(1/\varepsilon))). \end{aligned} \tag{2.24}$$

Hence

$$J(x_\varepsilon, \lambda_\varepsilon, v_\varepsilon) \le J(z_\varepsilon, \lambda^*_\varepsilon, 0) = \frac{A^{1-2/(2^*-\varepsilon)}}{Q(x_\circ)^{2/(2^*-\varepsilon)}}(1 + O(\varepsilon\ln(1/\varepsilon))). \tag{2.25}$$

In view of Lemma 2.1 and (2.18), we have

$$K(PU_{x_\varepsilon,\lambda_\varepsilon}) + \rho\|v_\varepsilon\|^2 + O(\varepsilon^{1+\gamma}) \le \frac{A^{1-2/(2^*-\varepsilon)}}{Q(x_\circ)^{2/(2^*-\varepsilon)}}(1 + O(\varepsilon\ln(1/\varepsilon))). \tag{2.26}$$

Consequently, it follows from Lemma A.1 and (2.26) that

$$\begin{aligned} \frac{A^{1-2/(2^*-\varepsilon)}}{Q(x_\varepsilon)^{2/(2^*-\varepsilon)}} &\left(1 + \frac{K_3 H(x_\varepsilon, x_\varepsilon)}{\lambda_\varepsilon^{N-2}} + K_2\varepsilon[\ln\lambda_\varepsilon + K_4]\right) \\ &+ \rho\|v_\varepsilon\|^2 + O(\varepsilon^{1+\gamma}) \le \frac{A^{1-2/(2^*-\varepsilon)}}{Q(x_\circ)^{2/(2^*-\varepsilon)}}(1 + O(\varepsilon\ln(1/\varepsilon))). \end{aligned} \tag{2.27}$$

Since $Q(x_\varepsilon) \le Q(x_\circ)$, (2.27) implies

$$\|v_\varepsilon\|^2 = O(\varepsilon\ln(1/\varepsilon)) \le \delta^2/2$$

for $\varepsilon > 0$ sufficiently small.

Now we prove that x_ε is an interior point of D_ε. This fact will be established in two steps.

Step 1. $x_\varepsilon \notin \{x : d(x, \partial\Omega) = \varepsilon^L\}$. Suppose to the contrary that $d(x_\varepsilon, \partial\Omega) = \varepsilon^L$. Then

$$\lambda_\varepsilon \ge \lambda^\varepsilon_{c_\circ}(x_\varepsilon) \ge c'_\circ\left(\frac{1}{d(x_\varepsilon, \partial\Omega)^{N-2}\varepsilon}\right)^{1/(N-2)} \ge \frac{C}{\varepsilon^L}.$$

Hence

$$\text{LHS of (2.27)} \geq \frac{A^{1-2/(2^*-\varepsilon)}}{Q(x_\circ)^{2/(2^*-\varepsilon)}}(1+K_2\varepsilon\ln\lambda_\varepsilon+O(\varepsilon)) \geq \frac{A^{1-2/(2^*-\varepsilon)}}{Q(x_\circ)^{2/(2^*-\varepsilon)}}(1+K_2L\varepsilon\ln(1/\varepsilon)+O(\varepsilon)). \tag{2.29}$$

Combining (2.27) and (2.29), we are led to

$$K_2L\varepsilon\ln(1/\varepsilon)+O(\varepsilon)\leq C\varepsilon\ln(1/\varepsilon),$$

where $C>0$ is a constant independent of L. So we get a contradiction if $L>0$ is chosen large enough.

Step 2. $x_\varepsilon \notin \partial B_{\varepsilon^\beta}(x_\circ)$. Again arguing indirectly suppose that $x_\varepsilon \in \partial B_{\varepsilon^\beta}(x_\circ)$. Then by the assumption on $Q(y)$, we have

$$\frac{1}{Q(x_\varepsilon)^{2/(2^*-\varepsilon)}} \geq \frac{1}{(Q(x_\circ)-a\varepsilon^{\beta(2+\alpha)})^{2/(2^*-\varepsilon)}} \geq \frac{1+a'\varepsilon^{\beta(2+\alpha)}}{Q(x_\circ)^{2/(2^*-\varepsilon)}}.$$

Hence, if we can choose $\beta>0$ satisfying

$$\beta(2+\alpha)<1, \tag{2.30}$$

then

$$\text{LHS of (2.27)} \geq \frac{A^{1-2/(2^*-\varepsilon)}}{Q(x_\circ)^{2/(2^*-\varepsilon)}}(1+a'\varepsilon^{\beta(2+\alpha)})(1+O(\varepsilon\ln(1/\varepsilon))) \geq \frac{A^{1-2/(2^*-\varepsilon)}}{Q(x_\circ)^{2/(2^*-\varepsilon)}}(1+a'\varepsilon^{\beta(2+\alpha)}). \tag{2.31}$$

Combining (2.27) and (2.31), we obtain

$$a'\varepsilon^{\beta(2+\alpha)}\leq O(\varepsilon\ln(1/\varepsilon)), \tag{2.32}$$

which is impossible. Thus it remains to prove that we can choose a $\beta>0$, such that (2.13) and (2.30) hold. We distinguish two cases: (i) $N\geq 5$ and (ii) $N=3,4$. In the case (i) since $\alpha\in[0,4/(N-4))$, we can choose $\beta\in(1/2-1/(N-2),1/2)$ satisfying $\beta(2+\alpha)<1$. Finally, if $N=3,4$, we can take $\beta>0$ sufficiently small such that (2.30) holds.

From Steps 1 and 2 we deduce x_ε is an interior point of D_ε. □

PROOF OF THEOREM 1.2. Theorem 1.2 can be derived essentially by the same method as used for Theorem 1.1. We only point out the necessary changes in the proof.

We consider the minimization problem

$$\inf\{J(x,\lambda,v): x\in\overline{B_\delta(x_\circ)},\ \lambda\in[\varepsilon^{-\beta},\varepsilon^{-L}],\ v\in E_{x,\lambda},\ \|v\|\leq\delta\}, \tag{2.33}$$

in place of (2.8), where $\beta < L$ are some positive constants to be determined later. Let $(x_\varepsilon, \lambda_\varepsilon, v_\varepsilon)$ be a minimizer of problem (2.33). From $J(x_\varepsilon, \lambda_\varepsilon, v_\varepsilon) \le J(x_0, \lambda_\varepsilon, 0)$, we easily derive that $x_\varepsilon \to x_\circ$, $\|v_\varepsilon\| \to 0$ as $\varepsilon \to 0$.

We now show that $L > 0$ and $\beta > 0$ can be chosen so that $\varepsilon^{-\beta} < \lambda_\varepsilon < \varepsilon^{-L}$. It follows from Lemma A.1 that

$$J(x_\varepsilon, \varepsilon^{-4}, 0) = \frac{A^{1-1/(2^*-\varepsilon)}}{Q(x_\circ)^{1/(2^*-\varepsilon)}}(1 + K_2\varepsilon[4\ln(1/\varepsilon) - K_4] + O(\varepsilon^{1+\sigma})). \tag{2.34}$$

On the other hand, by Lemma A.2, we have the estimate

$$\begin{aligned}
&\int_\Omega Q(y)|PU_{x,\lambda} + v|^{2^*-\varepsilon} \\
&= \int_{B_{\delta(x_0)}} Q(y)|PU_{x,\lambda} + v|^{2^*-\varepsilon} + O(\|v\|^{2^*-\varepsilon} + \lambda^{-N}) \\
&\le Q(x_\circ)\int_{B_{\delta(x_0)}} |PU_{x,\lambda} + v|^{2^*-\varepsilon} + O(\|v\|^{2^*-\varepsilon} + \lambda^{-N}) \\
&= Q(x_\circ)\int_\Omega |PU_{x,\lambda} + v|^{2^*-\varepsilon} + O(\|v\|^{2^*-\varepsilon} + \lambda^{-N}) \\
&\le Q(x_\circ)\bigg(\int_\Omega |PU_{x,\lambda}|^{2^*-\varepsilon} + \frac{(2^*-\varepsilon)(2^*-\varepsilon-1)}{2}\int_\Omega |PU_{x,\lambda}|^{2^*-\varepsilon-2}v^2\bigg) \\
&\quad + O\bigg(\varepsilon\ln\lambda + \frac{1}{\lambda^{\theta+(N-2)/2}}\bigg)\|v\| + O(\|v\|^{2^*-\varepsilon} + \lambda^{-N}) \\
&= Q(x_\circ)\bigg(A - (2^*-\varepsilon)\frac{BH(x,x)}{\lambda^{N-2}} - \varepsilon\bigg(A\ln\lambda^{(N-2)/2} - \int_{R^N} U^{2^*}\ln U\bigg) \\
&\quad + \frac{(2^*-\varepsilon)(2^*-\varepsilon-1)}{2}\int_\Omega |PU_{x,\lambda}|^{2^*-\varepsilon-2}v^2\bigg) \\
&\quad + O\bigg(\varepsilon\ln\lambda + \frac{1}{\lambda^{\theta+(N-2)/2}}\bigg)\|v\| + O(\|v\|^{2^*-\varepsilon} + \lambda^{-N}).
\end{aligned} \tag{2.35}$$

Clearly, (2.35) implies

$$\begin{aligned}
J(x,\lambda,v) &\ge \frac{A^{1-2/(2^*-\varepsilon)}}{Q(x_0)^{2/(2^*-\varepsilon)}}\bigg(1 + \rho'\|v\|^2 + \frac{K_3H(x,x)}{\lambda^{N-2}} \\
&\quad + K_2\varepsilon[\ln\lambda - K_4] + O(\varepsilon^{1+\sigma})\bigg) \\
&\ge \frac{A^{1-1/(2^*-\varepsilon)}}{Q(x_0)^{2/(2^*-\varepsilon)}}\bigg(1 + \frac{K_3H(x,x)}{\lambda^{N-2}} \\
&\quad + K_2\varepsilon[\ln\lambda - K_4] + O(\varepsilon^{1+\sigma})\bigg).
\end{aligned} \tag{2.36}$$

Using the inequality $J(x_\varepsilon, \lambda_\varepsilon, v_\varepsilon) \le J(x_0, \varepsilon^{-4}, 0)$, we deduce from (2.34) and (2.36) that

$$\frac{K_3H(x_\varepsilon, x_\varepsilon)}{\lambda_\varepsilon^{N-2}} + K_2\varepsilon\ln\lambda_\varepsilon \le 4K_2\varepsilon\ln\frac{1}{\varepsilon} + O(\varepsilon^{1+\sigma}). \tag{2.37}$$

As in the proof of the previous theorem we now proceed in two steps.

Step 1. $\lambda_\varepsilon < \varepsilon^{-L}$ for $L > 0$ sufficiently large. Suppose that $\lambda_\varepsilon = \varepsilon^{-L}$. Then it follows from (2.37) that

$$LK_2\varepsilon\ln(1/\varepsilon) \le 4K_2\varepsilon\ln(1/\varepsilon) + O(\varepsilon^{1+\sigma}),$$

which is impossible if $L > 0$ is large enough.

Step 2. $\lambda_\varepsilon = \varepsilon^{-\beta}$ is impossible if $\beta > 0$ is small enough. Assuming that $\lambda_\varepsilon = \varepsilon^{-\beta}$, we deduce from (2.37) that

$$\varepsilon^{(N-2)\beta} \le C\varepsilon\ln(1/\varepsilon).$$

This is a contradiction if $\beta > 0$ is small enough and this completes the proof. □

Now we give an example which shows that, in general, a solution for (1.1)–(1.3) concentrating on the minimum point of $Q(y)$ on the boundary may not exists.

EXAMPLE 2.2. Let $\phi(y)$ be a continuous function which attains its global maximum at $y = 0$ and $\phi(y)$ is decreasing in every direction. For each $x_0 \in \partial\Omega$, define

$$Q(y) = C - \phi(y - x_0),$$

where $C > 0$ is chosen large enough such that $Q(y)$ is positive in Ω. Then using the moving plane method of Gidas, Ni and Nirenberg [22] in the normal direction of $\partial\Omega$ at $x_\circ$, we see that the maximum point of every solution of (1.1)–(1.3) is away from a neighbourhood of $x_\circ$. This means that there is no solution concentrating on $x_\circ$.

To close this section we give the following nonexistence result:

THEOREM 2.3. *Suppose that $x_\circ \in \Omega$ is a critical point of $Q(y)$ satisfying one of the following conditions:*

(i) $N \ge 5$ *and* $\Delta Q(x_\circ) > 0$,

(ii) $N = 4$ *and* $K_3H(x_\circ, x_\circ) - K_1\Delta Q(x_\circ) < 0$, *where K_1 and K_3 are the constants in Lemma* A.1.

Then (1.1)–(1.3) *has no solution of the form* (1.8) *satisfying* (1.9)–(1.12).

PROOF. Suppose that there exists a solution of (1.1)–(1.3) of the form (1.8) and satisfying (1.9)–(1.12). We commence by showing that $\varepsilon\ln\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$. Indeed, multiplying (1.1) by $PU_{x_\varepsilon,\lambda_\varepsilon}$ and integrating over Ω, we get

$$\begin{aligned} \alpha_\varepsilon\int_\Omega |DPU_{x_\varepsilon,\lambda_\varepsilon}|^2 &= \int_\Omega Q(y)|\alpha_\varepsilon PU_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon|^{2^*-\varepsilon-1}PU_{x_\varepsilon,\lambda_\varepsilon} \\ &= Q(x_\varepsilon)\alpha_\varepsilon^{2^*-\varepsilon-1}\int_\Omega U_{x_\varepsilon,\lambda_\varepsilon}^{2^*-\varepsilon} + o(1), \end{aligned} \tag{2.38}$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Consequently, we have

$$\int_{R^N} |DU|^2 + o(1) = \lambda_\varepsilon^{-(N-2)\varepsilon/2} \int_{R^N} U^{2^*} + o(1), \tag{2.39}$$

which yields $\lambda_\varepsilon^{-(N-2)\varepsilon/2} \to 1$ as $\varepsilon \to 0$. Hence $\varepsilon \ln \lambda_\varepsilon \to 0$ as $\varepsilon \to 0$.

Next, we estimate v_ε. Multiplying (1.1) by v_ε and integrating over Ω, we get

$$\begin{aligned} \int_\Omega |Dv_\varepsilon|^2 &= \int_\Omega Q(y)|\alpha_\varepsilon PU_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon|^{2^*-\varepsilon-1} v_\varepsilon \\ &= \alpha_\varepsilon^{2^*-\varepsilon-1} \int_\Omega Q(y) PU_{x_\varepsilon,\lambda_\varepsilon}^{2^*-\varepsilon-1} v_\varepsilon \\ &\quad + (2^* - \varepsilon - 1)\alpha_\varepsilon^{2^*-\varepsilon-2} \int_\Omega Q(y) PU_{x_\varepsilon,\lambda_\varepsilon}^{2^*-\varepsilon-2} v_\varepsilon^2 + O(\|v_\varepsilon\|^{2+\theta_1}), \end{aligned} \tag{2.40}$$

where $\theta_1 > 0$ is a constant.

It follows from Appendix D in [34] that there exists a $\rho > 0$, such that

$$\int_\Omega |Dv_\varepsilon|^2 - (2^* - \varepsilon - 1)\alpha_\varepsilon^{2^*-\varepsilon-2} \int_\Omega Q(y) PU_{x_\varepsilon,\lambda_\varepsilon}^{2^*-\varepsilon-2} v_\varepsilon^2 \geq \rho \int_\Omega |Dv_\varepsilon|^2. \tag{2.41}$$

Combining (2.40) and (2.41) we get

$$\|v_\varepsilon\|^2 \leq O\bigg(\int_\Omega Q(y) PU_{x_\varepsilon,\lambda_\varepsilon}^{2^*-\varepsilon-1} v_\varepsilon \bigg).$$

From this, with the aid of Lemma A.2, we obtain

$$\|v_\varepsilon\| \leq O\bigg(\frac{|DQ(x_\varepsilon)|}{\lambda_\varepsilon} + \varepsilon \bigg). \tag{2.42}$$

Suppose that $N \geq 5$. Multiplying (1.1) by $\partial PU_{x_\varepsilon,\lambda_\varepsilon}/\partial\lambda$ and integrating over Ω, we get

$$\bigg\langle PU_{x_\varepsilon,\lambda_\varepsilon}, \frac{\partial PU_{x_\varepsilon,\lambda_\varepsilon}}{\partial \lambda} \bigg\rangle - \int_\Omega Q(y)(\alpha_\varepsilon PU_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon)^{2^*-\varepsilon-1} \frac{\partial PU_{x_\varepsilon,\lambda_\varepsilon}}{\partial \lambda} = 0.$$

Arguing as in Lemma B.2 (in fact, in the proof of Lemma B.2, we only use the assumption that $v_{x,\lambda} \in E_{x,\lambda}$ and the estimate such as (2.42)), we easily arrive at the following relation

$$\frac{2K_1 \Delta Q(x_\varepsilon)}{\lambda_\varepsilon^3} + \frac{K_2 \varepsilon}{\lambda_\varepsilon} + o\bigg(\frac{1}{\lambda_\varepsilon^3} + \frac{\varepsilon}{\lambda_\varepsilon} \bigg) + O\bigg(\frac{\varepsilon^2}{\lambda_\varepsilon} \bigg) = 0. \tag{2.43}$$

Since $\Delta Q(x_\varepsilon) > 0$ and $\varepsilon \ln \lambda_\varepsilon \to 0$ as $\varepsilon \to 0$, we get from (2.43) that

$$\frac{1}{\lambda_\varepsilon^3} + \frac{\varepsilon}{\lambda_\varepsilon} \leq 0,$$

which is impossible.

Finally, we consider the case $N = 4$. As in the case $N \geq 5$ we derive the following asymptotic relation

$$2\frac{K_1 \Delta Q(x_\varepsilon) - K_3 H(x_\varepsilon, x_\varepsilon)}{\lambda_\varepsilon^3} + \frac{K_2 \varepsilon}{\lambda_\varepsilon} + o\left(\frac{1}{\lambda_\varepsilon^3} + \frac{\varepsilon}{\lambda_\varepsilon}\right) + O\left(\frac{\varepsilon^2}{\lambda_\varepsilon}\right) = 0,$$

which contradicts the assumption (ii). □

3. Proof of Theorems 1.3 and 1.4

In this section, except in the proof of Theorem 1.4 we always assume that $Q(y)$ satisfies the conditions in Theorem 1.3.

Proposition 3.1. *There exists an $\varepsilon_\circ > 0$, such that for each $\varepsilon \in (0, \varepsilon_\circ]$, there is a C^1-map $v_\varepsilon = v_\varepsilon(x, \lambda)$, $x \in \{x : d(x, \partial\Omega) \geq d_\circ > 0\}$ and $\lambda > 0$ large, such that* (1.21) *is satisfied. Moreover,*

$$\|v_\varepsilon\| = O(\varepsilon^{(1+\sigma)/2}), \tag{3.1}$$

where $\sigma > 0$ is a constant.

Proof. The proof of Proposition 3.1 is standard, see [34]. Estimate (3.1) follows from Lemma A.2. □

Proposition 3.2. *There exists an $\varepsilon_\circ > 0$, such that for each $\varepsilon \in (0, \varepsilon_\circ]$, there exists a C^1-map $\lambda_\varepsilon = \lambda_\varepsilon(x) : B_{\varepsilon^{1/L}}(x_\circ) \to R^+$, such that* (1.20) *is satisfied. Moreover, $\lambda_\varepsilon = t_\varepsilon(x)\varepsilon^{-1/(N-2)}$ with*

$$|t_\varepsilon(x) - t_\circ(x)| = O(\varepsilon^{1/(N-2)}), \tag{3.2}$$

where

$$t_\circ(x) = \left(\frac{(N-2)K_3 H(x,x)}{K_2}\right)^{1/(N-2)}.$$

Proof. Using Lemma B.2 or Lemma C.2, we derive the following relation

$$\frac{\partial J}{\partial \lambda} = -\frac{(N-2)K_3 H(x,x)}{\lambda^{N-1}} + \frac{K_2 \varepsilon}{\lambda} + O(\varepsilon^{1+2/(N-2)}). \tag{3.3}$$

On the other hand by virtue of Lemma E.1 we have

$$B\left\langle \frac{\partial^2 PU_{x,\lambda}}{\partial \lambda^2}, v_\varepsilon \right\rangle + \sum_{j=1}^{N} G_j \left\langle \frac{\partial^2 PU_{x,\lambda}}{\partial \lambda \partial x_j}, v_\varepsilon \right\rangle = O(\varepsilon^{1+2/(N-2)}). \tag{3.4}$$

Consequently, equation (1.20) is equivalent to

$$-\frac{(N-2)K_3 H(x,x)}{\lambda^{N-1}} + \frac{K_2 \varepsilon}{\lambda} + O(\varepsilon^{1+2/(N-2)}) = 0. \tag{3.5}$$

Letting

$$\lambda_\varepsilon = t_\varepsilon \varepsilon^{-1/(N-2)}, \tag{3.6}$$

we deduce from (3.5) that

$$-\frac{(N-2)K_3H(x,x)}{t_\varepsilon^{N-1}}+\frac{K_2}{t_\varepsilon}+O(\varepsilon^{1/(N-2)})=0. \tag{3.7}$$

It is easy to see that (3.7) has a solution

$$t_\varepsilon(x)\in\biggl(\frac{3}{4}\biggl(\frac{(N-2)K_3H(x,x)}{K_2}\biggr)^{1/(N-2)},\frac{3}{2}\biggl(\frac{(N-2)K_3H(x,x)}{K_2}\biggr)^{1/(N-2)}\biggr).$$

Let

$$F(\lambda)=\frac{\partial J}{\partial\lambda}-B\biggl\langle\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda^2},v_\varepsilon\biggr\rangle-\sum_{j=1}^N G_j\biggl\langle\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda\partial x_j},v_\varepsilon\biggr\rangle.$$

Then it follows from Lemma D.1 and Lemma E.1 that

$$\begin{aligned}F'(\lambda)=&\frac{(N-1)(N-2)K_3H(x,x)}{\lambda^N}-\frac{K_2\varepsilon}{\lambda^2}+O(\varepsilon^{1+\tau+2/(N-2)})\\&-\frac{\partial B}{\partial\lambda}\biggl\langle\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda^2},v_\varepsilon\biggr\rangle-B\biggl\langle\frac{\partial^3 PU_{x,\lambda}}{\partial\lambda^3},v_\varepsilon\biggr\rangle-B\biggl\langle\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda^2},\frac{\partial v_\varepsilon}{\partial\lambda}\biggr\rangle\\&-\sum_{j=1}^N\biggl[\frac{\partial G_j}{\partial\lambda}\biggl\langle\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda\partial x_j},v_\varepsilon\biggr\rangle+G_j\biggl\langle\frac{\partial^3 PU_{x,\lambda}}{\partial\lambda^2\partial x_j},v_\varepsilon\biggr\rangle\\&+G_j\biggl\langle\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda\partial x_j},\frac{\partial v_\varepsilon}{\partial\lambda}\biggr\rangle\biggr]\\=&\frac{(N-1)(N-2)K_3H(x,x)}{\lambda^N}-\frac{K_2\varepsilon}{\lambda^2}+O(\varepsilon^{1+\tau+2/(N-2)})>0,\end{aligned}\tag{3.8}$$

for all $\lambda\in\bigl(\frac{3}{4}\bigl(\frac{(N-2)K_3H(x,x)}{K_2\varepsilon}\bigr)^{1/(N-2)},\frac{3}{2}\bigl(\frac{(N-2)K_3H(x,x)}{K_2\varepsilon}\bigr)^{1/(N-2)}\bigr)$. Consequently, the equation $F(\lambda)=0$ has a unique solution in

$$\biggl(\frac{3}{4}\biggl(\frac{(N-2)K_3H(x,x)}{K_2\varepsilon}\biggr)^{1/(N-2)},\frac{3}{2}\biggl(\frac{(N-2)K_3H(x,x)}{K_2\varepsilon}\biggr)^{1/(N-2)}\biggr).$$

Let $\lambda_\varepsilon(x)$ in the unique solution of (1.20) in

$$\biggl(\frac{3}{4}\biggl(\frac{(N-2)K_3H(x,x)}{K_2\varepsilon}\biggr)^{1/(N-2)},\frac{3}{2}\biggl(\frac{(N-2)K_3H(x,x)}{K_2\varepsilon}\biggr)^{1/(N-2)}\biggr).$$

Since all the terms in (1.20) are of C^1 with respect to x and λ, we see that $\lambda_\varepsilon(x)$ is a C^1 map in x.

Let

$$\phi(t)=-\frac{(N-2)K_3H(x,x)}{t^{N-1}}+\frac{K_2}{t}.$$

We then have

$$\phi(t_\varepsilon(x))=O(\varepsilon^{1/(N-2)}),\qquad \phi(t_0(x))=0. \tag{3.9}$$

Sine $\phi'(t_\circ(x))\neq 0$, it follows from (3.9) that

$$|t_\varepsilon(x)-t_\circ(x)|=O(\varepsilon^{1/(N-2)}).$$

□

To prove Theorems 1.3 and 1.4 we consider the maximization problem:

$$\sup\{J(x,\lambda_\varepsilon(x),v_\varepsilon(x,\lambda_\varepsilon(x))) : |x-x_0|\le \varepsilon^{1/L}\}.$$

Then the above problem has a maximiser $x_\varepsilon \in \{|x_\varepsilon - x_\circ| \le \varepsilon^{1/L}\}$. In order to prove that x_ε is a critical point, we only need to prove that $|x_\varepsilon - x_\circ| < \varepsilon^{1/L}$.

PROPOSITION 3.3. *Let x_ε be a maximiser of* (3.10). *Then there exists a $\sigma_2 > 0$, such that $|x_\varepsilon - x_\circ|^L = O(\varepsilon^{1+\sigma_2})$. In particular, if $\varepsilon > 0$ is small enough, x_ε is an interior point of $B_{\varepsilon^{1/L}}(x_\circ)$.*

PROOF. It follows from Lemma A.1, Propositions 3.1 and 3.2 that

$$\begin{aligned}(3.11)\quad J(x,\lambda_\varepsilon(x),v_\varepsilon(x,\lambda_\varepsilon(x))) &= J(x,\lambda_\varepsilon(x),0)+O(\varepsilon^{1+\sigma})\\ &= \frac{A^{1-1/(2^*-\varepsilon)}}{Q(x)^{2/(2^*-\varepsilon)}}\bigg[1+\frac{K_3H(x,x)}{\lambda_\varepsilon^{N-2}(x)}+K_2\varepsilon(\ln\lambda_\varepsilon(x)-K_4)+O(\varepsilon^{1+\sigma})\bigg].\end{aligned}$$

Letting

$$\lambda_\varepsilon(x) =: t_\varepsilon(x)\varepsilon^{-1/(N-2)} = (t_\circ(x)+O(\varepsilon^{1/(N-2)}))\varepsilon^{-1/(N-2)},$$

we deduce from (3.11) that

$$\begin{aligned}(3.12)\qquad & J(x,\lambda_\varepsilon(x),v_\varepsilon(x,\lambda_\varepsilon(x)))\\ &= \frac{A^{1-1/(2^*-\varepsilon)}}{Q(x)^{2/(2^*-\varepsilon)}}\bigg[1+\frac{K_2}{N-2}\varepsilon\ln\frac{1}{\varepsilon}-K_2K_4\varepsilon\\ &\quad+\varepsilon\bigg(\frac{K_3H(x,x)}{t_\varepsilon^{N-2}(x)}+K_2\ln t_\varepsilon(x)\bigg)+O(\varepsilon^{1+\sigma})\bigg]\\ &= \frac{A^{1-1/(2^*-\varepsilon)}}{Q(x)^{2/(2^*-\varepsilon)}}\bigg[1+\frac{K_2}{N-2}\varepsilon\ln\frac{1}{\varepsilon}-K_2K_4\varepsilon\\ &\quad+\varepsilon\bigg(\frac{K_3H(x,x)}{t_\circ^{N-2}(x)}+K_2\ln t_\circ(x)\bigg)\\ &\quad+\varepsilon O(|t_\varepsilon(x)-t_\circ(x)|^2)+O(\varepsilon^{1+\sigma})\bigg]\\ &= \frac{A^{1-1/(2^*-\varepsilon)}}{Q(x)^{2/(2^*-\varepsilon)}}\bigg[1+\frac{K_2}{N-2}\varepsilon\ln\frac{1}{\varepsilon}-K_2K_4\varepsilon\\ &\quad+\varepsilon\bigg(K_2+\frac{K_2}{N-2}\ln\frac{K_2H(x,x)}{K_3}\bigg)+O(\varepsilon^{1+\sigma_1})\bigg],\end{aligned}$$

where $\sigma_1 > 0$ is a constant. Since x_ε is a maximum of (3.10), we have

$$J(x_\varepsilon,\lambda_\varepsilon(x_\varepsilon),v_\varepsilon(x_\varepsilon,\lambda_\varepsilon(x_\varepsilon)))\ge J(x_\circ,\lambda_\varepsilon(x_\circ),v_\varepsilon(x_\circ),\lambda_\varepsilon(x_\circ)).$$

This, together with (3.12) and

$$Q(x_\varepsilon)\ge Q(x_\circ)+C|x-x_\circ|^L,$$

implies

$$|x-x_\circ|^L \le C\varepsilon(\ln H(x_\varepsilon,x_\varepsilon)-\ln H(x_\circ,x_\circ))+O(\varepsilon^{1+\sigma_1}) = O(\varepsilon|x_\varepsilon-x_\circ|)+O(\varepsilon^{1+\sigma_1}). \tag{3.13}$$

Hence $|x-x_\circ|^L = O(\varepsilon^{1+\sigma_2})$, where $\sigma_2>0$ is a constant. The assertion of Proposition 3.3 readily follows. □

Proof of Theorem 1.3. We only need to prove that

$$(x_\varepsilon,\lambda_\varepsilon(x_\varepsilon),v_\varepsilon(x_\varepsilon,\lambda_\varepsilon(x_\varepsilon)))$$

satisfies (1.19). Indeed, we have by straightforward calculations that

$$\begin{aligned}0&=\frac{\partial J}{\partial x_i}+\frac{\partial J}{\partial\lambda}\frac{\partial\lambda}{\partial x_i}+\left\langle\frac{\partial J}{\partial v},\frac{\partial v}{\partial x_i}+\frac{\partial v}{\partial\lambda}\frac{\partial\lambda}{\partial x_i}\right\rangle\\&=\frac{\partial J}{\partial x_i}+\left[B\left\langle\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda^2},v\right\rangle+\sum_{j=1}^N G_j\left\langle\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda\partial x_j},v\right\rangle\right]\frac{\partial\lambda}{\partial x_i}\\&\quad+B\left\langle\frac{\partial PU_{x,\lambda}}{\partial\lambda},\frac{\partial v}{\partial x_i}\right\rangle+\sum_{j=1}^N G_j\left\langle\frac{\partial PU_{x,\lambda}}{\partial x_j},\frac{\partial v}{\partial x_i}\right\rangle\\&\quad+\left[B\left\langle\frac{\partial PU_{x,\lambda}}{\partial\lambda},\frac{\partial v}{\partial\lambda}\right\rangle+\sum_{j=1}^N G_j\left\langle\frac{\partial PU_{x,\lambda}}{\partial x_j},\frac{\partial v}{\partial\lambda}\right\rangle\right]\frac{\partial\lambda}{\partial x_i}\\&=\frac{\partial J}{\partial x_i}-B\left\langle\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda\partial x_i},v\right\rangle-\sum_{j=1}^N G_j\left\langle\frac{\partial^2 PU_{x,\lambda}}{\partial x_i\partial x_j},v\right\rangle.\end{aligned}\tag{3.14}$$

This obviously shows that (1.19) holds. □

Proof of Theorem 1.4. Since all the estimates from Appendix C to Appendix E are valid for each $x\in\overline{B_\delta(x_\circ)}$ and $\lambda\in[c_\circ\varepsilon^{-1/(N-2)},c_1\varepsilon^{-1/(N-2)}]$ under the conditions of this theorem, we can prove Theorem 1.4 in exactly the same way as Theorem 1.3. □

Appendix A

Lemma A.1. *Suppose that λ satisfies $\varepsilon\ln\lambda\to 0$ as $\varepsilon\to 0$. Then*

$$\begin{aligned}K(PU_{x,\lambda})=\frac{A^{1-2/(2^*-\varepsilon)}}{Q(x)^{2/(2^*-\varepsilon)}}\bigg[&1-\frac{K_1\Delta Q(x)}{Q(x)\lambda^2}+\frac{K_3H(x,x)}{\lambda^{N-2}}+K_2\varepsilon[\ln\lambda-K_4]\\&+O\bigg(\sum_{j=3}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^j}+\frac{\varepsilon}{\lambda}+\frac{\varepsilon}{(\lambda d)^{N-2}}\\&+\frac{1}{(\lambda d)^N}+\varepsilon^2\ln^2\lambda\bigg)\bigg],\end{aligned}\tag{A.1}$$

where

$$(A.2)\qquad A=\int_{\mathbb{R}^N}U^{2^*},\quad K_1=\frac{1}{2^*NA}\int_{\mathbb{R}^N}|y|^2U^{2^*},\quad K_2=\frac{N-2}{2^*},$$

$$(A.3)\qquad K_3=\frac{1}{A}\int_{\mathbb{R}^N}U^{2^*-1},\qquad K_4=\frac{2}{2^*A}\int_{\mathbb{R}^N}U^{2^*}\ln U.$$

Proof. Let $\varphi_{x,\lambda}=U_{x,\lambda}-PU_{x,\lambda}$. It then follows from Proposition 1 in [34] that

$$0\le\varphi_{x,\lambda}\le\frac{H(x,x)}{\lambda^{(N-2)/2}}.$$

As in [3] and [34], we have

$$(A.4)\qquad \int_\Omega|DPU_{x,\lambda}|^2=\int_\Omega U_{x,\lambda}^{2^*-1}PU_{x,\lambda}=A-\frac{BH(x,x)}{\lambda^{N-2}}+O\bigg(\frac{1}{(\lambda d)^N}\bigg),$$

where $B=\int_\Omega U_{x,\lambda}^{2^*-1}$. We also have (see [34]):

$$\begin{aligned}(A.5)\qquad \int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon}&=\int_\Omega Q(y)|U_{x,\lambda}-\varphi_{x,\lambda}|^{2^*-\varepsilon}\\&=\int_\Omega Q(y)U_{x,\lambda}^{2^*-\varepsilon}-(2^*-\varepsilon)\int_\Omega Q(y)U_{x,\lambda}^{2^*-\varepsilon-1}\varphi_{x,\lambda}+O\bigg(\frac{1}{(\lambda d)^{N-1}}\bigg)\\&=\int_\Omega Q(y)U_{x,\lambda}^{2^*-\varepsilon}-(2^*-\varepsilon)\frac{BQ(x)H(x,x)}{\lambda^{N-2}}+O\bigg(\frac{1}{(\lambda d)^{N-1}}\bigg)\\&=\int_\Omega Q(y)U_{x,\lambda}^{2^*-\varepsilon}-2^*\frac{BQ(x)H(x,x)}{\lambda^{N-2}}+O\bigg(\frac{\varepsilon}{(\lambda d)^{N-2}}+\frac{1}{(\lambda d)^{N-1}}\bigg),\end{aligned}$$

and

$$\begin{aligned}(A.6)\qquad \int_\Omega Q(y)U_{x,\lambda}^{2^*-\varepsilon}&=\int_\Omega Q(y)U_{x,\lambda}^{2^*}-\varepsilon\int_\Omega Q(y)U_{x,\lambda}^{2^*}\ln U_{x,\lambda}+O(\varepsilon^2\ln^2\lambda)\\&=\int_\Omega Q(y)U_{x,\lambda}^{2^*}-\varepsilon Q(x)\bigg[A\ln\lambda^{(N-2)/2}-\int_{R^N}U^{2^*}\ln U\\&\qquad+O\bigg(\frac{1}{\lambda}+\frac{1}{(\lambda d)^N}\bigg)\bigg]+O(\varepsilon^2\ln^2\lambda).\end{aligned}$$

Using Taylor's expansion we write

$$\begin{aligned}(A.7)\qquad \int_\Omega Q(y)U_{x,\lambda}^{2^*}&=\int_\Omega Q(x)U_{x,\lambda}^{2^*}+\int_\Omega\langle DQ(x),y-x\rangle U_{x,\lambda}^{2^*}\\&\quad+\frac{1}{2}\int_\Omega\langle D^2Q(x)(y-x),y-x\rangle U_{x,\lambda}^{2^*}\\&\quad+O\bigg(\sum_{j=3}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^j}\bigg)+O\bigg(\frac{1}{\lambda^{N-1}}\bigg)\end{aligned}$$

$$
\begin{aligned}
&= Q(x)A + \frac{\Delta Q(x)}{2N\lambda^2}\int_{\mathbb{R}^N}|y|^2U^{2^*}\\
&\quad + O\bigg(\sum_{j=3}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^j}\bigg) + O\bigg(\frac{1}{\lambda^{N-1}} + \frac{1}{(\lambda d)^N}\bigg).
\end{aligned}
$$

To estimate the integral involving $DQ(x)$ we set $\Omega_{x,\lambda} = \{y : y/\lambda + x \in \Omega\}$. Then

$$
\begin{aligned}
\int_\Omega \langle DQ(x), y-x\rangle U_{x,\lambda}^{2^*} &= \frac{1}{\lambda}\int_{\Omega_{x,\lambda}}\langle DQ(x), y\rangle U^{2^*}\\
&= \frac{1}{\lambda}\int_{B(0,\lambda d)}\langle DQ(x), y\rangle U^{2^*} + \frac{1}{\lambda}\int_{\Omega_{x,\lambda}-B(0,\lambda d)}\langle DQ(x), y\rangle U^{2^*}\\
&= O\bigg(\frac{1}{\lambda}\int_{|y|\geq\lambda d}\frac{|y|}{(1+|y|^2)^N}\,dy\bigg) = O\bigg(\frac{1}{\lambda}\frac{1}{(\lambda d)^{N-1}}\bigg).
\end{aligned}
$$

Using the last relation and the radial symmetry of U we deduce from (A.5) that

$$
\begin{aligned}
\text{(A.8)}\quad &\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon}\\
&= Q(x)A - \varepsilon Q(x)\bigg(A\ln\lambda^{(N-2)/2} - \int_{\mathbb{R}^N}U^{2^*}\ln U\bigg)\\
&\quad + \frac{\Delta Q(x)}{2N\lambda^2}\int_{\mathbb{R}^N}|y|^2U^{2^*} - 2^*\frac{BQ(x)H(x,x)}{\lambda^{N-2}} + O\bigg(\sum_{j=3}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^j}\bigg)\\
&\quad + O\bigg(\frac{\varepsilon}{\lambda} + \frac{\varepsilon}{(\lambda d)^{N-2}} + \frac{1}{\lambda^{N-1}} + \frac{1}{(\lambda d)^N} + \varepsilon^2\ln^2\lambda\bigg).
\end{aligned}
$$

It is easy to see that Lemma A.1 follows from (A.4) and (A.8). □

Lemma A.2. *Let k be the biggest positive integer satisfying $k \leq (N-2)/2$. For any $v \in E_{x,\lambda}$, we have*

$$
\text{(A.9)}\quad \int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon-1}v = O\bigg(\sum_{j=1}^{k}\frac{|D^jQ(x)|}{\lambda^j} + \varepsilon + \frac{1}{(\lambda d)^{\theta+(N-2)/2}}\bigg)\|v\|,
$$

$$
\begin{aligned}
\text{(A.10)}\quad &\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon-2}\frac{\partial PU_{x,\lambda}}{\partial\lambda}v\\
&\qquad = O\bigg(\sum_{j=1}^{k}\frac{|D^jQ(x)|}{\lambda^{j+1}} + \frac{\varepsilon}{\lambda} + \frac{1}{\lambda(\lambda d)^{\theta+(N-2)/2}}\bigg)\|v\|,
\end{aligned}
$$

$$
\begin{aligned}
\text{(A.11)}\quad &\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon-2}\frac{\partial PU_{x,\lambda}}{\partial x_j}v\\
&\qquad = O\bigg(\sum_{j=1}^{k}\frac{|D^jQ(x)|}{\lambda^{j-1}} + \varepsilon\lambda + \frac{\lambda}{(\lambda d)^{\theta+(N-2)/2}}\bigg)\|v\|,
\end{aligned}
$$

where $\theta > 0$ is a positive constant.

PROOF. In fact, arguing as Rey [34] (see (3.20)–(3.22) there), we have

$$
\begin{aligned}
&\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon-1}v \qquad \text{(A.12)}\\
&= \int_\Omega Q(y)|U_{x,\lambda}|^{2^*-\varepsilon-1}v + \int_\Omega Q(y)(|PU_{x,\lambda}|^{2^*-\varepsilon-1} - |U_{x,\lambda}|^{2^*-\varepsilon-1})v \\
&= \int_\Omega Q(y)|U_{x,\lambda}|^{2^*-\varepsilon-1}v + O\bigg(\frac{\|v\|}{(\lambda d)^{\theta+(N-2)/2}}\bigg) \\
&= \lambda^{-(N-2)\varepsilon/2}\int_{\Omega_{x,\lambda}} Q\bigg(\frac{y}{\lambda}+x\bigg)|U|^{2^*-\varepsilon-1}v\bigg(\frac{y}{\lambda}+x\bigg) \\
&\quad + O\bigg(\frac{1}{(\lambda d)^{\theta+(N-2)/2}}\bigg)\|v\| \\
&= \lambda^{-(N-2)\varepsilon/2}\int_{\Omega_{x,\lambda}} Q\bigg(\frac{y}{\lambda}+x\bigg)|U|^{2^*-1}v(\frac{y}{\lambda}+x) \\
&\quad + O\bigg(\varepsilon + \frac{1}{(\lambda d)^{\theta+(N-2)/2}}\bigg)\|v\| \\
&= \lambda^{-(N-2)\varepsilon/2}\int_{\Omega_{x,\lambda}} \bigg(Q\bigg(\frac{y}{\lambda}+x\bigg) - Q(x)\bigg)|U|^{2^*-1}v\bigg(\frac{y}{\lambda}+x\bigg) \\
&\quad + O\bigg(\varepsilon + \frac{1}{(\lambda d)^{\theta+(N-2)/2}}\bigg)\|v\| \\
&= O\bigg(\sum_{j=1}^{k}\frac{|D^{(j)}Q(x)|}{\lambda^j} + \varepsilon + \frac{1}{(\lambda d)^{\theta+(N-2)/2}}\bigg)\|v\|.
\end{aligned}
$$

Since $|\partial PU_{x,\lambda}/\partial\lambda| \le C\lambda^{-1}U_{x,\lambda}$ and $|\partial PU_{x,\lambda}/\partial x_j| \le C\lambda U_{x,\lambda}$, we can prove (A.10) and (A.11) in a similar way. □

Appendix B

From now on, we always assume that $x \in \Omega$ satisfies $d = d(x,\partial\Omega) \ge d_\circ > 0$, and v_ε is the function obtained in Proposition 3.1.

Let us define

$$l(u) = \frac{\|u\|^2}{\int_\Omega Q(y)|u|^{2^*-\varepsilon}}.$$

In Lemma B.1 we establish a basic property of the functional l.

LEMMA B.1. *The functional l has the following expansion in λ:*

$$l(PU_{x,\lambda} + v_\varepsilon) = \frac{1}{Q(x)}\bigg[1 + O\bigg(\frac{1}{\lambda^{N-2}} + \sum_{j=1}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^j} + \varepsilon\ln\lambda\bigg)\bigg].$$

Proof. From Lemma A.2 and (A.8), we get

$$
\begin{aligned}
&l(PU_{x,\lambda}+v_\varepsilon) \\
&= \frac{\|PU_{x,\lambda}\|^2+\|v_\varepsilon\|^2}{\int_\Omega Q(y)PU_{x,\lambda}^{2^*-\varepsilon}+(2^*-\varepsilon)\int_\Omega Q(y)PU_{x,\lambda}^{2^*-\varepsilon-1}v_\varepsilon+O(\|v_\varepsilon\|^2)} \\
&= \frac{\|PU_{x,\lambda}\|^2}{\int_\Omega Q(y)PU_{x,\lambda}^{2^*-\varepsilon}}\bigg[1+O\bigg(\frac{1}{\lambda^{N-2+2\theta}}+\sum_{j=1}^{k}\frac{|D^{(j)}(x)|^2}{\lambda^{2j}}+\varepsilon^2\ln^2\lambda\bigg)\bigg] \\
&= \frac{1}{Q(x)}\bigg[1+O\bigg(\frac{1}{\lambda^{N-2}}+\sum_{j=1}^{N-2}\frac{|D^{(j)}(x)|}{\lambda^{j}}+\varepsilon\ln\lambda\bigg)\bigg].
\end{aligned}
\tag{B.1}
$$

□

Lemma B.2. *The derivative of the functional K satisfies*

$$
\begin{aligned}
&\bigg\langle DK(PU_{x,\lambda}+v_\varepsilon),\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle \\
&=\frac{A^{1-2/(2^*-\varepsilon)}}{Q(x)^{2/(2^*-\varepsilon)}}\bigg[\frac{2K_1\Delta Q(x)}{Q(x)\lambda^3}-\frac{(N-2)K_3H(x,x)}{\lambda^{N-1}}+\frac{K_2\varepsilon}{\lambda} \\
&\quad+O\bigg(\sum_{j=1}^{2}\frac{|D^{(j)}Q(x)|^2}{\lambda^{2j+1}}+\sum_{j=3}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^{j+1}}+\frac{\varepsilon}{\lambda^2}+\frac{1}{\lambda^{N+1}}+\frac{\varepsilon^2\ln^2\lambda}{\lambda}\bigg)\bigg].
\end{aligned}
\tag{B.2}
$$

Proof. We have

$$
\begin{aligned}
&\bigg\langle DK(PU_{x,\lambda}+v_\varepsilon),\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle \\
&\qquad=\frac{2}{(\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon})^{2/(2^*-\varepsilon)}}\bigg(\bigg\langle PU_{x,\lambda},\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle \\
&\qquad\quad-l(PU_{x,\lambda}+v_\varepsilon)\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-1}\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg).
\end{aligned}
\tag{B.3}
$$

By (B.5) in [34], we have

$$
\bigg\langle PU_{x,\lambda},\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle=\frac{(N-2)BH(x,x)}{2\lambda^{N-1}}+O\bigg(\frac{1}{\lambda^N}\bigg).
\tag{B.4}
$$

On the other hand it follows from Lemma A.2 and Proposition 3.1 that

$$
\begin{aligned}
&\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-1}\frac{\partial PU_{x,\lambda}}{\partial\lambda} \\
&\quad=\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon-1}\frac{\partial PU_{x,\lambda}}{\partial\lambda} \\
&\quad\quad+(2^*-\varepsilon-1)\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon-2}\frac{\partial PU_{x,\lambda}}{\partial\lambda}v_\varepsilon+O\bigg(\frac{\|v_\varepsilon\|^2}{\lambda}\bigg)
\end{aligned}
\tag{B.5}
$$

$$= \int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon-1}\frac{\partial PU_{x,\lambda}}{\partial\lambda} + O\bigg(\sum_{j=1}^{k}\frac{|D^jQ(x)|^2}{\lambda^{2j+1}} + \frac{(\varepsilon\ln\lambda)^2}{\lambda} + \frac{1}{\lambda^{2\theta+(N-1)}}\bigg).$$

Also, by Proposition 1 in [34], we have

$$\text{(B.6)}\quad \int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon-1}\frac{\partial PU_{x,\lambda}}{\partial\lambda}$$
$$= \int_\Omega Q(y)|U_{x,\lambda}|^{2^*-\varepsilon-1}\frac{\partial PU_{x,\lambda}}{\partial\lambda} - (2^*-\varepsilon-1)\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-\varepsilon-2}\varphi_{x,\lambda}\frac{\partial PU_{x,\lambda}}{\partial\lambda} + O\bigg(\frac{\|\varphi_{x,\lambda}\|_\infty^2}{\lambda}\int_\Omega U_{x,\lambda}^{2^*-\varepsilon-2}\bigg)$$
$$= \int_\Omega Q(y)|U_{x,\lambda}|^{2^*-\varepsilon-1}\frac{\partial PU_{x,\lambda}}{\partial\lambda} - (2^*-\varepsilon-1)\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-\varepsilon-2}\varphi_{x,\lambda}\frac{\partial PU_{x,\lambda}}{\partial\lambda} + O\bigg(\frac{1}{\lambda^{N-1+2\theta}}\bigg)$$

and

$$\text{(B.7)}\quad \int_\Omega Q(y)|U_{x,\lambda}|^{2^*-\varepsilon-1}\frac{\partial PU_{x,\lambda}}{\partial\lambda} = \int_\Omega Q(y)|U_{x,\lambda}|^{2^*-\varepsilon-1}\frac{\partial U_{x,\lambda}}{\partial\lambda} - \int_\Omega Q(y)|U_{x,\lambda}|^{2^*-\varepsilon-1}\frac{\partial \varphi_{x,\lambda}}{\partial\lambda} + O\bigg(\frac{1}{\lambda^N}\bigg).$$

In view of the symmetry of $U_{x,\lambda}$ and $\partial U_{x,\lambda}/\partial\lambda$, we have

$$\text{(B.8)}\quad \int_\Omega Q(y)|U_{x,\lambda}|^{2^*-\varepsilon-1}\frac{\partial U_{x,\lambda}}{\partial\lambda}$$
$$= Q(x)\int_\Omega |U_{x,\lambda}|^{2^*-\varepsilon-1}\frac{\partial U_{x,\lambda}}{\partial\lambda} + \int_\Omega \langle DQ(x), y-x\rangle|U_{x,\lambda}|^{2^*-\varepsilon-1}\frac{\partial U_{x,\lambda}}{\partial\lambda} + \frac{1}{2}\int_\Omega\langle D^2Q(x)(y-x), y-x\rangle|U_{x,\lambda}|^{2^*-\varepsilon-1}\frac{\partial U_{x,\lambda}}{\partial\lambda} + O\bigg(\sum_{j=3}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^{j+1}} + \frac{1}{\lambda^N}\bigg)$$
$$= \frac{Q(x)}{2^*-\varepsilon}\frac{\partial}{\partial\lambda}\int_\Omega |U_{x,\lambda}|^{2^*-\varepsilon} + \frac{\Delta Q(x)}{2N}\frac{1}{2^*-\varepsilon}\frac{\partial}{\partial\lambda}\int_\Omega |y|^2|U_{x,\lambda}|^{2^*-\varepsilon} + O\bigg(\sum_{j=3}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^{j+1}} + \frac{1}{\lambda^N}\bigg)$$

$$= -\frac{N-2}{22^*}Q(x)A\frac{\varepsilon}{\lambda} - \frac{\Delta Q(x)}{2^*N\lambda^3}\int_{R^N}|y|^2U^{2^*} + O\bigg(\sum_{j=3}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^{j+1}} + \frac{1}{\lambda^N} + \frac{\varepsilon^2\ln\lambda}{\lambda}\bigg).$$

Following the proof of (B.13) in [34], it is easy to show that

$$\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-\varepsilon-1}\frac{\partial\varphi_{x,\lambda}}{\partial\lambda} = -\frac{N-2}{2}\frac{BQ(x)H(x,x)}{\lambda^{N-1}} + O\bigg(\frac{1}{\lambda^N} + \frac{\varepsilon\ln\lambda}{\lambda^{N-1}}\bigg). \tag{B.9}$$

Substituting (B.8) and (B.9) into (B.7), we obtain

$$\begin{aligned}\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-\varepsilon-1}\frac{\partial PU_{x,\lambda}}{\partial\lambda} &= -\frac{N-2}{22^*}Q(x)A\frac{\varepsilon}{\lambda} - \frac{\Delta Q(x)}{2^*N\lambda^3}\int_{R^N}|y|^2U^{2^*} + \frac{N-2}{2}\frac{BQ(x)H(x,x)}{\lambda^{N-1}} \\ &\quad + O\bigg(\sum_{j=3}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^{j+1}} + \frac{1}{\lambda^N} + \frac{\varepsilon^2\ln\lambda}{\lambda}\bigg).\end{aligned} \tag{B.10}$$

By Proposition 1 in [34], we have

$$\begin{aligned}(2^*-\varepsilon-1)&\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-\varepsilon-2}\varphi_{x,\lambda}\frac{\partial PU_{x,\lambda}}{\partial\lambda} \\ &= (2^*-\varepsilon-1)\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-\varepsilon-2}\varphi_{x,\lambda}\frac{\partial U_{x,\lambda}}{\partial\lambda} \\ &\quad - (2^*-\varepsilon-1)\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-\varepsilon-2}\varphi_{x,\lambda}\frac{\partial \varphi_{x,\lambda}}{\partial\lambda} \\ &= (2^*-\varepsilon-1)\frac{1}{\lambda^{(N-2)/2}}\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-\varepsilon-2}H(y,x)\frac{\partial U_{x,\lambda}}{\partial\lambda} \\ &\quad + O\bigg(\frac{1}{\lambda^{N-1+2\theta}}\bigg) \\ &= \frac{Q(x)H(x,x)}{\lambda^{(N-2)/2}}\frac{\partial\lambda^{-(N-2)(1+\varepsilon)/2}}{\partial\lambda}\int_{R^N}U^{2^*-\varepsilon-1} + O\bigg(\frac{1}{\lambda^{N-1+2\theta}}\bigg) \\ &= -\frac{N-2}{2}\frac{Q(x)H(x,x)}{\lambda^{N-1}}B + O\bigg(\frac{\varepsilon\ln\lambda}{\lambda^{N-1}} + \frac{1}{\lambda^{N-1+2\theta}}\bigg).\end{aligned} \tag{B.11}$$

Combining (B.6), (B.10) and (B.11) we obtain

$$\begin{aligned}\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon-1}\frac{\partial PU_{x,\lambda}}{\partial\lambda} &= -\frac{N-2}{22^*}Q(x)A\frac{\varepsilon}{\lambda} - \frac{\Delta Q(x)}{2^*N\lambda^3}\int_{R^N}|y|^2U^{2^*} + \frac{(N-2)BQ(x)H(x,x)}{\lambda^{N-1}} \\ &\quad + O\bigg(\sum_{j=3}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^{j+1}} + \frac{1}{\lambda^{N-1+2\theta}} + \frac{\varepsilon^2\ln\lambda}{\lambda}\bigg).\end{aligned} \tag{B.12}$$

Inserting (B.12) into (B.5), we get

$$\begin{aligned}(\text{B.13})\qquad &\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-1}\frac{\partial PU_{x,\lambda}}{\partial\lambda}\\ &= -\frac{N-2}{22^*}Q(x)A\frac{\varepsilon}{\lambda}-\frac{\Delta Q(x)}{2^*N\lambda^3}\int_{R^N}|y|^2U^{2^*}+\frac{(N-2)BQ(x)H(x,x)}{\lambda^{N-1}}\\ &\quad+O\bigg(\sum_{j=3}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^{j+1}}\bigg)\\ &\quad+O\bigg(\sum_{j=1}^{k}\frac{|D^jQ(x)|^2}{\lambda^{2j+1}}+\frac{(\varepsilon\ln\lambda)^2}{\lambda}+\frac{1}{\lambda^{2\theta+(N-1)}}\bigg).\end{aligned}$$

Then Lemma B.2 follows from (B.3), (B.4), (B.13) and Lemma B.1. □

Appendix C

In this section, we will estimate $\|\partial v_\varepsilon/\partial\lambda\|$, where v_ε is the map obtained in Proposition 3.1. We assume that $Q(y)$ satisfies the assumptions of Theorem 1.3 and

$$(\text{C.1})\qquad |x-x_\circ|\le\varepsilon^{1/L},\quad \lambda\in[c_\circ\varepsilon^{-1/(N-2)},c_1\varepsilon^{-1/(N-2)}].$$

Lemma C.1. *Suppose that $Q(y)$ satisfies the assumptions of Theorem* 1.3. *Then*

$$(\text{C.2})\qquad \|v_\varepsilon\|=O(\varepsilon^{(1+\sigma)/2}),$$

where $\sigma>0$ is a constant.

Proof. In view of Lemma A.2, we only need to check

$$(\text{C.3})\qquad \frac{|D^{(j)}Q(x)|}{\lambda^j}=O(\varepsilon^{1+\sigma}).$$

Indeed, by the assumptions imposed on $Q(y)$, we see that if $\sigma>0$ is sufficiently small, then

$$\frac{|D^{(j)}Q(x)|}{\lambda^j}\le C\frac{|x-x_\circ|^{L-j}}{\lambda^j}\le C|x-x_\circ|^{L(1+\sigma)}+C\lambda^{-L(1+\sigma)j/(L\sigma+j)}=O(\varepsilon^{1+\sigma}).$$

□

Lemma C.2. *Suppose that $Q(y)$ satisfies the assumptions of Theorem* 1.3. *Then we have the following estimates:*

$$(\text{C.4})\qquad \langle DK(PU_{x,\lambda}+v_\varepsilon),PU_{x,\lambda}\rangle=O(\varepsilon^{1-\sigma}),$$

$$(\text{C.5})\qquad \bigg\langle DK(PU_{x,\lambda}+v_\varepsilon),\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle=O(\varepsilon^{1+\sigma+1/(N-2)}),$$

$$(\text{C.6})\qquad \bigg\langle DK(PU_{x,\lambda}+v_\varepsilon),\frac{\partial PU_{x,\lambda}}{\partial x_j}\bigg\rangle=O(\varepsilon^{1-\sigma-1/(N-2)}).$$

Proof. Lemma B.2, together with (C.3), gives (C.5). The relation (C.4) follows from

$$
\begin{aligned}
\text{(C.7)}\ &\langle DK(PU_{x,\lambda}+v_\varepsilon), PU_{x,\lambda}\rangle \\
&= O\bigg(\|PU_{x,\lambda}\|^2 - \frac{1}{Q(x)}(1+O(\varepsilon\ln 1/\varepsilon)) \\
&\qquad \cdot \int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-2}(PU_{x,\lambda}+v_\varepsilon)PU_{x,\lambda}\bigg) \\
&= O\bigg(\|PU_{x,\lambda}\|^2 - \frac{1}{Q(x)}(1+O(\varepsilon\ln 1/\varepsilon))\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon} + O(\|v_\varepsilon\|^2)\bigg) \\
&= O\bigg(\|PU_{x,\lambda}\|^2 - \int_\Omega PU_{x,\lambda}^{2^*} + O(\varepsilon\ln 1/\varepsilon)\bigg) = O(\varepsilon^{1-\sigma}).
\end{aligned}
$$

Finally, noting $|\partial PU_{x,\lambda}/\partial x_j| \le C\lambda PU_{x,\lambda}$, we can prove (C.6) in a similar manner. □

Proposition C.3. *Let v_ε be a function from Proposition* 3.1. *Then*

$$
\left\|\frac{\partial v_\varepsilon}{\partial\lambda}\right\| = O(\varepsilon^{1+\sigma/2+1/(N-2)}). \tag{C.8}
$$

To prove Proposition C.3 we need Lemma C.4 below. We write the following decomposition

$$
\frac{\partial v_\varepsilon}{\partial\lambda} = w + \alpha PU_{x,\lambda} + \beta\frac{\partial PU_{x,\lambda}}{\partial\lambda} + \sum_{j=1}^N \gamma_j \frac{\partial PU_{x,\lambda}}{\partial x_j}, \tag{C.9}
$$

where α, β and γ_j are chosen in such a way that $w \in E_{x,\lambda}$.

Lemma C.4. *Let α, β and γ_j be coefficients from* (C.9). *Then*

$$
\alpha = O\bigg(\frac{\|v_\varepsilon\|}{\lambda^2}\bigg), \qquad \beta = O(\|v_\varepsilon\|), \qquad \gamma_j = O\bigg(\frac{\|v_\varepsilon\|}{\lambda^2}\bigg).
$$

Proof. We know that α, β and γ_j satisfy

$$
\alpha\|PU_{x,\lambda}\|^2 + \beta\bigg\langle\frac{\partial PU_{x,\lambda}}{\partial\lambda}, PU_{x,\lambda}\bigg\rangle + \sum_{j=1}^N \gamma_j\bigg\langle\frac{\partial PU_{x,\lambda}}{\partial x_j}, PU_{x,\lambda}\bigg\rangle = 0, \tag{C.10}
$$

$$
\begin{aligned}
\text{(C.11)}\quad \alpha\bigg\langle PU_{x,\lambda}, \frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle + \beta\Big\|\frac{\partial PU_{x,\lambda}}{\partial\lambda}\Big\|^2 + \sum_{j=1}^N \gamma_j\bigg\langle\frac{\partial PU_{x,\lambda}}{\partial x_j}, \frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle& \\
= -\bigg\langle v_\varepsilon, \frac{\partial^2 PU_{x,\lambda}}{\partial\lambda^2}\bigg\rangle = O\bigg(\frac{\|v_\varepsilon\|}{\lambda^2}\bigg),&
\end{aligned}
$$

$$\text{(C.12)}\quad \alpha\left\langle PU_{x,\lambda}, \frac{\partial PU_{x,\lambda}}{\partial x_i}\right\rangle + \beta\left\langle \frac{\partial PU_{x,\lambda}}{\partial \lambda}, \frac{\partial PU_{x,\lambda}}{\partial x_i}\right\rangle + \sum_{j=1}^{N}\gamma_j\left\langle \frac{\partial PU_{x,\lambda}}{\partial x_j}, \frac{\partial PU_{x,\lambda}}{\partial x_i}\right\rangle = -\left\langle v_\varepsilon, \frac{\partial^2 PU_{x,\lambda}}{\partial\lambda\partial x_i}\right\rangle = O(\|v_\varepsilon\|).$$

Solving the above system we get the desired result. □

For a fixed $w_0 \in E_{x_\circ,\lambda_\circ}$, let $\pi(x,\lambda)$ be the orthogonal projection of $w_\circ$ into $E_{x,\lambda}$. Then

$$\text{(C.13)}\quad w_\circ = \pi(x,\lambda) + a(x,\lambda)PU_{x,\lambda} + b(x,\lambda)\frac{\partial PU_{x,\lambda}}{\partial\lambda} + \sum_{j=1}^{N} g_j(x,\lambda)\frac{\partial PU_{x,\lambda}}{\partial x_j}.$$

Lemma C.5. *The map $\pi(x,\lambda)$ is C^1 with respect to x and λ, and*

$$\text{(C.14)}\quad a(x_\circ,\lambda_\circ) = 0, \qquad \frac{\partial a(x_\circ,\lambda_\circ)}{\partial\lambda} = O\bigg(\frac{\|w_\circ\|}{\lambda^2}\bigg),$$

$$\text{(C.15)}\quad b(x_\circ,\lambda_\circ) = 0, \qquad \frac{\partial b(x_\circ,\lambda_\circ)}{\partial\lambda} = O(\|w_\circ\|),$$

$$\text{(C.16)}\quad g_j(x_\circ,\lambda_\circ) = 0, \qquad \frac{\partial g_j(x_\circ,\lambda_\circ)}{\partial\lambda} = O\bigg(\frac{\|w_\circ\|}{\lambda^2}\bigg).$$

Proof. It is clear that $a(x,\lambda)$, $b(x,\lambda)$ and $g_j(x,\lambda)$ satisfy

$$\text{(C.17)}\quad a\|PU_{x,\lambda}\|^2 + b\left\langle \frac{\partial PU_{x,\lambda}}{\partial\lambda}, PU_{x,\lambda}\right\rangle + \sum_{j=1}^{N} g_j\left\langle \frac{\partial PU_{x,\lambda}}{\partial x_j}, PU_{x,\lambda}\right\rangle = \langle w_\circ, PU_{x,\lambda}\rangle,$$

$$\text{(C.18)}\quad a\left\langle PU_{x,\lambda}, \frac{\partial PU_{x,\lambda}}{\partial\lambda}\right\rangle + b\left\|\frac{\partial PU_{x,\lambda}}{\partial\lambda}\right\|^2 + \sum_{j=1}^{N} g_j\left\langle \frac{\partial PU_{x,\lambda}}{\partial x_j}, \frac{\partial PU_{x,\lambda}}{\partial\lambda}\right\rangle = \left\langle w_\circ, \frac{\partial PU_{x,\lambda}}{\partial\lambda}\right\rangle,$$

$$\text{(C.19)}\quad a\left\langle PU_{x,\lambda}, \frac{\partial PU_{x,\lambda}}{\partial x_i}\right\rangle + b\left\langle \frac{\partial PU_{x,\lambda}}{\partial\lambda}, \frac{\partial PU_{x,\lambda}}{\partial x_i}\right\rangle + \sum_{j=1}^{N} g_j\left\langle \frac{\partial PU_{x,\lambda}}{\partial x_j}, \frac{\partial PU_{x,\lambda}}{\partial x_i}\right\rangle = \left\langle w_\circ, \frac{\partial PU_{x,\lambda}}{\partial x_i}\right\rangle.$$

Solving the above system we easily see that $a(x,\lambda)$, $b(x,\lambda)$ and $g_j(x,\lambda)$ are C^1 with respect to x and λ.

On the other hand from the fact that $w_\circ \in E_{x_\circ,\lambda_\circ}$, we easily deduce

$$a(x_\circ,\lambda_\circ) = b(x_\circ,\lambda_\circ) = g_j(x_\circ,\lambda_\circ) = 0.$$

Differentiating (C.17)–(C.19) with respect to λ, we get

$$\text{(C.20)}\quad \frac{\partial a(x_\circ,\lambda_\circ)}{\partial\lambda}\|PU_{x_\circ,\lambda_\circ}\|^2+\frac{\partial b(x_\circ,\lambda_\circ)}{\partial\lambda}\bigg\langle\frac{\partial PU_{x_\circ,\lambda_\circ}}{\partial\lambda},PU_{x_\circ,\lambda_\circ}\bigg\rangle + \sum_{j=1}^{N}\frac{\partial g_j(x_\circ,\lambda_\circ)}{\partial\lambda}\bigg\langle\frac{\partial PU_{x_\circ,\lambda_\circ}}{\partial x_j},PU_{x_\circ,\lambda_\circ}\bigg\rangle = \bigg\langle w_\circ,\frac{\partial PU_{x_\circ,\lambda_\circ}}{\partial\lambda}\bigg\rangle = O\bigg(\frac{\|w_\circ\|}{\lambda}\bigg),$$

$$\text{(C.21)}\quad \frac{\partial a(x_\circ,\lambda_\circ)}{\partial\lambda}\bigg\langle PU_{x_\circ,\lambda_\circ},\frac{\partial PU_{x_\circ,\lambda_\circ}}{\partial\lambda}\bigg\rangle+\frac{\partial b(x_\circ,\lambda_\circ)}{\partial\lambda}\bigg\|\frac{\partial PU_{x_\circ,\lambda_\circ}}{\partial\lambda}\bigg\|^2 + \sum_{j=1}^{N}\frac{\partial g_j(x_\circ,\lambda_\circ)}{\partial\lambda}\bigg\langle\frac{\partial PU_{x_\circ,\lambda_\circ}}{\partial x_j},\frac{\partial PU_{x_\circ,\lambda_\circ}}{\partial\lambda}\bigg\rangle = \bigg\langle w_\circ,\frac{\partial^2 PU_{x_\circ,\lambda_\circ}}{\partial\lambda^2}\bigg\rangle = O\bigg(\frac{\|w_\circ\|}{\lambda^2}\bigg),$$

$$\text{(C.22)}\quad \frac{\partial a(x_\circ,\lambda_\circ)}{\partial\lambda}\bigg\langle PU_{x_\circ,\lambda_\circ},\frac{\partial PU_{x_\circ,\lambda_\circ}}{\partial x_i}\bigg\rangle+\frac{\partial b(x_\circ,\lambda_\circ)}{\partial\lambda}\bigg\langle\frac{\partial PU_{x_\circ,\lambda_\circ}}{\partial\lambda},\frac{\partial PU_{x_\circ,\lambda_\circ}}{\partial x_i}\bigg\rangle + \sum_{j=1}^{N}\frac{\partial g_j(x_\circ,\lambda_\circ)}{\partial\lambda}\bigg\langle\frac{\partial PU_{x_\circ,\lambda_\circ}}{\partial x_j},\frac{\partial PU_{x_\circ,\lambda_\circ}}{\partial x_i}\bigg\rangle = \bigg\langle w_\circ,\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda\partial x_i}\bigg\rangle = O(\|w_\circ\|).$$

Solving the above system we get the desired estimate. □

Proof of Proposition C.3. In view of Lemma C.4, we only need to estimate $\|w\|$. Let $\pi(x',\lambda')$ be the orthogonal projection of $w\in E_{x,\lambda}$ onto $E_{x',\lambda'}$. By (1.21), we have

$$\text{(C.23)}\quad \langle DK(PU_{x',\lambda'}+v_\varepsilon(x',\lambda')),\pi(x',\lambda')\rangle=0.$$

Differentiating (C.23) with respect to λ' and letting $(x',\lambda')=(x,\lambda)$, we get

$$\text{(C.24)}\quad D^2K(PU_{x,\lambda}+v_\varepsilon)\bigg(\frac{\partial PU_{x,\lambda}}{\partial\lambda}+\frac{\partial v_\varepsilon}{\partial\lambda},w\bigg) + \bigg\langle DK(PU_{x,\lambda}+v_\varepsilon),\frac{\partial\pi(x,\lambda)}{\partial\lambda}\bigg\rangle=0.$$

It follows from Lemmas C.2 and C.5 that

$$\text{(C.25)}\quad \bigg\langle DK(PU_{x,\lambda}+v_\varepsilon),\frac{\partial\pi(x,\lambda)}{\partial\lambda}\bigg\rangle = \frac{\partial a}{\partial\lambda}\langle DK(PU_{x,\lambda}+v_\varepsilon),PU_{x,\lambda}\rangle+\frac{\partial b}{\partial\lambda}\bigg\langle DK(PU_{x,\lambda}+v_\varepsilon),\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle + \sum_{j=1}^{N}\frac{\partial g_j}{\partial\lambda}\bigg\langle DK(PU_{x,\lambda}+v_\varepsilon),\frac{\partial PU_{x,\lambda}}{\partial x_j}\bigg\rangle O(\varepsilon^{1-\sigma+1/(N-2)}\|w\|).$$

Combining (C.24) and (C.25) and taking Lemmas C.2 and C.4 into account we obtain

$$\text{(C.26)}\qquad \begin{aligned} D^2K(PU_{x,\lambda}+v_\varepsilon)(w,w) &= -D^2K(PU_{x,\lambda}+v_\varepsilon)\\ &\quad\cdot\bigg(\frac{\partial PU_{x,\lambda}}{\partial\lambda}+\alpha PU_{x,\lambda}+\beta\frac{\partial PU_{x,\lambda}}{\partial\lambda}+\sum_{j=1}^{N}\gamma_j\frac{\partial PU_{x,\lambda}}{\partial x_j},w\bigg)\\ &\quad+O(\varepsilon^{1-\sigma+1/(N-2)})\|w\|\\ &= -D^2K(PU_{x,\lambda}+v_\varepsilon)\bigg(\frac{\partial PU_{x,\lambda}}{\partial\lambda}+\beta\frac{\partial PU_{x,\lambda}}{\partial\lambda},w\bigg)\\ &\quad+O(\varepsilon^{(1+\sigma)/2+1/(N-2)})\|w\|. \end{aligned}$$

We now claim that

$$\text{(C.27)}\qquad D^2K(PU_{x,\lambda}+v_\varepsilon)(w,w)\ge\rho\|w\|^2,$$

for some constant $\rho>0$ and

$$\text{(C.28)}\qquad D^2K(PU_{x,\lambda}+v_\varepsilon)\bigg(\frac{\partial PU_{x,\lambda}}{\partial\lambda},w\bigg)=O(\varepsilon^{(1+\sigma)/2+1/(N-2)}\|w\|).$$

Then obviously (C.27) and (C.28) imply that $\|w\|=O(\varepsilon^{(1+\sigma)/2+1/(N-2)})$, and Proposition C.3 follows.

To prove (C.27)–(C.28), let us write

$$\text{(C.29)}\quad \begin{aligned} D^2K(u)(\varphi,\psi) &= \frac{2\langle\varphi,\psi\rangle}{(\int_\Omega Q(y)|u|^{2^*-\varepsilon})^{2/(2^*-\varepsilon)}}\\ &\quad-\frac{4\langle u,\varphi\rangle}{(\int_\Omega Q(y)|u|^{2^*-\varepsilon})^{2/(2^*-\varepsilon)+1}}\int_\Omega Q(y)|u|^{2^*-\varepsilon-1}\psi\\ &\quad-\frac{4\langle u,\psi\rangle}{(\int_\Omega Q(y)|u|^{2^*-\varepsilon})^{2/(2^*-\varepsilon)+1}}\int_\Omega Q(y)|u|^{2^*-\varepsilon-1}\varphi\\ &\quad+\frac{2}{2^*-\varepsilon}\bigg(\frac{2}{2^*-\varepsilon}+1\bigg)\frac{\int_\Omega Q(y)|u|^{2^*-\varepsilon-1}\varphi\int_\Omega Q(y)|u|^{2^*-\varepsilon-1}\psi}{(\int_\Omega Q(y)|u|^{2^*-\varepsilon})^{2/(2^*-\varepsilon)+2}}\\ &\quad-2(2^*-\varepsilon-1)\frac{\|u\|^2}{(\int_\Omega Q(y)|u|^{2^*-\varepsilon})^{2/(2^*-\varepsilon)+1}}\int_\Omega Q(y)|u|^{2^*-\varepsilon-2}\varphi\psi. \end{aligned}$$

Verification of (C.27). First, we observe that

$$\text{(C.30)}\qquad \langle PU_{x,\lambda}+v_\varepsilon,w\rangle=\langle v_\varepsilon,w\rangle=O(\varepsilon^{(1+\sigma)/2}\|w\|),$$

By Lemma A.2, we see

$$\text{(C.31)}\quad \begin{aligned} &\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-1}w\\ &\qquad=\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon-1}w+O(\|v_\varepsilon\|\|w\|)=O(\varepsilon^{(1+\sigma)/2}\|w\|). \end{aligned}$$

It follows from (C.30) and (C.31) that

$$
\begin{aligned}
\text{(C.32)}\quad D^2K(PU_{x,\lambda}+v_\varepsilon)(w,w) &= \frac{2}{(\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon})^{2/(2^*-\varepsilon)+1}} \\
&\cdot\bigg(\|w\|^2-(2^*-\varepsilon-1)l(PU_{x,\lambda}+v_\varepsilon)\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-2}w^2\bigg) \\
&+O(\varepsilon^{1+\sigma}\|w\|^2).
\end{aligned}
$$

Thus (C.27) follows from (C.32) and Appendix D in [34].

Verification of (C.28). We have

$$
\text{(C.33)}\qquad \bigg\langle\frac{\partial PU_{x,\lambda}}{\partial\lambda},w\bigg\rangle=0,\qquad \big\langle PU_{x,\lambda}+v_\varepsilon,w\big\rangle=O(\|v_\varepsilon\|\|w\|),
$$

$$
\text{(C.34)}\qquad \bigg\langle PU_{x,\lambda}+v_\varepsilon,\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle=\bigg\langle PU_{x,\lambda},\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle=O\bigg(\frac{1}{\lambda^{N-1}}\bigg).
$$

Also, as in the proof of Lemma B.2, we have

$$
\text{(C.35)}\qquad \int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-1}\frac{\partial PU_{x,\lambda}}{\partial\lambda}=O(\varepsilon^{1+1/(N-2)})
$$

and

$$
\begin{aligned}
\text{(C.36)}\quad &\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-2}\frac{\partial PU_{x,\lambda}}{\partial\lambda}w \\
&\qquad=\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon-2}\frac{\partial PU_{x,\lambda}}{\partial\lambda}w+O\bigg(\frac{\|v_\varepsilon\|\|w\|}{\lambda}\bigg) \\
&\qquad=O(\varepsilon^{(1+\sigma)2+1/(N-2)})\|w\|.
\end{aligned}
$$

Combining (C.33)–(C.33) we get (C.28) and this complete the proof of Proposition C.23. □

Appendix D

Lemma D.1. *The derivative of the functional K satisfies*

$$
\begin{aligned}
\text{(D.1)}\quad &\frac{\partial}{\partial\lambda}\bigg\langle DK(PU_{x,\lambda}+v_\varepsilon),\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle \\
&=\frac{A^{1-2/(2^*-\varepsilon)}}{Q(x)^{2/(2^*-\varepsilon)}}\bigg[\frac{(N-1)(N-2)K_3H(x,x)}{\lambda^N}-\frac{K_2\varepsilon}{\lambda^2}+O(\varepsilon^{1+\sigma+2/(N-2)})\bigg].
\end{aligned}
$$

Proof. By straightforward calculations we have

$$
\begin{aligned}
\text{(D.2)}\qquad &\frac{\partial}{\partial\lambda}\bigg\langle DK(PU_{x,\lambda}+v_\varepsilon),\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle \\
&\qquad=D^2K(PU_{x,\lambda}+v_\varepsilon)\bigg(\frac{\partial PU_{x,\lambda}}{\partial\lambda}+\frac{\partial v_\varepsilon}{\partial\lambda},\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg) \\
&\qquad\quad+\bigg\langle DK(PU_{x,\lambda}+v_\varepsilon),\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda^2}\bigg\rangle.
\end{aligned}
$$

First, we estimate $D^2K(PU_{x,\lambda}+v_\varepsilon)(\partial PU_{x,\lambda}/\partial\lambda, \partial v_\varepsilon/\partial\lambda)$. By Proposition C.3, we have

$$\text{(D.3)}\qquad \begin{aligned} \left\langle PU_{x,\lambda}+v_\varepsilon, \frac{\partial v_\varepsilon}{\partial\lambda}\right\rangle &= O(\varepsilon^{1+\sigma+1/(N-2)}),\\ \left\langle PU_{x,\lambda}+v_\varepsilon, \frac{\partial PU_{x,\lambda}}{\partial\lambda}\right\rangle &= O(\varepsilon^{(N-1)/(N-2)}). \end{aligned}$$

As in the proof of Lemma B.2, we also have

$$\text{(D.4)}\qquad \int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-2}(PU_{x,\lambda}+v_\varepsilon)\frac{\partial PU_{x,\lambda}}{\partial\lambda} = O(\varepsilon^{1+1/(N-2)}).$$

On the other hand we have

$$\text{(D.5)}\qquad \begin{aligned} &\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-2}(PU_{x,\lambda}+v_\varepsilon)\frac{\partial v_\varepsilon}{\partial\lambda}\\ &= \int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon-1}\frac{\partial v_\varepsilon}{\partial\lambda} + O\biggl(\|v_\varepsilon\|\biggl\|\frac{\partial v_\varepsilon}{\partial\lambda}\biggr\|\biggr)\\ &= \int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-1}\frac{\partial v_\varepsilon}{\partial\lambda} + O(\varepsilon^{1+\sigma+1/(N-2)})\\ &= Q(x)\int_\Omega |PU_{x,\lambda}|^{2^*-1}\frac{\partial v_\varepsilon}{\partial\lambda} + O(\varepsilon^{1+\sigma+1/(N-2)})\\ &= Q(x)\int_\Omega U_{x,\lambda}^{2^*-1}\frac{\partial v_\varepsilon}{\partial\lambda} + O(\varepsilon^{1+\sigma+1/(N-2)}) = O(\varepsilon^{1+\sigma+1/(N-2)}). \end{aligned}$$

Combining (D.3)–(D.5) we obtain

$$\text{(D.6)}\qquad \begin{aligned} &D^2K(PU_{x,\lambda}+v_\varepsilon)\biggl(\frac{\partial PU_{x,\lambda}}{\partial\lambda}, \frac{\partial v_\varepsilon}{\partial\lambda}\biggr)\\ &= \frac{2}{(\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon})^{2/(2^*-\varepsilon)}}\biggl[\biggl\langle\frac{\partial PU_{x,\lambda}}{\partial\lambda}, \frac{\partial v_\varepsilon}{\partial\lambda}\biggr\rangle\\ &\quad - (2^*-\varepsilon-1)l(PU_{x,\lambda}+v_\varepsilon)\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-2}\frac{\partial PU_{x,\lambda}}{\partial\lambda}\frac{\partial v_\varepsilon}{\partial\lambda}\biggr]\\ &\quad + O(\varepsilon^{1+\sigma+3/(N-2)}). \end{aligned}$$

We now observe that

$$\frac{\partial v_\varepsilon}{\partial\lambda} = w + \alpha PU_{x,\lambda} + \beta\frac{\partial PU_{x,\lambda}}{\partial\lambda} + \sum_{j=1}^N \gamma_j\frac{\partial PU_{x,\lambda}}{\partial x_j}$$

and

$$\left\langle\frac{\partial PU_{x,\lambda}}{\partial\lambda}, w\right\rangle = 0.$$

Consequently, it follows from Lemma A.2 and Proposition C.3 that

$$\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-2}\frac{\partial PU_{x,\lambda}}{\partial\lambda}w = O(\varepsilon^{1+\sigma+2/(N-2)}).$$

Hence

$$\begin{aligned}(\mathrm{D.7})\quad D^2K(PU_{x,\lambda}+v_\varepsilon)\bigg(\frac{\partial PU_{x,\lambda}}{\partial\lambda},\frac{\partial v_\varepsilon}{\partial\lambda}\bigg) &= O\bigg(\alpha\bigg[\bigg\langle\frac{\partial PU_{x,\lambda}}{\partial\lambda},PU_{x,\lambda}\bigg\rangle\\
&- dl(PU_{x,\lambda}+v_\varepsilon)\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-2}\frac{\partial PU_{x,\lambda}}{\partial\lambda}PU_{x,\lambda}\bigg]\\
&+\beta\bigg[\bigg\|\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\|^2\\
&- dl(PU_{x,\lambda}+v_\varepsilon)\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-2}\bigg|\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg|^2\bigg]\\
&+\sum_{j=1}^N\gamma_j\bigg[\bigg\langle\frac{\partial PU_{x,\lambda}}{\partial\lambda},\frac{\partial PU_{x,\lambda}}{\partial x_j}\bigg\rangle\\
&- dl(PU_{x,\lambda}+v_\varepsilon)\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-2}\frac{\partial PU_{x,\lambda}}{\partial\lambda}\frac{\partial PU_{x,\lambda}}{\partial x_j}\bigg]\bigg)\\
&+O(\varepsilon^{1+\sigma+2/(N-2)}),\end{aligned}$$

where $d=2^*-\varepsilon-1$. Since $\partial PU_{x,\lambda}/\partial\lambda$ is a solution of

$$-\Delta u=(2^*-1)U_{x,\lambda}^{2^*-1}u,$$

it follows from Appendix F in [21], Lemma B.1 and Lemma C.4 that

$$(\mathrm{D.8})\qquad D^2K(PU_{x,\lambda}+v_\varepsilon)\bigg(\frac{\partial PU_{x,\lambda}}{\partial\lambda},\frac{\partial v_\varepsilon}{\partial\lambda}\bigg)=O(\varepsilon^{1+\sigma+2/(N-2)}).$$

We now estimate

$$D^2K(PU_{x,\lambda}+v_\varepsilon)\bigg(\frac{\partial PU_{x,\lambda}}{\partial\lambda},\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg)+\bigg\langle DK(PU_{x,\lambda}+v_\varepsilon),\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda^2}\bigg\rangle.$$

It is easy to check

$$\begin{aligned}(\mathrm{D.9})\quad & D^2K(PU_{x,\lambda}+v_\varepsilon)\bigg(\frac{\partial PU_{x,\lambda}}{\partial\lambda},\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg)+\bigg\langle DK(PU_{x,\lambda}+v_\varepsilon),\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda^2}\bigg\rangle\\
&=\bigg(\frac{2}{\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon}}\bigg)^{2/(2^*-\varepsilon)}\bigg\{\bigg[\bigg\|\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\|^2\\
&\quad-(2^*-\varepsilon-1)l(PU_{x,\lambda}+v_\varepsilon)\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-2}\bigg|\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg|^2\bigg]\\
&\quad+\bigg\langle PU_{x,\lambda}+v_\varepsilon,\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda^2}\bigg\rangle\\
&\quad-l(PU_{x,\lambda}+v_\varepsilon)\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-1}\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda^2}\bigg\}\\
&\quad+O(\varepsilon^{1+\sigma+2/(N-2)})\\
&=\bigg(\frac{2}{\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon}}\bigg)^{2/(2^*-\varepsilon)}\end{aligned}$$

$$
\begin{aligned}
&\cdot\bigg[\frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\|PU_{x,\lambda}\|^2-\frac{1}{2^*-\varepsilon}\frac{\partial^2}{\partial\lambda^2}\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon}\\
&+\bigg\langle v_\varepsilon,\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda^2}\bigg\rangle-l(PU_{x,\lambda}+v_\varepsilon)\int_\Omega Q(y)v_\varepsilon\frac{\partial^2}{\partial\lambda^2}PU_{x,\lambda}^{2^*-\varepsilon-1}\bigg]\\
&+O(\varepsilon^{1+\sigma+2/(N-2)})\\
=&\bigg(\frac{2}{\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon}}\bigg)^{2/(2^*-\varepsilon)}\\
&\cdot\bigg[\frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\|PU_{x,\lambda}\|^2-\frac{1}{2^*-\varepsilon}\frac{\partial^2}{\partial\lambda^2}\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon}\\
&+\int_\Omega v_\varepsilon\frac{\partial^2}{\partial\lambda^2}PU_{x,\lambda}^{2^*-1}-l(PU_{x,\lambda}+v_\varepsilon)\int_\Omega Q(y)v_\varepsilon\frac{\partial^2}{\partial\lambda^2}PU_{x,\lambda}^{2^*-\varepsilon-1}\bigg]\\
&+O(\varepsilon^{1+\sigma+2/(N-2)})\\
=&\bigg(\frac{2}{\int_\Omega Q(y)|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon}}\bigg)^{2/(2^*-\varepsilon)}\\
&\cdot\bigg[\frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\|PU_{x,\lambda}\|^2-\frac{1}{2^*-\varepsilon}\frac{\partial^2}{\partial\lambda^2}\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-\varepsilon}\bigg]\\
&+O(\varepsilon^{1+\sigma+2/(N-2)}).
\end{aligned}
$$

and Lemma D.1 readily follows. □

Lemma D.2. *We have*

$$\frac{\partial}{\partial\lambda}\bigg\langle DK(PU_{x,\lambda}+v_\varepsilon),\frac{\partial PU_{x,\lambda}}{\partial x_j}\bigg\rangle=O(\varepsilon^{1-\sigma}).$$

Proof. The proof of Lemma D.2 is similar to that of Lemma D.1 and therefore is omitted. □

Lemma D.3. *We have*

$$\frac{\partial}{\partial\lambda}\langle DK(PU_{x,\lambda}+v_\varepsilon),PU_{x,\lambda}\rangle=O(\varepsilon^{1-\sigma+1/(N-2)}).$$

Proof. By Lemma C.2, we have

$$
\begin{aligned}
\text{(D.10)}\quad&\frac{\partial}{\partial\lambda}\langle DK(PU_{x,\lambda}+v_\varepsilon),PU_{x,\lambda}\rangle\\
&=D^2K(PU_{x,\lambda}+v_\varepsilon)\bigg(\frac{\partial PU_{x,\lambda}}{\partial\lambda}+\frac{\partial v_\varepsilon}{\partial\lambda},PU_{x,\lambda}\bigg)\\
&\quad+\bigg\langle DK(PU_{x,\lambda}+v_\varepsilon),\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle\\
&=D^2K(PU_{x,\lambda}+v_\varepsilon)\bigg(\frac{\partial PU_{x,\lambda}}{\partial\lambda}+\frac{\partial v_\varepsilon}{\partial\lambda},PU_{x,\lambda}\bigg)+O(\varepsilon^{1+\sigma+1/(N-2)}).
\end{aligned}
$$

As in the proof of (D.8), we get

$$D^2K(PU_{x,\lambda}+v_\varepsilon)\bigg(\frac{\partial v_\varepsilon}{\partial\lambda},PU_{x,\lambda}\bigg)=O(\varepsilon^{1+\sigma+1/(N-2)}). \tag{D.11}$$

We also have

$$\begin{aligned}D^2K(PU_{x,\lambda}+v_\varepsilon)&\bigg(\frac{\partial PU_{x,\lambda}}{\partial\lambda},PU_{x,\lambda}\bigg)\\ &=O\bigg(\bigg\langle\frac{\partial PU_{x,\lambda}}{\partial\lambda},PU_{x,\lambda}\bigg\rangle\\ &\quad-(2^*-\varepsilon-1)\int_\Omega|PU_{x,\lambda}+v_\varepsilon|^{2^*-\varepsilon-2}\frac{\partial PU_{x,\lambda}}{\partial\lambda}PU_{x,\lambda}\bigg)+O(\varepsilon^{1+1/(N-2)})\\ &=O\bigg(\frac{\|v_\varepsilon\|^2}{\lambda}+\varepsilon^{1-\sigma+1/(N-2)}\bigg)=O(\varepsilon^{1-\sigma+1/(N-2)}).\end{aligned} \tag{D.12}$$

Hence, Lemma D.3 follows from (D.10)–(D.12). □

Appendix E

Let A, B and G_j be the constants in (1.21), that is,

$$\frac{\partial J}{\partial v}=APU_{x,\lambda}+B\frac{\partial PU_{x,\lambda}}{\partial\lambda}+\sum_{j=1}^N G_j\frac{\partial PU_{x,\lambda}}{\partial x_j}.$$

Lemma E.1. *Let A, B and G_j be the constants in* (1.21). *Then*

$$A=O(\varepsilon^{1-\sigma}),\quad B=O(\varepsilon^{1-1/(N-2)}),\quad G_j=O(\varepsilon^{1-\sigma+1/(N-2)}), \tag{E.1}$$

$$\frac{\partial A}{\partial\lambda}=O(\varepsilon^{1-\sigma+1/(N-2)}),\quad \frac{\partial B}{\partial\lambda}=O(\varepsilon^{1-\sigma}),\quad \frac{\partial G_j}{\partial\lambda}=O(\varepsilon^{1-\sigma+2/(N-2)}). \tag{E.2}$$

Proof. By Lemma C.2 we see that A, B and G_j satisfy

$$\begin{aligned}A\|PU_{x,\lambda}\|^2+B\bigg\langle\frac{\partial PU_{x,\lambda}}{\partial\lambda},PU_{x,\lambda}\bigg\rangle+\sum_{j=1}^N G_j\bigg\langle\frac{\partial PU_{x,\lambda}}{\partial x_j},PU_{x,\lambda}\bigg\rangle&\\ =\bigg\langle\frac{\partial J}{\partial v},PU_{x,\lambda}\bigg\rangle=O(\varepsilon^{1-\sigma}),&\end{aligned} \tag{E.3}$$

$$\begin{aligned}A\bigg\langle PU_{x,\lambda},\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle+B\bigg\|\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\|^2+\sum_{j=1}^N G_j\bigg\langle\frac{\partial PU_{x,\lambda}}{\partial x_j},\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle&\\ =\bigg\langle\frac{\partial J}{\partial v},\frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle=O(\varepsilon^{1+1/(N-2)}),&\end{aligned} \tag{E.4}$$

$$\text{(E.5)}\quad A\left\langle PU_{x,\lambda},\frac{\partial PU_{x,\lambda}}{\partial x_i}\right\rangle+B\left\langle\frac{\partial PU_{x,\lambda}}{\partial\lambda},\frac{\partial PU_{x,\lambda}}{\partial x_i}\right\rangle + \sum_{j=1}^N G_j\left\langle\frac{\partial PU_{x,\lambda}}{\partial x_j},\frac{\partial PU_{x,\lambda}}{\partial x_i}\right\rangle=\left\langle\frac{\partial J}{\partial v},\frac{\partial PU_{x,\lambda}}{\partial x_i}\right\rangle=O(\varepsilon^{1-\sigma-1/(N-2)}).$$

Solving (E.3)–(E.5) and taking Appendix F in [21] into account, we obtain (E.1). Differentiating (E.3)–(E.5) with respect to λ, in view of Lemmas D.1–D.3, we get

$$\text{(E.6)}\quad \frac{\partial A}{\partial\lambda}\|PU_{x,\lambda}\|^2+\frac{\partial B}{\partial\lambda}\left\langle\frac{\partial PU_{x,\lambda}}{\partial\lambda},PU_{x,\lambda}\right\rangle + \sum_{j=1}^N\frac{\partial G_j}{\partial\lambda}\left\langle\frac{\partial PU_{x,\lambda}}{\partial x_j},PU_{x,\lambda}\right\rangle=O(\varepsilon^{1-\sigma+1/(N-2)}),$$

$$\text{(E.7)}\quad \frac{\partial A}{\partial\lambda}\left\langle PU_{x,\lambda},\frac{\partial PU_{x,\lambda}}{\partial\lambda}\right\rangle+\frac{\partial B}{\partial\lambda}\left\|\frac{\partial PU_{x,\lambda}}{\partial\lambda}\right\|^2 + \sum_{j=1}^N\frac{\partial G_j}{\partial\lambda}\left\langle\frac{\partial PU_{x,\lambda}}{\partial x_j},\frac{\partial PU_{x,\lambda}}{\partial\lambda}\right\rangle=O(\varepsilon^{1-\sigma+2/(N-2)}),$$

$$\text{(E.8)}\quad \frac{\partial A}{\partial\lambda}\left\langle PU_{x,\lambda},\frac{\partial PU_{x,\lambda}}{\partial x_i}\right\rangle+\frac{\partial B}{\partial\lambda}\left\langle\frac{\partial PU_{x,\lambda}}{\partial\lambda},\frac{\partial PU_{x,\lambda}}{\partial x_i}\right\rangle + \sum_{j=1}^N\frac{\partial G_j}{\partial\lambda}\left\langle\frac{\partial PU_{x,\lambda}}{\partial x_j},\frac{\partial PU_{x,\lambda}}{\partial x_i}\right\rangle=O(\varepsilon^{1-\sigma}).$$

Solving the system (E.6)–(E.8) we get (E.2). □

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