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ON SOLUTIONS OF TWO-POINT BOUNDARY VALUE PROBLEMS INSIDE ISOLATING SEGMENTS

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Abstract. We consider a two-point boundary value problem

$$\dot{x} = f(t, x), \quad x(a) = g(x(b)).$$

We assume that in the extended space of the equation there exist an isolating segment, a set such that f properly behaves on its boundary. We give a formula for the fixed point index of the composition of g with the translation operator in a neighbourhood of the set of the initial points of solutions contained in the isolating segment. We apply that formula to results on existence of solutions of some planar boundary value problem associated to equations of the form $\dot{z} = \overline{z}^q + \dots$ and $\dot{z} = e^{it}\overline{z}^q + \dots$.

1. Introduction

In this paper we consider the equation

$$\dot{x} = f(t, x), \tag{1}$$

where $f : \mathbb{R} \times M \to TM$ is a continuous time-dependent vector-field on a smooth manifold M. We assume the uniqueness of the Cauchy problem (1),

$$x(t_0) = x_0. \tag{2}$$

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Let $g : M \to M$ be a continuous map and let $a < b$ be real numbers. We associate to (1) the two-point boundary value condition

$$x(a) = g(x(b)). \tag{3}$$

In particular, if $g = \mathrm{id}$ then (3) is the *periodic condition* and in the case $M = \mathbb{R}^n$ and $g \in GL_n(\mathbb{R})$, (3) is called the *Floquet condition* (see [10]). If $g = -\mathrm{id}$ then it is called the *anti-periodic condition*. A great number of papers is devoted to the periodic condition; for results on other conditions of the form (3) see, for example, [1], [3], [4], [9], [10]. (Actually, in those papers the boundary condition

$$z(b) = g(z(a))$$

is considered frequently; results on the latter condition can be easily transformed to the corresponding results on (3).)

Let u denote the *evolutionary operator* of (3), i.e. the collection of maps $\{u_{s,t}\}_{s,t\in\mathbb{R}}$ defined by the following rule: for $x_0 \in M$, $u_{t_0,t}(x_0)$ is equal to the value at t of the solution of the Cauchy problem (1), (2) provided that value exists, and x_0 does not belong to the domain of $u_{(t_0,t)}$ in the opposite case. It follows that the map $t \mapsto u_{(a,t)}(x)$ is a solution of the boundary value problem (1), (3) if and only if x is a fixed point of the map $g \circ u_{(a,b)}$.

In the present paper we use the *fixed point index* in order to get results on existence and properties of such fixed points. The definition and properties of the index can be found in [2]. We use slightly different notation then the one from that book: for a continuous map $\phi : D \to X$ on an ENR (Euclidean Neighbourhood Retract) X and D open in X let $\mathrm{Fix}(\phi)$ denote the set of fixed points of ϕ. A compact subset K of $\mathrm{Fix}(\phi)$ is called an *isolated set of fixed points* if there exists an open set U in D such that $\mathrm{Fix}(\phi) \cap U = K$. To such ϕ and K we associate the fixed point index, an integer number denoted here by $\mathrm{ind}(\phi, K)$ (it is equal to $I_{\phi|_U}$ in the notation from [2]). In particular, if $X = \mathbb{R}^n$ then $\mathrm{ind}(\phi, K)$ is equal to the Brouwer degree $\deg(0, \mathrm{id} - \phi, U)$.

The main result of this paper (Theorem 1) provides a formula for the fixed point index of $g \circ u_{(a,b)}$ in the set of the initial points of those solutions of (1), (3) which are contained in an isolating segment. It is a generalization of [11, Theorem 7.1], where the periodic problem was considered. In Section 3 we apply Theorem 1 to some natural boundary value problems associated to planar equations. In order to simplify presentation of the obtained results, we consider equations of the form $\dot{z} = \overline{z}^q + \ldots$ and $\dot{z} = e^{it}\overline{z}^q + \ldots$ only. It is not difficult to extend those results to a more general class of equations. In the case of periodic problem, such results were established in [5], [6], [8], [11], [12] and in the case of Floquet boundary problem, some of them were established in [9], [10].

2. The main theorem

In the sequel $\pi_1 : \mathbb{R} \times M \to \mathbb{R}$ and $\pi_2 : \mathbb{R} \times M \to M$ denote the projections. For a set $Z \subset \mathbb{R} \times M$ we put

$$Z_t := \{z \in X : (t, z) \in Z\}.$$

A compact ENR $W \in [a, b] \times M$ is called an *isolated segment* for the problem (1), (3) provided there are two compact ENRs W^- and W^+ contained in W such that

$$\partial W_a = W_a^- \cup W_a^+, \tag{4}$$

$$\begin{aligned} &W^- \cap ([a, b) \times M) \\ &\qquad = \{(t, x) \in W : t \in [a, b),\ \exists\{t_n\},\ t < t_n \to t : u_{(t,t_n)}(x) \notin W_{t_n}\}, \end{aligned} \tag{5}$$

$$\begin{aligned} &W^+ \cap ((a, b] \times M) \\ &\qquad = \{(t, x) \in W : t \in (a, b],\ \exists\{t_n\},\ t > t_n \to t : u_{(t_n,t)}(x) \notin W_{t_n}\}, \end{aligned} \tag{6}$$

there is a homeomorphism $h : [a, b] \times M \to [a, b] \times M$ satisfying

$$\pi_1 \circ h = \pi_1, \tag{7}$$

$$h([a, b] \times W_a) = W, \quad h([a, b] \times W_a^{\pm}) = W^{\pm}, \tag{8}$$

and g satisfies

$$(g(W_b^-) \cup \overline{g(W_b) \setminus W_a}) \cap W_a \subset W_a^-, \tag{9}$$

$$g(\operatorname{int} W_b) \cap W_a^+ \subset W_a^-. \tag{10}$$

We call W^- the *proper exit set.*

Proposition 1. *If* (4) *and* (9) *are satisfied, and g is homeomorphism then* (10) *also holds.*

Proof. Indeed, let x be a point in left-hand side of (10). Then $x \in \operatorname{int} g(W_b) \cap \partial W_a$ by (4). It follows there exists a sequence $\{y_n\}$ in $\operatorname{int} g(W_b) \setminus W_a$ such that $y_n \to x$. Thus $x \in \overline{g(W_b) \setminus W_a}$ and, by (9), $x \in W_a^-$. □

Define

$$F_{W,g} := \{x \in M : \forall t \in [a, b] : u_{(a,t)}(x) \in W_t, g \circ u_{(a,b)}(x) = x\}.$$

$F_{W,g}$ consists of the initial points of those solutions of the problem (1), (3) which graphs over $[a, b]$ are contained in W.

PROPOSITION 2. *$F_{W,g}$ is an isolated set of fixed points of $g \circ u_{(a,b)}$ (hence, in particular,* $\operatorname{ind}(g \circ u_{(a,b)}, F_{W,g})$ *is defined) and*

$$u_{(a,t)}(F_{W,g}) \subset \operatorname{int} W_t \tag{11}$$

for every $t \in [a,b]$.

PROOF. At first note that

$$\forall\, t \in [a,b] : \partial W_t = W_t^- \cup W_t^+. \tag{12}$$

Indeed, this is a consequence of (4) and properties of the homeomorphism h. Note also that (11) is always satisfied for $t \in (a,b)$ since (5), (6), and (12) are valid.

We prove that

$$u_{(a,b)}(F_{W,g}) \subset \operatorname{int} W_b. \tag{13}$$

Indeed, let $x \in F_{W,g}$. Then $u_{(a,b)}(x) \notin W_b^+$ since $x \in F_{W,g}$ and (6) holds. Moreover, $u_{(a,b)}(x) \notin W_b^-$, because in the opposite case

$$x = g(u_{(a,b)}(x)) \in g(W_b^-) \cap W_a \subset W_a^-$$

by (9), and $x \in W_a^-$ contradicts to (5). Thus (13) is proved.

It remains to prove

$$F_{W,g} \subset \operatorname{int} W_a, \tag{14}$$

which also implies that $F_{W,g}$ is an isolated set of fixed points. Assume on the contrary $x \in F_{W,g} \cap \partial W_a$. It follows by (4) and (5) that $x \in W_a^+$. Since (10) and (13) imply $x \in W_a^-$, a contradiction, (14) is proved. □

The homeomorphism h from the definition of segment induces

$$m : (W_a, W_a^-) \to (W_b, W_b^-), \quad x \mapsto \pi_2 h(b, \pi_2 h^{-1}(a,x)),$$

called a *monodromy homeomorphism*. In the quotient space it has the form

$$\begin{gathered} m^{\#} : (W_a/W_a^-, [W_a^-]) \to (W_b/W_b^-, [W_b^-]), \\ m^{\#}([x]) := [\pi_2 h(b, \pi_2 h^{-1}(a,x))]. \end{gathered}$$

It is easy to check that m (hence also $m^{\#}$) is uniquely determined (up to homotopy class) by the segment (see [11]). The map g induces

$$\begin{gathered} g^{\dagger} : (W_b/W_b^-, [W_b^-]) \to (W_a/W_a^-, [W_a^-]), \\ g^{\dagger}([y]) := \begin{cases} [g(y)] & \text{if } g(y) \in W_a \setminus W_a^-, \\ [W_a^-] & \text{elsewhere.} \end{cases} \end{gathered}$$

It follows easily by (9) that $g^\dagger$ is continuous (see [13, Lemma 1]). Thus we are able to define the Lefschetz number, an invariant of the segment W and the map g, as

$$\Lambda_{W,g} := \sum_{i=0}^{\infty} (-1)^i \operatorname{tr} H_i(g^\dagger \circ m^\#),$$

where homologies are taken over $\mathbb{Q}$.

Theorem 1. *Let W be an isolating segment for the problem* (1), (3). *Then*

$$\operatorname{ind}(g \circ u_{(a,b)}, F_{W,g}) = \Lambda_{W,g}.$$

Corollary 1. *If W is an isolating segment for the problem* (1), (3) *and*

$$\Lambda_{W,g} \neq 0$$

then $F_{W,g}$ is nonempty; in particular (1), (3) *has a solution.*

Proof of Theorem 1. Define a map

$$\sigma : W_a \ni x \to \sup\{t \in [a,b] : \forall s \in [a,t] : u_{(a,s)}(x) \in W_s\} \in [a,b].$$

It is continuous by the Ważewski Theorem. For each t the quotient space W_t/W_t^- is treated as a pointed space with the base point $[W_t^-]$ and we assume that 1 is the base point of the circle S^1. Let a map

$$\psi : W_a/W_a^- \vee S^1 \to W_b/W_b^- \vee S^1$$

be defined as

$$\psi([x], z) := \begin{cases} ([u_{(a,b)}(x)], 1) & \text{if } \sigma(x) = b,\ z = 1, \\ ([W_b^-], ze^{[1/(b-a)]\pi i (b - \sigma(x))}) & \text{if } \sigma(x) < b \text{ or } z \neq 1. \end{cases}$$

It follows

$$\operatorname{Fix}((g^\dagger \vee \mathrm{id}_{S^1}) \circ \psi) = \{([x], 1) \in W_a/W_a^- \vee S^1 : x \in F_{W,g}\}. \tag{15}$$

It follows by Proposition 2 that $F_{W,g}$ is contained in the interior of W_a. Observe that $g \circ u_{(a,b)}$ restricted to some neighbourhood of $F_{W,g}$ in $\operatorname{int} W_a$ is conjugated to $(g^\dagger \vee \mathrm{id}_{S^1}) \circ \psi$ restricted to the corresponding neighbourhood of $\{([x], 1) \in W_a/W_a^- \vee S^1 : x \in F_{W,g}\}$. By properties of the fixed point index, the Lefschetz Fixed Point Theorem, and (15) it suffices to prove that the Lefschetz number of $(g^\dagger \vee \mathrm{id}_{S^1}) \circ \psi$ is equal to $\Lambda_{W,g}$. To this aim we introduce a homotopy

$$\Psi : (W_a/W_a^- \vee S^1) \times [0,1] \to W_b/W_b^- \vee S^1$$

defined as

$$\Psi(([x], z), \lambda) := \begin{cases} ([m_\lambda u_{(a, a+\lambda(b-a))}(x)], 1) & \text{if } \sigma(x) \geq a + \lambda(b-a),\ z = 1, \\ ([W_b^-], ze^{[1/(b-a)]\pi i (a + \lambda(b-a) - \sigma(x))}) & \text{elsewhere,} \end{cases}$$

where
$$m_\lambda(y) := \pi_2 h(b, \pi_2 h^{-1}(a + \lambda(b-a), y)).$$
Then $\Psi(\,\cdot\,, 0) = (g^\dagger \circ m^\#) \vee \mathrm{id}_{S^1}$ and $\Psi(\,\cdot\,, 1) = (g^\dagger \vee \mathrm{id}_{S^1}) \circ \psi$, hence
$$\Lambda((g^\dagger \vee \mathrm{id}_{S^1}) \circ \psi) = \Lambda((g^\dagger \circ m^\#) \vee \mathrm{id}_{S^1}) = \Lambda_{W,g}$$
and Theorem 1 is proved. □

3. On some planar boundary value problems

In this section we present some results on boundary value problems of planar equations. We consider the problem (1), (3) in $M = \mathbb{C}$ with g beeing one of the following maps:
$$g_m : \mathbb{C} \ni z \to e^{\pi i m/(q+1)} z \in \mathbb{C},$$
$$\overline{g}_m : \mathbb{C} \ni z \to e^{\pi i m/(q+1)} \overline{z} \in \mathbb{C},$$
$$h_{\lambda,\mu} : \mathbb{C} \ni z \to \lambda \Re z + \mu \Im z \in \mathbb{C},$$
(where $m \in \mathbb{Z}$, $0 \le m \le 2q+1$, and $\lambda, \mu \in \mathbb{R}$), i.e. we associate the boundary value conditions
$$z(a) = e^{\pi i m/(q+1)} z(b), \tag{16}$$
$$z(a) = e^{\pi i m/(q+1)} \overline{z(b)}, \tag{17}$$
$$z(a) = \lambda \Re z(b) + \mu \Im z(b), \tag{18}$$
where $a < b$, to the equation
$$\dot{z} = f(t, z).$$
In [9], [10], (16) is called the Floquet condition. In particular, for $m = 0$ it coincides with the periodic condition
$$z(a) = z(b)$$
and for $m = q+1$ it is the anti-periodic condition
$$z(a) = -z(b).$$
In the case $|\lambda| = |\mu| = 1$ we will not write results on (18) separately; they can be deduced from results on (16) and (17).

In the sequel $p : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is a continuous map, smooth with respect to the second variable and q is a positive integer. We begin with the equation
$$\dot{z} = \overline{z}^q + p(t, z), \tag{19}$$
where $z \in \mathbb{C}$. We build isolating segments associated to the boundary value problems (19), (16), (19), (17), and (19), (18). To this purpose define the linear segment
$$J := \{\lambda e^{-\pi i/2(q+1)} + (1-\lambda) e^{\pi i/2(q+1)} \in \mathbb{C} : \lambda \in [0,1]\}$$

and define D as the regular $(2q+1)$-gon in the complex plane centered at 0 such that J is its side. It follows, in particular, that

$$\partial D = \bigcup_{r=0}^{2q+1} e^{\pi i r/(q+1)} J.$$

Denote by u the evolutionary operator of (19).

Proposition 3. *If*

$$\lim_{|z|\to\infty} \frac{p(t,z)}{z^q} = 0 \quad (\textit{uniformly in } t \in [a,b]) \tag{20}$$

then there exists an $\varepsilon_\infty > 0$ *such that if* $\varepsilon \geq \varepsilon_\infty$ *then the set*

$$U := [a,b] \times \varepsilon D \tag{21}$$

is an isolating segment both for (19), (16) *and for* (19), (17) *provided* m *is even. It is also an isolating segment for* (19), (18) *if*

$$q = 1 \bmod 4, \quad |\lambda| \geq \frac{1}{\tan \pi/2(q+1)}, \quad |\mu| \leq \tan\frac{\pi}{2(q+1)}. \tag{22}$$

Moreover, the proper exit set of U *is given by*

$$U^- = \bigcup_{k=0}^{q} [a,b] \times \varepsilon e^{2\pi i k/(q+1)} J, \tag{23}$$

and

$$\begin{aligned} F_{U,g_m} &= \mathrm{Fix}(g_m \circ u_{(a,b)}), \\ F_{U,\overline{g}_m} &= \mathrm{Fix}(\overline{g}_m \circ u_{(a,b)}), \\ F_{U,h_{\lambda,\mu}} &= \mathrm{Fix}(h_{\lambda,\mu} \circ u_{(a,b)}). \end{aligned}$$

The interior of εD is thus an *autonomous bound set* for the problems (19), (16) and (19), (17); see [10, Definition 2.1].

Proposition 4. *If*

$$\lim_{|z|\to 0} \frac{p(t,z)}{z^q} = 0 \quad (\textit{uniformly in } t \in [a,b]) \tag{24}$$

then there exists an $\varepsilon_0 > 0$ *such that if* $0 < \varepsilon \leq \varepsilon_0$ *the set* U *given by* (21) *is an isolating segment both for* (19), (16) *and for* (19), (17) *provided* m *is even, and, if* (22) *is satisfied, also for* (19), (18). *Its proper exit set is given by* (23). *Moreover,*

$$F_{U,g_m} = F_{U,\overline{g}_m} = F_{U,h_{\lambda,\mu}} = \{0\}.$$

THEOREM 2. *If the hypotheses of Proposition* 3 *or Proposition* 4 *are satisfied then*

$$\operatorname{ind}(g_m \circ u_{(a,b)}, F_{U,g_m}) = \begin{cases} -q & \text{if } m = 0, \\ 1 & \text{if } m \neq 0, \end{cases}$$

$$\operatorname{ind}(\overline{g}_m \circ u_{(a,b)}, F_{U,\overline{g}_m}) = \begin{cases} 0 & \text{if } q \text{ even}, \\ -1 & \text{if } q \text{ odd, } m/2 \text{ even}, \\ 1 & \text{if } q \text{ odd, } m/2 \text{ odd}, \end{cases}$$

$$\operatorname{ind}(h_{\lambda,\mu} \circ u_{(a,b)}, F_{U,h_{\lambda,\mu}}) = -\operatorname{sgn} \lambda.$$

COROLLARY 2. *Assume that* (20) *is satisfied. Then*

(a) *the problem* (19), (16) *has a solution provided m is even,*
(b) *the problem* (19), (17) *has a solution provided q is odd and m is even,*
(c) *the problem* (19), (18) *has a solution provided* (22) *is satisfied.*

Now we consider the equation

$$\dot{z} = e^{it}\overline{z}^q + p(t,z). \tag{25}$$

We associate to (25) the two-point boundary value conditions (16), (17), and (18), but we restrict the values of a and b to $a = 0$ and $b = n\pi$, $n = 1, 2, \ldots$, i.e. we consider the conditions

$$z(0) = e^{\pi i m/(q+1)} z(n\pi), \tag{26}$$

$$z(0) = e^{\pi i m/(q+1)} \overline{z(n\pi)}, \tag{27}$$

$$z(0) = \lambda \Re z(n\pi) + \mu \Im z(n\pi). \tag{28}$$

Let v denote the evolutionary operator of (25).

PROPSITION 5. *Assume that $q \geq 2$ and*

$$\lim_{|z|\to\infty} \frac{p(t,z)}{z^q} = 0 \quad (\textit{uniformly in } t \in [0, n\pi]). \tag{29}$$

Then there exists an $\varepsilon_\infty > 0$ such that if $\varepsilon \geq \varepsilon_\infty$ then the set

$$V := \{(t,z) \in [0, n\pi] \times \mathbb{C} : z \in \varepsilon e^{it/(q+1)} D\} \tag{30}$$

is an isolating segment both for (25), (26) *and for* (25), (27) *provided $m + n$ is even. It is also an isolating segment for* (25), (28) *if one of the following* *conditions is satisfied:*

(31) $\quad n$ is even, $q = 1 \bmod 4$, $|\lambda| \geq \dfrac{1}{\tan \pi/2(q+1)}$, $|\mu| \leq \tan \dfrac{\pi}{2(q+1)}$,

(32) $\quad n$ is odd, $q = 3 \bmod 4$, $|\lambda| \geq \dfrac{1}{\tan \pi/2(q+1)}$, $|\mu| \leq \tan \dfrac{\pi}{2(q+1)}$,

(33) $\quad n$ is odd, $q = 3 \bmod 4$, $|\lambda| \leq \tan \dfrac{\pi}{2(q+1)}$, $|\mu| \geq \dfrac{1}{\tan \pi/2(q+1)}$.

Moreover, the proper exit set of V *is given by*

$$V^- = \bigcup_{k=0}^{q} \{(t,z) \in [0,n\pi] \times \mathbb{C} : z \in \varepsilon e^{i(2k\pi+t)/(q+1)} J\}, \tag{34}$$

and

$$\begin{aligned} F_{V,g_m} &= \mathrm{Fix}(g_m \circ v_{(0,n\pi)}), \\ F_{V,\overline{g}_m} &= \mathrm{Fix}(\overline{g}_m \circ v_{(0,n\pi)}), \\ F_{V,h_{\lambda,\mu}} &= \mathrm{Fix}(h_{\lambda,\mu} \circ v_{(0,n\pi)}). \end{aligned}$$

Consider now the equation

$$\dot{z} = \frac{1}{q+1} iz + e^{it}\overline{z}^q + p(t,z). \tag{35}$$

We denote the evolutionary operator of (35) also by v.

Proposition 6. *If*

$$\lim_{|z|\to 0} \frac{p(t,z)}{z^q} = 0 \quad (\textit{uniformly in } t \in [0,n\pi]) \tag{36}$$

then there exists an $\varepsilon_0 > 0$ *such that if* $0 < \varepsilon \leq \varepsilon_0$ *then the set* V *given by* (30) *is an isolating segment both for* (35), (26) *and for* (35), (27) *provided* $m+n$ *is even, and also for* (35), (28) *provided one among the conditions* (31), (32), *and* (33) *holds. The set* V^- *is given by* (34). *Moreover,*

$$F_{V,g_m} = F_{V,\overline{g}_m} = F_{V,h_{\lambda,\mu}} = \{0\}.$$

Theorem 3. *If the hypotheses of Proposition* 5 *or Proposition* 6 *are satisfied then*

$$\mathrm{ind}(g_m \circ v_{(0,n\pi)}, F_{V,g_m}) = \begin{cases} -q & \textit{if } m+n = 0 \bmod 2(q+1), \\ 1 & \textit{if } m+n \neq 0 \bmod 2(q+1), \end{cases}$$

$$\mathrm{ind}(\overline{g}_m \circ v_{(0,n\pi)}, F_{V,g_m}) = \begin{cases} 0 & \textit{if } q \textit{ even}, \\ -1 & \textit{if } q \textit{ odd}, (m-n)/2 \textit{ even}, \\ 1 & \textit{if } q \textit{ odd}, (m-n)/2 \textit{ odd}. \end{cases}$$

Moreover, if (31) *holds,* $q = 4r + 1$ *and* $k = n/2 \bmod q + 1$ *then*

$$\operatorname{ind}(h_{\lambda,\mu} \circ v_{(0,n\pi)}) = \begin{cases} -\operatorname{sgn}\lambda & \text{if } 0 \le k \le r \text{ or } 3r+2 \le k \le 4r+1, \\ \operatorname{sgn}\lambda & \text{if } r+1 \le k \le 3r+1, \end{cases}$$

and if (32) *or* (33) *holds,* $q = 4r + 3$, *and* $k = (n-1)/2 \bmod q + 1$ *then*

$$\operatorname{ind}(h_{\lambda,\mu} \circ v_{(0,n\pi)}) = \begin{cases} -\operatorname{sgn}\eta & \text{if } 0 \le k \le r \text{ or } 3r+3 \le k \le 4r+3, \\ \operatorname{sgn}\eta & \text{if } r+1 \le k \le 3r+2, \end{cases}$$

where η *stands for* λ *if* (32) *is satisfied, and* η *stands for* μ *if* (33) *is satisfied.*

Corollary 3. *Assume that* (29) *is satisfied and* $q \ge 2$. *Then*

(a) *the problem* (25), (26) *has a solution provided* $m + n$ *is even,*
(b) *the problem* (25), (27) *has a solution provided* $m + n$ *is even and* q *is odd,*
(c) *the problem* (25), (28) *has a solution provided one among the conditions* (31), (32), *and* (33) *holds.*

Remark 1. Both [9, Theorem 2] and [10, Theorem 5.1] extend Corollaries 2(a) and 3(a) (m even) to a more general class of equations. Proofs of those theorems are based on a suitably modified Mawhin Theorem (see [7, Theorem IV.13]; it provides also the absolute value of the coincidence degree, hence also the absolute value of the fixed point index associated to the considered problem). In [9], calculations of a priori bounds are based on Hölder inequalities, while in [10] the bound set $\operatorname{int}(\varepsilon D)$ is used.

Theorems 2 and 3, and Propositions 3, 4, 5 and 6 can be applied to results on existence of nonzero solutions of boundary value problems. For example, they imply that the problems

$$\begin{cases} \dot z &= \overline{z}^5 + e^{it}\overline{z}^2 + iz/3, \\ z(0) &= e^{5\pi i/3} z(\pi), \end{cases} \qquad \begin{cases} \dot z &= e^{it}\overline{z}^5 + \overline{z}^2, \\ z(0) &= e^{2\pi i/3}\overline{z(2\pi)}, \end{cases}$$

$$\begin{cases} \dot z &= e^{it}\overline{z}^7 + \overline{z}^5, \\ z(0) &= 6\Re z(5\pi), \end{cases} \qquad \begin{cases} \dot z &= e^{it}\overline{z}^7 + e^{it}\overline{z}^3 + iz/4, \\ z(0) &= 6\Im z(3\pi), \end{cases}$$

have nonzero solutions. Indeed, in each case the fixed point indices related to a segment associated to the lower order terms and to a segment associated to the upper order term are different. Actually, theorems generalizing the above examples can be easily formulated and proved basing on the same argument. In the case of the periodic problem such theorems are given in [12].

Remark 2. If the condition (36) is satisfied, Theorem 3 delivers essential information on the problem (35), (26) only if $m + n = 0 \bmod 2(q + 1)$ — Proposition 7 given below can be applied to the other cases. On the other hand, Proposition 7 does not apply to the problem (35), (27) because 1 is an eigenvalue of the map $z \mapsto e^{\pi i(m-n)/(q+1)}\overline{z}$.

Proposition 7. *Let A and B be real $n \times n$ matrices. Let $p : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map, smooth with respect to the second variable. Assume that*

$$\lim_{x\to 0} \frac{p(t,x)}{|x|} = 0 \quad (\textit{uniformly in } t \in [a,b]).$$

Let v denote the evolutionary operator of

$$\dot{x} = Ax + p(t,x).$$

If 1 is not an eigenvalue of $Be^{A(b-a)}$ then $\{0\}$ is an isolated set of fixed points of $B \circ v_{(a,b)}$ and

$$\operatorname{ind}(B \circ v_{(a,b)}, \{0\}) = \operatorname{sgn} \det I - Be^{A(b-a)}.$$

If $Be^{A(b-a)}$ does not have eigenvalues in the real half-line $[1,\infty)$ then $\operatorname{ind}(B \circ v_{(a,b)}, \{0\}) = 1$.

Proof. The differential of $v_{(a,t)}$ at 0 is represented as the solution of the Cauchy problem

$$\dot{X} = AX, \quad X(a) = I$$

hence the differential of $\mathrm{id} - B \circ v_{(a,b)}$ at 0 is nondegenerate. It follows that $\{0\}$ is isolated and the result is a consequence of properties of the fixed point index. $\square$

4. Proofs of the main results of Section 3

Proof of Propositions 3 and 4. Let $\varepsilon > 0$. The triple

$$(U, U^-, U^+) := \Big([a,b] \times \varepsilon D, \bigcup_{k=0}^{q} [a,b] \times \varepsilon e^{2\pi i k/(q+1)} J, \bigcup_{k=0}^{q} [a,b] \times \varepsilon e^{\pi i(2k+1)/(q+1)} J\Big)$$

satisfies (4) and the identity satisfies (7) and (8). If m is even then the homeomorphisms g_m and $\overline{g}_m$ transform (U_a, U_a^-, U_a^+) into itself. It follows that (9) and (10) are satisfied for $g = g_m$ and $g = \overline{g}_m$.

Assume the condition (22). Since $q = 1 \bmod 4$, the sides εJ and $-\varepsilon J$ are contained in U_a^-, and the sides $\varepsilon i J$ and $-\varepsilon i J$ are contained in U_a^+. $h_{\lambda,\mu}(U_a)$ stretches U_a in the horizontal direction and squeezes in the vertical direction such that

$$h_{\lambda,\mu}(U_a^-) \cap \operatorname{int} U_a = \emptyset, \quad h_{\lambda,\mu}(U_a) \cap \partial U_a \subset \varepsilon J \cup -\varepsilon J,$$

hence (9) and (10) are satisfied for $g = h_{\lambda,\mu}$. (Actually, by Proposition 1, the condition (10) should be verified only for $h_{\lambda,0}$.)

Moreover, direct calculations show that (5) and (6) are satisfied for $U^{\pm}$ (for an arbitrary $\varepsilon > 0$) and the evolutionary operator associated to the equation

$$\dot{z} = \overline{z}^q.$$

If we add to the right-hand side of that equation a perturbation term p satisfying (20) then there exists an $\varepsilon_\infty > 0$ such that (6) and (5) are still satisfied provided $\varepsilon \geq \varepsilon_\infty$ (see [10]–[12]). Analogously, they are satisfied if the perturbation term satisfies (24) and $0 < \varepsilon \leq \varepsilon_0$ for some ε_0. Thus U is an isolating segment for the considered problems in the required range of values of ε.

Let g be one of the maps g_m, $\overline{g}_m$, and $h_{\lambda,\mu}$. We prove that

$$\mathrm{Fix}(g \circ u_{(a,b)}) \subset \{z \in \mathbb{C} : u_{(a,t)}(z) \in \varepsilon_\infty D \ \forall t \in [a,b]\}.$$

Indeed, if $z_0 = g(u_{(a,b)})(z_0)$ and $u_{(a,t)}(z_0) \notin \varepsilon_\infty D$ for some $t \in [a,b]$ then there exist an $\varepsilon^* > \varepsilon$ and a $t_0 \in [a,b]$ such that $u_{(a,t)}(z_0) \in \varepsilon^* D$ for all $t \in [a,b]$ and $u_{(a,t_0)}(z_0) \in \partial(\varepsilon^* D)$, which contradicts to (11) in Proposition 2. It follows $F_{U,g} = \mathrm{Fix}(g \circ u_{(a,b)})$ provided $\varepsilon \geq \varepsilon_\infty$, hence Proposition 3 is proved.

By Proposition 2, in order to finish the proof of Proposition 4 it suffices to observe that

$$\{z \in \mathbb{C} : z = g(u_{(a,b)})(z),\ u_{(a,t)}(z) \in \mathrm{int}(\varepsilon_0 D) \ \forall t \in [a,b]\} = \{0\}.$$

If there is a $z_0 \neq 0$ in the left-hand side of the above equation, then there is an $0 < \varepsilon_* < \varepsilon_0$ and $t_0 \in [a,b]$ such that $u_{(a,t)}(z_0) \in \varepsilon_* D$ for all $t \in [a,b]$ and $u_{(a,t_0)}(z_0) \in \partial(\varepsilon_* D)$, which contradicts to Proposition 2. □

Proof of Propositions 5 and 6. We apply the same argument as in the previous proof. Here we have

$$\begin{aligned}
V &:= \bigcup_{t \in [0,n\pi]} \{t\} \times \varepsilon e^{it/(q+1)} D,\\
V^- &:= \bigcup_{t \in [0,n\pi]} \{t\} \times \bigcup_{k=0}^{q} \varepsilon e^{i(2k\pi+t)/(q+1)} J,\\
V^+ &:= \bigcup_{t \in [0,n\pi]} \{t\} \times \bigcup_{k=0}^{q} \varepsilon e^{i((2k+1)\pi+t)/(q+1)} J,
\end{aligned}$$

The homeomorphism $(t,z) \mapsto (t, e^{it/(q+1)}z)$ satisfies (7) and (8). The properties (4), (9), and (10) easily follow for $g = g_m$ and $g = \overline{g}_m$ provided $m+n$ is even. If (31) is satisfied than the same holds for $g = h_{\lambda,\mu}$ by the argument from the previous proof.

If $q = 3 \bmod 4$ and n is odd then all the sets εJ, $-\varepsilon J$, $\varepsilon i J$, and $-\varepsilon i J$ are contained in V_0^- and in $V_{n\pi}^+$. It follows that both (32) and (33) guarantee (9) and (10) for $g = h_{\lambda,\mu}$.

It can be shown that (5) and (6) hold for $V^{\pm}$ (for an arbitrary $\varepsilon > 0$) and the evolutionary operator associated to

$$\dot{z} = e^{it}\overline{z}^{q} + \frac{1}{q+1} iz,$$

hence perturbations of the above equation leads to the existence of the required ε_∞ and ε_0. The remaining parts of Propositions 5 and 6 can be proved by the argument in the proof of the analogous parts of Propositions 3 and 4. $\square$

Proof of Theorem 2. We will apply Theorem 1 to the segment U with the proper exit set U^- given by (21) and (23). In particular,

$$(U_t, U_t^-) = \left(\varepsilon D, \bigcup_{k=0}^{q} \varepsilon e^{2\pi ik/(q+1)} J\right)$$

for each $t \in [a, b]$. We choose the identity as a monodromy homeomorphism of U. For $k = 0, \dots, q-1$ let α_k be a 1-dimensional singular simplex in U_a connecting $\varepsilon e^{2\pi ik/(q+1)} J$ to $\varepsilon e^{2\pi i(k+1)/(q+1)} J$ (see Figure 1).

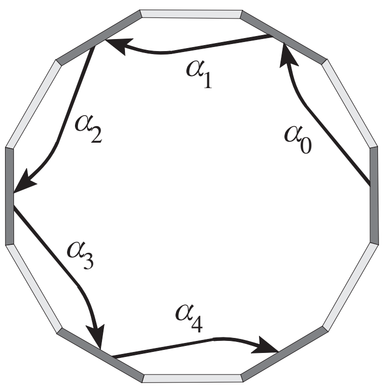

Figure 1. Singular simplexes $\alpha_0, \dots, \alpha_4$ $(q = 5)$

Let $[\alpha_k]$ denote the homology class in $H_1(U_a/U_a^-, [U_a^-])$ generated by α_k. Then

$$[\alpha] := ([\alpha_0], \dots, [\alpha_{q-1}])$$

is a basis of $H_1(U_a/U_a^-, [U_a^-])$. Let A^k denote the matrix of the homomorphism

$$H_1(g_{2k}^{\dagger}) : H_1(U_a/U_a^-, [U_a^-]) \to H_1(U_a/U_a^-, [U_a^-])$$

in that basis. Then $A^0 = I$ and, for $0 < k \le q$,

$$A^k := (a^k_{i,j})_{i,j=0,\dots,q-1} = \begin{bmatrix} 0 & \dots & 0 & -1 & 1 & & \\ \vdots & & \vdots & \vdots & & \ddots & 1 \\ 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ 1 & & & & & & \\ & \ddots & & \vdots & & & \\ & & 1 & -1 & & & \end{bmatrix},$$

where $a^k_{i,q-k} = -1$ for $i = 0, \dots, q-1$, $a^k_{i,i-k} = 1$ for $i = k, \dots, q-1$, $a^k_{i,q-k+1+i} = 1$ for $i = 0, \dots, k-2$, and $a^k_{i,j} = 0$ elsewhere. We have

$$\Lambda_{U,g_{2k}} = -\operatorname{tr} H_1(g^\dagger_{2k})$$

and it is clear that $\operatorname{tr} A^k$ is equal to q if $k = 0$ and it is equal to -1 if $0 < k \le q$, hence the first assertion of Theorem 2 follows.

Let B denote the matrix (in the basis $[\alpha]$) of the endomorphism of $H_1(U_a/U_a^-, [U_a^-])$ generated by the conjugacy $z \mapsto \overline{z}$:

$$B := (b_{i,j})_{i,j=0,\dots,q-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & -1 & \dots & 0 \\ 1 & -1 & 0 & \dots & 0 \end{bmatrix},$$

where $b_{i,0} = 1$ for $i = 0, \dots, q-1$, $b_{i,q-i} = -1$ for $i = 1, \dots, q-1$, and $b_{i,j} = 0$ elsewhere. We conclude by elementary calculations that $\operatorname{tr} A^k B$ is equal to 0 if q is even, it is equal to 1 if q is odd and k is even, and it is equal to -1 if both q and k are odd. Since $A^k B$ is the matrix of $H_1(\overline{g}^\dagger_{2k})$, the second assertion follows.

Assume (22) and assume that $\lambda > 0$. Let $q = 4r + 1$. The matrix C of $H_1(h^\dagger_{\lambda,\mu})$ in $[\alpha]$ has the form

$$C := (c_{i,j})_{i,j=0,\dots,q-1} = \begin{bmatrix} \dots & 1 & \dots & -1 & \dots \\ & \vdots & & \vdots & \\ \dots & 1 & \dots & -1 & \dots \\ \dots & 0 & \dots & 0 & \dots \\ & \vdots & & \vdots & \end{bmatrix},$$

where $c_{i,r} = 1$ and $c_{i,3r+1} = -1$ for $i = 0, \dots, 2r$ and $c_{i,j} = 0$ elsewhere. Since $\operatorname{tr} C$ is equal to 1, the third assertion follows in the considered case. The matrix $-C$ is used in a proof of the case $\lambda < 0$. $\square$

Proof of Theorem 3. We apply Theorem 1 to V and V^- defined by (30) and (34), respectively. For $j = 0, 1, \dots,$

$$(V_{j\pi}, V_{j\pi}^-) = \begin{cases} \left(\varepsilon D, \bigcup_{k=0}^{q} \varepsilon e^{2\pi ik/(q+1)} J\right) & \text{if } j \text{ is even,} \\ \left(\varepsilon e^{\pi i/(q+1)} D, \bigcup_{k=0}^{q} \varepsilon e^{\pi i(2k+1)/(q+1)} J\right) & \text{if } j \text{ is odd.} \end{cases}$$

We choose g_n as a monodromy homeomorphism of V. It follows that

$$\begin{aligned} \Lambda_{V,g_m} &= -\operatorname{tr} H_1(g_{m+n}^{\dagger}), \\ \Lambda_{V,\overline{g}_m} &= -\operatorname{tr} H_1(\overline{g}_{m-n}^{\dagger}), \end{aligned}$$

hence the first and the second assertion follow from the proof of analogous assertions of Theorem 2.

Now we calculate the fixed point indices related to the boundary condition (28). We have

$$\Lambda_{V,h_{\lambda,\mu}} = -\operatorname{tr}(H_1(w_{\lambda,\mu}^{\dagger}) \circ H_1(g_n^{\#})). \tag{37}$$

We use notation from the previous proof. Whenever the choice of sign is essential, we assume that λ (μ, respectively) is positive — the opposite case follows by the same argument. Let $0 \le k \le q$. At first we assume that (31) holds, $q = 4r + 1$ and $k = n/2 \bmod q + 1$. Then both $H_1(w_{\lambda,\mu}^{\dagger})$ and $H_1(g_n^{\#})$ are endomorphisms of $H_1(V_0/V_0^-, [V_0^-])$, and in the basis $[\alpha]$ they have matrix representations C and A^k, respectively. A calculation of $\operatorname{tr} CA^k$ and (37) provides the result.

In the sequel we assume that $k = n - 1/2 \bmod q + 1$. For $j = 0, \dots, q-1$ put $\beta_j := e^{\pi i/(q+1)}\alpha_j$; the induced homology classes form the basis

$$[\beta] = ([\beta_0], \dots, [\beta_{q-1}])$$

of $H_1(V_\pi/V_\pi^-, [V_\pi^-])$. It follows that in the bases $[\alpha]$ and $[\beta]$ the homomorphism

$$H_1(g_n^{\#}) : H_1(V_0/V_0^-, [V_0^-]) \to H_1(V_\pi/V_\pi^-, [V_\pi^-])$$

is represented by the matrix A^k.

Let $q = 4r + 3$. Assume that (32) holds. In the bases $[\beta]$ and $[\alpha]$ the homomorphism

$$H_1(h_{\lambda,\mu}^{\dagger}) : H_1(V_\pi/V_\pi^-, [V_\pi^-]) \to H_1(V_0/V_0^-, [V_0^-])$$

is represented by the matrix

$$D := (d_{i,j})_{i,j=0,\dots,q-1} = \begin{bmatrix} \dots & 1 & \dots & -1 & \dots \\ & \vdots & & \vdots & \\ \dots & 1 & \dots & -1 & \dots \\ \dots & 0 & \dots & 0 & \dots \\ & \vdots & & \vdots & \end{bmatrix},$$

where $d_{i,r} = 1$ and $d_{i,3r+2} = -1$ for $i = 0, \dots, 2r+1$ and $d_{i,j} = 0$ elsewhere. A calculation of $\operatorname{tr} DA^k$ and (37) provide the required result.

Finally, assume that (33) holds. In this case the homomorphism $H_1(h^{\dagger}_{\lambda,\mu})$ is represented in the bases $[\beta]$ and $[\alpha]$ by the matrix

$$E := (e_{i,j})_{i,j=0,\dots,q-1} = \begin{bmatrix} & \vdots & \\ \dots & 0 & \dots \\ \dots & 1 & \dots \\ & \vdots & \\ \dots & 1 & \dots \\ \dots & 0 & \dots \\ & \vdots & \end{bmatrix},$$

where $e_{i,2r+1} = 1$ for $i = r+1, \dots, 3r+2$ and $e_{i,j} = 0$ elsewhere. By (37) and a calculation of $\operatorname{tr} EA^k$ the last assertion follows. $\square$

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