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EXISTENCE OF ENTIRE SOLUTIONS FOR SEMILINEAR ELLIPTIC PROBLEMS ON $\mathbb{R}^N$

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Abstract. In this paper, we consider the existence of positive and negative entire solutions of semilinear elliptic problem

(P) $$-\Delta u + u = g(x,u), \quad u \in H^1(\mathbb{R}^N)$$

where $N \geq 2$ and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function with superlinear growth and $g(x,0) = 0$ on $\mathbb{R}^N$.

1. Introduction

Our purpose in this paper is to show the existence of positive and negative solutions of the problem

(P) $$\begin{cases} -\Delta u + u = g(x,u) & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $N \geq 2$ and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function with superlinear growth and $g(x,0) = 0$ on $\mathbb{R}^N$. Throughout this paper, we fix a positive number p such that $p > 1$ when $N = 2$ and $1 < p < (N+2)/(N-2)$ when $N \geq 3$. Let

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$g : \mathbb{R} \to \mathbb{R}$ be a continuous odd function satisfying the following condition:

$$(\mathrm{g0}) \quad \begin{cases} (1) \text{ there exists } C > 0 \text{ such that} \\ \qquad 0 \le |g(t)| \le C|t|^p \quad \text{for all } t \in \mathbb{R}; \\ (2) \text{ there exists a number } 0 < \theta < 1/2 \text{ such that} \\ \qquad \theta g(t)t \ge G(t) = \int_0^t g(t)\,dt \quad \text{for all } t \ge 0, \\ (3) \ \ g(t)/t \text{ is increasing on } [0, \infty) \text{ with} \lim_{t\to\infty} g(t)/t = \infty. \end{cases}$$

It is well known that problem

$$(\mathrm{P}_0) \qquad -\Delta u + u = g(u), \quad u \in H^1(\mathbb{R}^N)$$

has a positive solution (cf. [1]). In case that $g(t) = |t|^{p-1}t$, it is also known that the positive solution of problem (P_0) is unique up to translation (cf. Kwong [6]). The uniqueness of the positive solution of problem (P_0) for more general function g has been studied by several authors (cf. [9], [12]). The positive solution u of problem (P_0) is characterized as the ground state solution. That is if we consider a functional I defined by

$$I(v) = \int_{\mathbb{R}_N} \frac{1}{2}(|\nabla v|^2 + |v|^2)\,dx - \int_{R_N}\int_0^{v(x)} g(t)\,dt\,dx \quad \text{for } v \in H^1(\mathbb{R}^N),$$

then $c = I(u)$ is the minimal positive critical level of I. On the other hand, the existence of positive entire solution of problem (P) in the case that $g(x,t) = Q(x)|t|^{p-1}t$ has been studied by several authors (cf. [1]–[4]), where $Q : \mathbb{R}^N \to \mathbb{R}$ is a continuous function. That is the existence of positive solutions of problem

$$(\mathrm{P}_Q) \qquad \begin{cases} -\Delta u + u = Q(x)|u|^{p-1}u \quad \text{for } x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

was considered under the assumption that $Q(x)$ satisfies $Q(x) \to \overline{Q}$ as $|x| \to \infty$. In case that $Q(x) \ge \overline{Q}$ in $\mathbb{R}^N$, the existence of a solution of (P_Q) was established by Lions [8] using the concentrate compactness method. Lions's result was improved by Zhu [13] and Cao [2]. The case that $Q(x)|t|^{p-1}t$ is replaced by a more general function $g(x,t)$, the existence of positive solutions was proved in [5].

The method employed so far for problem (P_Q) is, as well as for problem (P_0), to find the ground state solution. Then one has to impose conditions for the existence of the ground state solution, such as $Q(x) \ge \overline{Q}$ on $\mathbb{R}^N$. Our method employed in this paper is based on the calculation of homology groups for the level sets of functionals and we need not conditions for the existence of the ground state solution.

We impose the following conditions on $g \in C(\mathbb{R};\mathbb{R})$ and $g \in C(\mathbb{R}^N \times \mathbb{R};\mathbb{R})$:

(g1) the positive solution of problem (P_0) is unique up to translation,

(g2) for each $x \in \mathbb{R}^N$, $g(x,t)/t$ is increasing on $[0,\infty)$,
(g3) $\lim_{|x|\to\infty} g(x,t)/g(t) = 1$ uniformly on closed bounded subsets of $(0,\infty)$,
(g4) there exists $\rho > 0$ such that

$$|g(x,t) - g(t)| \le \rho |g(t)| \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \in R.$$

We can now state our main result.

Theorem 1.1. *Assume that* (g0)–(g3) *hold. Then there exists a positive number ρ_0 such that if* (g4) *holds with $0 < \rho < \rho_0$, problem* (P) *possesses at least one positive and one negative solution.*

Remark 1.2. In the case that $g(t) = |t|^{p-1}t$, we proved the existence of positive solution of (P) by using the singular homology groups for level sets of functionals associate with problem (P) in [5]. The argument in [5] deeply depends on the shape of the function $|t|^{p-1}t$, and also needs assumtions on the derivatives $g_t(x,t)$, $g_{tt}(x,t)$.

In case that $g(x,t)$ is given by the form $g(x,t) = Q(x)g(t)$, conditions (g3) and (g4) are rewritten as

$$\lim_{|x|\to\infty} Q(x) = 1, \tag{g3'}$$

$$|Q(x) - 1| \le \rho \quad \text{for all } x \in \mathbb{R}^N. \tag{g4'}$$

That is we have

Corollary 1.3. *Assume that* (g0), (g1) *and* (g3') *hold. Then there exists a positive number ρ_0 such that if* (g4') *holds with $0 < \rho < \rho_0$, problem*

$$\begin{cases} -\Delta u + u = Q(x)g(u) & \textit{for } x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$\overline{\mathrm{P}}_Q$}$$

possesses at least one positive and one negative solution.

2. Preliminaries

We denote by D^n and S^{n-1} the unit disk and unit sphere of n-dimensional Euclidian space. For simplicity, we put $H = H^1(\mathbb{R}^N)$. By $|\cdot|_q$ we denote the norm of $L^q(\mathbb{R}^N)$, $(q > 1)$. $|\cdot|$ stands for the norm of $H^1(\mathbb{R}^N)$ defined by $|z|^2 = |\nabla z|_2^2 + |z|_2^2$ for $z \in H$. $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the inner products in $L^2(\mathbb{R}^N)$. For each function $v : \mathbb{R}^N \to R$, we put $v^+(x) = \max\{v(x), 0\}$ and $v^-(x) = \min\{v(x), 0\}$ for $x \in \mathbb{R}^N$. We put $A = -\Delta + I$. For each $a \in \mathbb{R}$ and each functional $F : H \to \mathbb{R}$, we denote by F_a the set $F_a = \{v \in H : F(v) \le a\}$. We call a real number d a critical value of a functional F if there exists a sequence $\{v_n\} \subset H$ such that $\lim_{n\to\infty} F(v_n) = d$ and $\lim_{n\to\infty} |\nabla F(v_n)| = 0$. For a pair of topological space (X,Y) with $Y \subset X$, we denote by $H_*(X,Y)$ the relative singular homology groups (cf. Spanier [10]). For $z \in H$, $D \subset H$ and $x \in \mathbb{R}^N$, we denote by z_x and D_x,

$$z_x(y) = z(y-x) \quad \text{for } y \in \mathbb{R}^N \text{ and } \quad D_x = \{z_x : z \in D\}.$$

Let $u \in H$ be the unique positive solution of problem (P_0). Then $c = I(u)$ is the minimal positive critical value of I. From the invariance of functional I under translation, we have that for each $x \in \mathbb{R}^N$, the function u_x is a solution of I with $I(u_x) = c$. It is also known that there exist no critical value of I in $(0, 2c) \setminus \{c\}$. Then as a direct consequence of the concentrate compactness lemma (cf. [7], [8], [13]), we have that

$$(*) \qquad \begin{cases} \{v_n\} \subset H,\ \lim_{n\to\infty} \|\nabla I(v_n)\| = 0 \text{ and } \lim_{n\to\infty} I(v_n) = c \\ \text{implies that there exist} \{x_n\} \subset \mathbb{R}^N \text{ and } \{i_n\} \subset \{0,1\} \\ \text{such that } \lim_{n\to\infty} \| v_n - (-1)^{i_n} u_{x_n} \| = 0. \end{cases}$$

We define a functional J^∞ on $H^1(\mathbb{R}^N)$ by

$$J^\infty(v) = \int_{R_N} \frac{1}{2}(|\nabla v|^2 + |v|^2)\, dx - \int_{R_N} G(x, v(x))\, dx,$$

for $v \in H^1(\mathbb{R}^N)$. We put

$$M = \left\{ v \in H \setminus \{0\} : |v|^2 = \int_{\mathbb{R}^N} g(v(x))v(x)\, dx \right\}.$$

Noting that

$$c = I(u) = \min \left\{ I(v) : |v|^2 = \int_{\mathbb{R}^N} g(u(x))u(x)\, dx \right\}, \tag{2.1}$$

we have that

$$I(v) \geq c \quad \text{on } M. \tag{2.2}$$

It is also easy to see from (3) of (g0) that

$$M \cap \{\lambda v : v \in H \setminus \{0\}, \lambda \geq 0\} \text{ is a unique point,} \tag{2.3}$$

$$I(v) = \max\{I(\lambda v) : \lambda \geq 0\} \quad \text{for each } v \in M \tag{2.4}$$

and each critical point of I is contained in M (cf. [13]).

We will work on a neighbourhood V_1 of the set $M \cap I_{3c/2}$ and try to find solution of (P) in V_1. For this purpose, we transform the functional J^∞ outside of V_1.

The following results is well known.

Lemma 2.1. *For each $\varepsilon > 0$ with $\varepsilon < c$, there exists $V_\varepsilon \subset M$ such that*

$$I_{c+\varepsilon} \cap M = V_\varepsilon \cup -V_\varepsilon, \quad V_\varepsilon \cap -V_\varepsilon = \phi.$$

Proof. For completeness, we give a proof. Let $p : [0,1] \to M$ be a path such that $p(0) = u$ and $p(1) = -u$. Since u is positive and $-u$ is negative, there exists $t_0 \in (0,1)$ such that

$$|p(t_0)^+|^2 = \int_{\mathbb{R}^N} g(p(t_0)^+)p(t_0)^+ \, dx$$

and

$$|p(t_0)^-|^2 = \int_{\mathbb{R}^N} g(p(t_0)^-)p(t_0)^- \, dx.$$

Then, by (2.1), we have that

$$I(p(t_0)) = I(p(t_0)^+) + I(p(t_0)^-) \geq 2c.$$

Let $0 < \varepsilon < c$ and V_ε be the component of $I_{c+\varepsilon} \cap M$ containing u. Then from the observation above, we find that $V_\varepsilon \cap V_{-\varepsilon} = \phi$. Suppose that there exists a component V of $I_{c+\varepsilon} \cap M$ which is disjoint from $V_\varepsilon \cup V_{-\varepsilon}$. It is easy to see that

$$K_0 = \{u_x : x \in \mathbb{R}^N\} \subset \mathrm{int} V_\varepsilon. \tag{2.5}$$

Let $\{u_n\} \subset V$ be a sequence such that $\lim_{n\to\infty} I(u_n) = \inf\{I(v) : v \in V\}$. Then it follows that $\lim_{n\to\infty} \nabla I(u_n) = 0$. Since c is the unique critical value in $(0, 2c)$, we have that $\lim_{n\to\infty} I(u_n) = c$. Then by $(*)$, we have that there exist $\{x_n\} \subset \mathbb{R}^N$ and $\{i_n\} \subset \{0,1\}$ such that $\lim_{n\to\infty} |v_n - (-1)^{i_n} u_{x_n}| = 0$. This implies by (2.5) that $u_n \in V_\varepsilon \cup -V_\varepsilon$ for n sufficiently large. This is a contradiction. Thus we have that $I_{c+\varepsilon} \cap M = V_\varepsilon \cup -V_\varepsilon$. $\square$

Here we put

$$X_{1/2} = \{\mu v \in M, \ \mu \geq 1/2\}$$

Then $M \subset \mathrm{int}\, X_{1/2}$. Let $V_0, \ V_1$ be bounded neighbourhoods of $V_{3c/2} (\subset M \cap I_{3c/2})$ such that

$$V_0 \subset \mathrm{int}\, V_1 \subset X_{1/2} \quad \text{and} \quad V_1 \subset I^{-1}[c/2, 3c/2].$$

Then we have that

$$\delta_0 = \inf\{|\nabla I(v)| : v \in I^{-1}[c/2, 3c/2] \setminus V_0\} > 0.$$

We next define a functional J. Let $\alpha(x) : H \to [0,1]$ be a continuous function such that

$$\alpha(x) = \begin{cases} 1 & \text{for } x \in V_1^c, \\ 0 & \text{for } x \in V_0, \end{cases}$$

and we put

$$J(v) = \alpha(v)I(v) + (1 - \alpha(x))J^\infty(v) \quad \text{for all } v \in H.$$

Then from the definition, $J \equiv J^\infty$ on V_0 and $J \equiv I$ on V_1^c. Here we note that

$$(2.6)\qquad \lim_{\rho\to 0}|I(v) - J^\infty(v)| = \lim_{\rho\to 0}|\nabla I(v) - \nabla J^\infty(v)| = 0 \quad \text{uniformly on } V_1.$$

Then there exists $\rho_1 > 0$ such that if $\rho \le \rho_1$,

$$|I(v) - J(v)| < c/2 \quad \text{on } V_1$$

and

$$|\nabla J^\infty(v) - \nabla I(v)| < \delta_0/2 \quad \text{on } V_1.$$

Therefore we have that

$$|\nabla J(v)| > \delta_0/2 \quad \text{for all } v \in I^{-1}[c/2, 3c/2] \setminus V_0.$$

This implies that

$$\text{if } \rho \le \rho_1,\ |\nabla J(v)| < \delta_0/2 \text{ and } 3c/2 > J(v) > 0,\ \text{ then } v \in V_0$$

and therefore $J(v) = J^\infty(v)$. This implies that if we find a critical point v of J with $2c > J(v) > 0$, then v is a critical point of J^∞ in V_0.

3. Homology groups

Our purpose in this section is to calculate homology groups $H_*(I_{c+\varepsilon}, I_{c-\varepsilon})$ for $0 < \varepsilon < c/2$. To calculate the homology groups $H_*(I_{c+\varepsilon}, I_{c-\varepsilon})$, we will find subsets K and U of V_0 satisfying

(a) $K \subset \operatorname{int} U$,
(b) $\pm K_0 = \pm\{u_x : x \in \mathbb{R}^N\} \subset \operatorname{int} K$,
(c) there exists $\varepsilon_1 > 0$ such that $I_{c/2}$ is a strong deformation retract of $I_{c+\varepsilon} \setminus K$ for $0 < \varepsilon < \varepsilon_1$.

In fact, for U and K satisfying (a), (b) and (c) , we have the following lemma.

LEMMA 3.1. *Suppose that U and K satisfy* (a), (b) *and* (c). *Then for each* $0 < \varepsilon < \varepsilon_1$.

$$H_*(I_{c+\varepsilon}, I_{c-\varepsilon}) = H_*(U \cap I_{c+\varepsilon}, (U \setminus K) \cap I_{c+\varepsilon})$$

PROOF. Assume that (a), (b) and (c) hold. Then by the exactness of singular homology groups (cf. [3], [10]) to the triple $(I_{c+\varepsilon}, I_{c+\varepsilon} \setminus K, I_{c/2})$:

$$\cdots \to H_q(I_{c+\varepsilon} \setminus K, I_{c/2}) \to H_q(I_{c+\varepsilon}, I_{c/2}) \\ \to H_q(I_{c+\varepsilon}, I_{c+\varepsilon} \setminus K) \to H_{q-1}(I_{c+\varepsilon} \setminus K, I_{c/2}) \to$$

and the fact that $H_q(I_{c+\varepsilon} \setminus K, I_{c/2}) \cong 0$, we find

$$H_*(I_{c+\varepsilon}, I_{c/2}) \cong H_*(I_{c+\varepsilon}, I_{c+\varepsilon} \setminus K).$$

Recalling that the interval $[c/2, c - \varepsilon]$ contains no critical value, we have

$$H_*(I_{c+\varepsilon}, I_{c/2}) \cong H_*(I_{c+\varepsilon}, I_{c-\varepsilon})$$

and then

$$H_*(I_{c+\varepsilon}, I_{c-\varepsilon}) \cong H_*(I_{c+\varepsilon}, I_{c+\varepsilon} \setminus K).$$

Since $U \cap I_{c+\varepsilon}$ is an neighbourhood of $K \cap I_{c+\varepsilon}$ in $I_{c+\varepsilon}$, we have by the excision property of homology groups that

$$H_*(I_{c+\varepsilon}, I_{c+\varepsilon} \setminus K) \cong H_*(U \cap I_{c+\varepsilon}, (U \setminus K) \cap I_{c+\varepsilon}).$$

Then the assertion follows. $\square$

We will define subsets U and K of V_0 satisfying (a), (b) and (c).

LEMMA 3.2. *For each neighbourhood V of $K_0 \cup -K_0$ in M, there exists $\varepsilon_V > 0$ such that*

$$I^M_{c+\varepsilon} \subset V \quad \text{for each } 0 < \varepsilon < \varepsilon_V,$$

where I^M denotes the restriction of I on M.

Lemma 3.2 is a direct consequence from $(*)$ and then we omit the proof.

LEMMA 3.3. *For each $0 < \varepsilon < c/2$,*

$$I^M_{c+\varepsilon} \cong \{u\} \cup \{-u\}.$$

PROOF. Let $0 < \varepsilon < c/2$. Recalling that $\pm K_0 = \{\pm u_x : x \in \mathbb{R}^N\} \subset \operatorname{int} I_{c+\varepsilon}$, we find that there exists a neighbourhood U_1 of K_0 such that

$$K_0 \subset \operatorname{int} U_1 \subset \operatorname{int} I_{c+\varepsilon}, \quad U_1 \cap -U_1 = \phi \text{ and } U_1 \cong \{u_x : x \in \mathbb{R}^N\} \cong \{u\}.$$

By Lemma 3.2, we can choose $k \geq 1$ so large that

$$I^M_{c+\varepsilon/k} \subset \operatorname{int}(U_1 \cup -U_1). \tag{3.1}$$

Similarly, we choose a neighbourhood U_2 of K_0 such that

$$K_0 \subset \operatorname{int} U_2 \subset \operatorname{int} I_{c+\varepsilon/k}, \quad U_2 \cap -U_2 = \phi \text{ and } U_2 \cong \{u_x : x \in \mathbb{R}^N\} \cong \{u\}.$$

Then we find that

$$U_1 \cup -U_1 \cong U_2 \cup -U_2 \cong I^M_{c+\varepsilon/k}.$$

Let $\gamma_1 : [0,1] \times I^M_{c+\varepsilon} \to I^M_{c+\varepsilon/k}$ be the strong deformation retraction from $I^M_{c+\varepsilon}$ onto $I^M_{c+\varepsilon/k}$. Also let γ_2 be the the strong deformation retraction from $U_1 \cup -U_1$ onto $U_2 \cup -U_2$. We put

$$\gamma(t,v) = \begin{cases} \gamma_1(2t, v) & \text{for } t \in [0, 1/2], \\ \gamma_2(2t-1, \gamma_1(1,v)) & \text{for } t \in [1/2, 1], \end{cases}$$

for each $v \in I^M_{c+\varepsilon}$. Then γ is a strong deformation retraction from $I^M_{c+\varepsilon}$ to $U_2 \cup -U_2$. This proves the assertion. $\square$

We next define U and K. Here we fix positive numbers r_1, r_2 with $r_1 > r_2$. We assume that r_1 is so small that

$$c/2 < I(v + \lambda v) \quad \text{for all } v \in K_0 \text{ and } \lambda \in \mathbb{R} \text{ with } |\lambda| \leq 2r_1. \tag{3.2}$$

By (2.4), we have that there exists $\widetilde{\varepsilon} > 0$

$$I(v + \lambda v) < I(v) - 2\widetilde{\varepsilon} \quad \text{for } v \in K_0 \text{ and } r_2 \leq |\lambda| \leq r_1. \tag{3.3}$$

Then we can choose a neighbourhood $\widetilde{V}$ of $K_0 \cup -K_0$ in M such that for each $v \in \widetilde{V}$,

$$c/2 < I(v+\lambda v) \quad \text{for } |\lambda| \leq r_1 \quad \text{and} \quad I(v+\lambda v) < c-\widetilde{\varepsilon} \quad \text{for } r_2 \leq |\lambda| \leq r_1. \tag{3.4}$$

By Lemma 3.2, we can choose a positive number $\varepsilon_0 < c/2$ so small that $I^M_{c+2\varepsilon_0} \subset \widetilde{V}$. Then by (3.4), we have that

$$c/2 < I(v + \lambda v) \quad \text{for all } v \in I^M_{c+2\varepsilon_0} \text{ and } |\lambda| \leq r_1 \tag{3.5}$$

and

$$I(v + \lambda v) < c - \widetilde{\varepsilon} \quad \text{for all } v \in I^M_{c+2\varepsilon_0} \text{ and } r_2 \leq |\lambda| \leq r_1. \tag{3.6}$$

We put $\widetilde{U} = I^M_{c+2\varepsilon_0}$ and $\widetilde{K} = I^M_{c+\varepsilon_0}$. Then it follows that

$$\inf\{I(v) : v \in \widetilde{U} \setminus \widetilde{K}\} \geq c + \varepsilon_0. \tag{3.7}$$

Now we set

$$U = \{v + \lambda v : v \in \widetilde{U}, |\lambda| \leq r_1\}, \quad K = \{v + \lambda v : v \in \widetilde{K}, |\lambda| \leq r_2\}.$$

Then it is obvious that U and K satisfies (a) and (b). Moreover, we have

LEMMA 3.4. *For each $0 < \varepsilon < \varepsilon_0$, $I_{c/2}$ is a strong deformation retract of $I_{c+\varepsilon} \setminus K$.*

PROOF. Let V be a closed subset of U such that

$$K \subset \operatorname{int} V \subset V \subset \operatorname{int} U.$$

We first define a pseudogradient vector field Φ on U by

$$\Phi(z) = \lambda v \quad \text{for } z = v + \lambda v,\ v \in \widetilde{U},\ |\lambda| \leq r_1.$$

Then recalling that $I(v + t\lambda v)$ is decreasing as t increases on $[0, 1]$, we have

$$\langle \nabla I(z), \Phi(z) \rangle \leq 0. \tag{3.8}$$

In (3.8), the equality holds if and only if $z \in \widetilde{U}$. Then by (3.7), we have that for each $0 < \varepsilon < \varepsilon_0$,

$$\sup\{\langle \nabla I(v), \Phi(v) \rangle : v \in (U \setminus K) \cap I_{c+\varepsilon}\} < 0. \tag{3.9}$$

On the other hand, we denote by Ψ a pseudogradient vector field on H associate with functional I (cf. [3]). Since $\inf\{|\nabla I(v)| : v \in V_0 \setminus K\} > 0$, we have that

$$\sup\{\langle \nabla I(v), -\Psi(v)\rangle : v \in V_0 \setminus K\} < 0. \tag{3.10}$$

Define $h(v) = d(x,V)/(d(x,V) + d(x,U^c))$ for $v \in H$. Then $h(v) = 0$ on V and $h(v) = 1$ on U^c. We now set

$$\Gamma(v) = -h(v)\Psi(v) + (1 - h(v))\Phi(v) \quad \text{for } v \in H. \tag{3.11}$$

Let consider the ordinary differential equation

$$\frac{d\eta}{dt} = \Gamma(\eta), \quad \eta(0,v) = v.$$

Then we have by (3.9) and (3.10) that there exists a positive number δ and

$$\begin{aligned} &I(\eta(t,v)) - I(\eta(0,v)) \\ &\qquad = \int_0^t \langle \nabla I(\eta(\tau,v), -h(v)\Psi(v) + (1-h(v))\Phi(v)\rangle\, dt < -\delta t \end{aligned} \tag{3.12}$$

for $t > 0$. It also follows from the definition of Γ that

$$\text{if } v \in K^c, \text{ then } \eta(t,v) \in K^c \quad \text{for all } t > 0. \tag{3.13}$$

Therefore, from (3.12) and (3.13), we have that there exists $m > 0$ such that for any $v \in I_{c+\varepsilon} \setminus K$, $\eta(t,v) \in I_{c/2}$ for all $t > m$. Then we can construct a deformation retraction from $I_{c+\varepsilon} \setminus K$ onto $I_{c/2}$ from η by a standard argument. $\square$

Lemma 3.5. *For each* $0 < \varepsilon < 2\varepsilon_0$,

$$U \cap I_{c+\varepsilon} \cong U \cong \{u\} \cup \{-u\}.$$

Proof. Let $0 < \varepsilon < 2\varepsilon_0$. Since $\widetilde{U} = I^M_{c+2\varepsilon_0}$, there exists a strong deformation retraction γ from $\widetilde{U}$ onto $I^M_{c+\varepsilon}$. Let $z \in U \cap I_{c+\varepsilon}$ with $z = v + w$, $v \in \widetilde{U}$ and $w = \lambda v$ for some $\lambda \in R$. For each $v(t) = \gamma(t,v)$, $t \in [0,1]$, we put

$$\alpha_t = \min\{s \in [0,1] : v(t) + s \cdot \operatorname{sgn}(\lambda) v(t) \in I_{c+\varepsilon}\},$$

where we put $\operatorname{sgn}(\lambda) = 0$ if $\lambda = 0$. We note that if $\alpha_0 = 0$, then $\alpha_t = 0$ for all $t \in [0,1]$. We put now

$$w(t) = (\lambda \alpha_t/\alpha_0) v(t)$$

where we put $\alpha_t/\alpha_0 = 1$ when $\alpha_0 = 0$. Since $\lambda/\alpha_0 \geq 1$, we have that $v(t) + w(t) \in U \cap I_{c+\varepsilon}$ for $t \in [0,1]$ and that

$$v(1) + w(1) \in U_1 = \{v + \lambda v : v \in I^M_{c+\varepsilon},\ |\lambda| \leq r_1\} \subset I_{c+\varepsilon}.$$

We put now

$$\eta(t,z) = v(t) + w(t) \quad \text{for } z \in U \text{ and } t \in [0,1].$$

Then from the argument above, we have that η is a strong deformation retraction from $U \cap I_{c+\varepsilon}$ onto U_1. Then since

$$U_1 \cong I^M_{c+\varepsilon} \times D^1 \cong \{u, -u\} \times D^1 \cong \{u\} \cup \{-u\},$$

the assertion follows. □

For each $v \in \widetilde{U}$. We put

$$U_v = \{v + \lambda v : |\lambda| \le r_1\}, \quad K_v = \begin{cases} \{v + \lambda v : |\lambda| \le r_2\} & \text{if } v \in \widetilde{K}, \\ \{\phi\} & \text{if } v \notin \widetilde{K}. \end{cases}$$

Then

LEMMA 3.6. *Let $0 < \varepsilon < \varepsilon_0$. Then, for each $v \in \widetilde{U}$,*

$$(U_v \setminus K_v) \cap I_{c+\varepsilon} \cong v + \{-r_1 v, r_1 v\} \cong S^0. \tag{3.14}$$

PROOF. Let $v \in \widetilde{U}$. If $v \in \widetilde{K}$, then from the definition, we have that

$$U_v \setminus K_v = \{v + \lambda v : r_2 \le |\lambda| \le r_1\} \cong S^0.$$

Since $\{v + \lambda v : r_2 \le |\lambda| \le r_1\} \subset I_c$ by (3.6), we have that

$$(U_v \setminus K_v) \cap I_{c+\varepsilon} \cong U_v \setminus K_v \cong S^0.$$

Suppose that $v \notin \widetilde{K}$. Then

$$U_v \setminus K_v = U_v = \{v + \lambda v : |\lambda| \le r_1\}.$$

Here we recall that $I(v) > c + \varepsilon_0$. Then since for $\lambda \in \mathbb{R}$ with $|\lambda| = r_1$, the mapping $t \to I(v + t\lambda v)$ is decreasing on $[0, 1]$ with $I(v + \lambda v) < c$, we find that $\{v + t\lambda v : t \in [0, 1]\} \cap I_c$ is an interval which does not contains 0. Therefore

$$(U_v \setminus K_v) \cap I_{c+\varepsilon} \cong v + \{-r_1 v, r_1 v\} \cong S^0.$$

□

LEMMA 3.7. *For $0 < \varepsilon < \varepsilon_0$,*

$$H_*(U \cap I_{c+\varepsilon}, (U \setminus K) \cap I_{c+\varepsilon}) = H_*(D^1, S^0) \oplus H_*(D^1, S^0).$$

PROOF. Let $0 < \varepsilon < \varepsilon_0$. By Lemma 3.5 and the definition, we have that

$$U \cap I_{c+\varepsilon} \cong U \cong \widetilde{U} \times D^1 \cong \{u\} \times D^1 \cup \{-u\} \times D^1.$$

On the other hand, by Lemma 3.6, we have that

$$(U \setminus K) \cap I_{c+\varepsilon} \cong \widetilde{U} \times S^0 \cong \{u\} \times S^0 \cup \{-u\} \times S^0.$$

Then the assertion follows. □

By Lemma 3.1 and Lemma 3.7, we have

PROPOSITION 3.8. *For each* $0 < \varepsilon < c$

$$H_n(I_{c+\varepsilon}, I_{c-\varepsilon}) = \begin{cases} 2 & \text{for } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

4. Proof of Theorem 1.1

In this section, we calculate the homology groups for J and prove Theorem 1.1. To find a positive(negative) solution of (P), we may assume without any loss of generality that $g(x,-t) = -g(x,t)(g(x,t) = -g(x,-t))$ for all $t \geq 0$ and $x \in \mathbb{R}^N$. Then, in the following, we assume that $g(x,-t) = -g(x,t)$ holds for $t \geq 0$ and $x \in \mathbb{R}^N$. From (2.6), we have that there exists $\rho_2 > 0$ such that if $0 < \rho < \rho_2$, then

$$H_*(I_{c+\varepsilon}, I_{c/2}) \cong H_*(J_{c+\varepsilon}, J_{c/2}) \quad \text{for } 0 < \varepsilon < c/2. \tag{4.1}$$

We next define a manifold $\mathcal{M}$ by

$$\mathcal{M} = \left\{ v \in H \setminus \{0\} : |v|^2 = \int_{\mathbb{R}^N} g(x, v(x))v(x)\, dx \right\}.$$

By (g2) and (g3), we can see that the following assertion holds.

(g2') For each $x \in \mathbb{R}^N$, $g(x,t)/t$ is increasing on $[0,\infty]$ and

$$\lim_{t\to\infty} g(x,t)/t = \infty \text{ uniformly in } \mathbb{R}^N.$$

Then by (g2'), we can see that for each $v \in H \setminus \{0\}$, the set $\{\lambda v : \lambda \geq 0\}$ intersect to $\mathcal{M}$ at exactly one point. It is also obvious that each critical point of J is in $\mathcal{M}$. By (2.6), we may assume that ρ_2 is so small that

$$\inf\{J(v) : v \in \mathcal{M}\} > 3c/4. \tag{4.2}$$

By the inequality (4.2), we have that if $v \in H$ is a critical point of J with $J(v) < 3c/2$, then v is positive or negative. In fact, if v is a sign changing solution of (P), it follows that

$$|v^+|^2 = \int_{\mathbb{R}^N} g(x, v^+(x))v^+(x)\, dx \text{ and } |v^-|^2 = \int_{\mathbb{R}^N} g(x, v^-(x))v^-(x)\, dx.$$

Then since $J(v) = J(v^+) + J(v^-) < 3c/2$, we find that $J(v^+) < 3/4$ or $J(v^-) < 3c/4$ holds. This contradicts to (4.2).

Now assume that $0 < \rho < \min\{\rho_1, \rho_2\}$. Then from the definition of ρ_1, there exists no critical point of J with critical value in $(0, c/2] \cup [3c/2, 2c)$. Then as a direct consequence from Lions's concentrate compactness lemma, we have

Lemma 4.1. *Let $\{u_n\} \subset H$ be a sequence such that*

$$\lim_{n\to\infty} \nabla J(u_n) = 0 \text{ and } 0 < \lim_{n\to\infty} J(u_n) < 3c/2 \text{ exists.}$$

Then there exists a subsequence (still denoted by $\{u_n\}$) for which the one of the following conditions holds:

(a) *there exists a critical point $\overline{u}$ of J and $u_n \to \overline{u}$ as $n \to \infty$,*
(b) *there exist a sequence $\{x_n\} \subset \mathbb{R}^N$ and a sequence $\{i_n\} \subset \{0,1\}$ such that*

$$\begin{aligned} &J(u_n) \to c && \text{as } n \to \infty, \\ &u_n - (-1)^{i_n} u_{x_n} \to 0 && \text{as } n \to \infty, \\ &|x_n| \to \infty && \text{as } n \to \infty \text{ for } i = 1,2. \end{aligned}$$

Lemma 4.1 is just a modification of Proposition 2.1 of [13] (cf. also [7], [14]). Then we omit the proof.

We will prove Theorem by contradiction. That is we assume in the following that J possesses no critical point with critical value in $(0, 2c)$.

For each $x \in R$, we define a positive number α_x by $\alpha_x u_x \in \mathcal{M}$. From condition (g3), we have that

$$\lim_{|x|\to\infty} \alpha_x = 1. \tag{4.3}$$

for $r > 0$, we put

$$K_r = \{\alpha_x u_x : x \in \mathbb{R}^N,\ |x| \geq r\}.$$

Then $K_r \cong S^{N-1}$ for $r > 0$, and

$$\lim_{r\to\infty} \sup\{J(v) : v \in K_r\} = c. \tag{4.4}$$

Lemma 4.2. *For each $\varepsilon > 0$ with $\varepsilon < c/2$, there exists $r_\varepsilon > 0$ and*

$$J^{\mathcal{M}}_{c+\varepsilon} \cong K_{r_\varepsilon} \cup -K_{r_\varepsilon} \cong S^{N-1} \amalg S^{N-1},$$

where $\amalg$ denotes disjoint union of sets.

Proof. We first see that

$$\inf\{J(v) : v \in \mathcal{M}\} = c \text{ and } J(v) > c \quad \text{for all } v \in \mathcal{M}. \tag{4.5}$$

Let $\{u_n\} \subset \mathcal{M}$ such that

$$\lim_{n\to\infty} J(u_n) = c_0 = \inf\{J(v) : v \in \mathcal{M}\} \leq c.$$

Then it follows that $\lim_{n\to\infty} \nabla J(u_n) = 0$. Since we are assuming that J possesses no critical point with critical value in $(0, 2c)$, (b) of Lemma 4.1 holds. That is $c_0 = c$. If c is attained by a element v of $\mathcal{M}$, v is a critical point of J. This contradicts to our assumption. Then we have that the second assertion of (4.5) holds. Now let $\varepsilon > 0$ with $\varepsilon < c/2$. Then we can choose a positive number γ_1 such that

$$K_{\gamma_1} \cup -K_{\gamma_1} \subset \operatorname{int} J^{\mathcal{M}}_{c+\varepsilon}.$$

Here we choose neighbourhoods $V_{+,1}$ of K_{γ_1} and $V_{-,1}$ of of $-K_{\gamma_1}$ such that

$$V_{+,1} \cap V_{-,1} = \phi, \quad V_{+,1} \cong K_{\gamma_1} \cong -K_{\gamma_1} \cong V_{-,1}$$

and

$$V_{+,1} \cup V_{-,1} \subset \operatorname{int} J^{\mathcal{M}}_{c+\varepsilon}.$$

By (b) of Lemma 4.1, we have that there exist $k > 1$, $\gamma_2 > \gamma_1$ such that

$$K_{\gamma_2} \cup -K_{\gamma_2} \subset J^{\mathcal{M}}_{c+\varepsilon/k} \subset V_{+,1} \cup V_{-,1}.$$

Then since $J^{\mathcal{M}}_{c+\varepsilon/k} \cong J^{\mathcal{M}}_{c+\varepsilon}$ and $K_{\gamma_2} \cup -K_{\gamma_2} \cong V_{+,1} \cup V_{-,1} \cong K_{\gamma_1} \cup -K_{\gamma_1}$, we obtain that

$$J^{\mathcal{M}}_{c+\varepsilon} \cong K_{\gamma_1} \cup -K_{\gamma_1} \cong S^{N-1} \amalg S^{N-1}. \qquad \square$$

Again by (2.6) , we can choose a positive number $\rho_0 < \min\{\rho_1, \rho_2\}$ so small that if $\rho < \rho_0$, (3.5) and (3.6) hold with I and M replaced by J and $\mathcal{M}$, respectively.

Now we assume that $\rho < \rho_0$ and put $\widetilde{\mathcal{K}} = J^{\mathcal{M}}_{c+\varepsilon}$ and $\widetilde{\mathcal{U}} = J^{\mathcal{M}}_{c+2\varepsilon}$. We also set

$$\mathcal{U} = \{v + \lambda v : v \in \widetilde{\mathcal{U}}, |\lambda| \le r_1\}, \quad \mathcal{K} = \{v + w : v \in \widetilde{\mathcal{U}}, w|\lambda| \le r_2\}.$$

Then by a parallel argument as in the proof of Lemma 3.4, we can see that $J_{c/2}$ is a strong deformation retract of $J_{c+\varepsilon} \setminus \mathcal{K}$ for each $0 < \varepsilon < \varepsilon_0$. That is we have

$$H_*(J_{c+\varepsilon}, J_{c/2}) = H_*(\mathcal{U} \cap J_{c+\varepsilon}, (\mathcal{U} \setminus \mathcal{K}) \cap J_{c+\varepsilon}), \tag{4.6}$$

for each $0 < \varepsilon < \varepsilon_0$.

We also have, by Lemma 4.2, that

LEMMA 4.3. *For each* $0 < \varepsilon < 2\varepsilon_0$*,*

$$\mathcal{U} \cap J_{c+\varepsilon} \cong \mathcal{U} \cong K_r \cup -K_r \cong S^{N-1} \amalg S^{N-1} \quad \textit{for all } r > 0.$$

The proof of Lemma 4.3 is the same as that of Lemma 3.5. Then we omit the proof.

As in Section 3, we put

$$\mathcal{U}_v = \{v + \lambda v : |\lambda| \le r_1\}, \quad \mathcal{K}_v = \begin{cases} \{v + \lambda v : |\lambda| \le r_2\} & \text{if } v \in \widetilde{\mathcal{K}}, \\ \{\phi\} & \text{if } v \notin \widetilde{\mathcal{K}}, \end{cases}$$

for each $v \in \widetilde{U}$. Then, by the same argument as in Section 3, we have

LEMMA 4.4. *Let* $0 < \varepsilon < \varepsilon_0$. *Then for each* $v \in \widetilde{\mathcal{U}}$,

$$(\mathcal{U}_v \setminus \mathcal{K}_v) \cap I_{c+\varepsilon} \cong v + \{-r_1 v, r_1 v\} \cong S^0. \tag{4.7}$$

Then, using Lemmas 4.3 and 4.4, we obtain

LEMMA 4.5. *For each* $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} &H_*(\mathcal{U} \cap J_{c+\varepsilon}, (\mathcal{U} \setminus \mathcal{K}) \cap J_{c+\varepsilon}) \\ &\qquad = H_*(S^{N-1} \times D^1, S^{N-1} \times S^0) \oplus H_*(S^{N-1} \times D^1, S^{N-1} \times S^0). \end{aligned}$$

Thus we obtain, by (4.6) and Lemma 4.5, that

PROPOSITION 4.6.

$$H_n(J_{c+\varepsilon}, J_{c/2}) = \begin{cases} 2 & \textit{for } n = 1 \textit{ or } n = N, \\ 0 & \textit{otherwise.} \end{cases}$$

We can now complete the proof of Theorem.

PROOF OF THEOREM 1.1. By (4.6), we have that if $\rho \leq \rho_0$, then for each $0 < \varepsilon < c$,

$$H_*(J_{c+\varepsilon}, J_{c/2}) \cong H_*(I_{c+\varepsilon}, I_{c/2}) \cong H_*(I_{c+\varepsilon}, I_{c-\varepsilon}). \tag{4.8}$$

But we can see from Proposition 3.8 and Proposition 4.6 that the equality does not holds. This is a contradiction. Thus we obtain that there exists a positive solution of (P). The existence of negative solution is obtained by the same way.□

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