Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 12, 1998, 367–373

PROPERTIES OF MINIMAL INVARIANT SETS FOR NONEXPANSIVE MAPPINGS

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In 1965 F. E. Browder [3] and D. Göhde [6] proved that each nonempty bounded and convex subset of a uniformly convex Banach space has the fixed point property for nonexpansive mappings. Also in 1965 W. A. Kirk [8] came to the same conclusion for weakly compact convex subsets of any Banach space under additional assumption that the set has the so-called normal structure. This condition is much weaker than uniform convexity of the space under concern. Since then the problem of finding weaker and weaker conditions implying existence of fixed points for nonexpansive mappings has been the subject of study by many authors. The central themes of these investigations can be found in the book by the author and W. A. Kirk [5].

Many proofs and reasonings in this theory are based on the analysis of a "bizarre" object called "the minimal invariant set".

Let C be a nonempty, weakly compact, convex subset of a Banach space X. Suppose the mapping $T : C \to C$ is *nonexpansive*, i.e. such that

$$\|Tx - Ty\| \leq \|x - y\|$$

holds for all $x, y \in C$.

The set C can contain many "smaller" closed, convex (thus weakly compact) subsets D which are also T-*invariant*, $T(D) \subset D$. Using Zorn's Lemma one

1991 *Mathematics Subject Classification.* Primary 47H09, 47H10.
Key words and phrases. Nonexpansive mappings, minimal invariant sets.
The author was partially sponsored by the KBN grant 2P03A02915.

can easily prove that the family of such sets contains minimal elements with respect to the order generated by inclusion. These are "*minimal invariant sets*". Obviously any set consisting of one element, a fixed point of T ($x = Tx$), is minimal.

Till 1981 it was not known whether singletons are the only possible minimal invariant sets. In other words it was not known whether weak compactness alone is sufficient for C to have the fixed point property for nonexpansive mappings.

The solution to this problem is due to D. Alspach [2].

Example. Let $X = L^1(0,1)$ and let $C = \{f \in L^1 : 0 \le f \le 2\}$. Define the isometry $T : C \to C$ by

$$(Tf)(t) = \begin{cases} \min\{2f(2t), 2\} & \text{if } 0 \le t \le 1/2, \\ \min\{2f(2t-1) - 2, 0\} & \text{if } 1/2 < t \le 1. \end{cases}$$

Thus C is weakly compact and $T : C \to C$ is nonexpansive. Two constant functions 0 and 2 are fixed points of T. On the other hand, for any $a \in (0,2)$,

$$C_a = \left\{ f \in C : \int_0^1 f = a \right\}$$

is a convex closed and T-invariant subset of C. None of C_a contains a fixed point of T, thus it has to contain a minimal invariant set which is not a singleton.

Actually Alspach's paper contains the proof of it for C_1 but it does not contain any kind of explicit description of any minimal invariant set contained in C_1.

According to our knowledge, till now no "constructive" examples of minimal invariant sets consisting of more than one point are known. Investigations of minimal invariant sets exhibited several "bizarre" properties of this object. In 1975 (six years before Alspach) the present author [4] listed eleven of such properties. The most important among them are the following.

Property 1. *If K is minimal then $K = \operatorname{Conv} T(K)$.*

Property 2. *If K is minimal and $\{x_n\}$ is a sequence of points in K such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ then for any $z \in K$,*

$$\lim_{n\to\infty} \|x_n - z\| = \operatorname{diam} K.$$

Since such sequence $\{x_n\}$ always exists, we have the following consequence of Property 2.

Property 3. *If K is minimal then, for any $z \in K$,*

$$\sup_{x\in K} \|x - z\| = \operatorname{diam} K.$$

In other words all points of K are "diametral" (K is a *diametral set*).

Property 2 was independently discovered in 1976 by L. Karlovitz [7] and later became very useful as a technical tool in proving fixed point theorems via nonstandard (ultraproduct) methods (see [1], [5]).

Since in the presented Alspach's example we have a set C containing many minimal invariant subsets, it is natural to ask about properties of this family.

In what follows we shall consider the standard setting of C being a weakly compact and convex set and $T : C \to C$ being nonexpansive. We shall deal only with closed and convex subsets of C. If $D \subset C$ is closed and convex (thus weakly closed) then for any $z \in C$ there exists at least one point $x \in D$ such that $\|x - z\| = \operatorname{dist}(z, D)$. Moreover, the set of such points x is closed and convex. This set is called the *metric projection* of z onto D and is denoted by $\operatorname{Proj}_D(z)$. Obviously

$$\operatorname{Proj}_D(z) = \bigcap_{\varepsilon>0} D \cap B(z, r+\varepsilon),$$

where $r = \operatorname{dist}(z, D)$ and $B(z, r+\varepsilon)$ denotes the closed ball centered at z and of radius $r+\varepsilon$.

This obvious fact will be practically the only tool for our investigations. For $D \subset C$, we denote by $B(D, r)$ the closed r-neighbourhood of D, $B(D, r) = C \cap \bigcup_{x\in D} B(x, r)$, and for sets D_1, D_2 let $H(D_1, D_2)$ be the Hausdorff distance between them. We shall call our findings "Observations". The first two are obvious.

OBSERVATION 1. *If $D \subset C$ is T-invariant then for any $r \geq 0$, $B(D, r)$ is T-invariant.*

OBSERVATION 2. *If D_1, D_2 are T-invariant, $D_1 \cap D_2 \neq \emptyset$ then $D_1 \cap D_2$ is T-invariant.*

The third follows.

OBSERVATION 3. *If D is invariant and K is minimal invariant then* $\operatorname{dist}(x, D)$ *is constant on K.*

In other words $K \subset S(D, r)$, where $r = \operatorname{dist}(x, D)$ for any $x \in K$ and $S(D, r) = \partial B(D, r) = \{z : \operatorname{dist}(z, D) = r\}$.

PROOF. Suppose that we have two points $x_1, x_2 \in K$ with $\operatorname{dist}(x_1, D) = r_1 < r_2 = \operatorname{dist}(x_2, D)$. Then the set

$$K \cap B(D, (r_1 + r_2)/2)$$

would be a closed invariant convex subset of K which contradicts minimality of K. $\square$

As a consequence we have.

Observation 4. *If K_0, K_1 are minimal invariant then for any $x \in K_0$ and any $y \in K_1$,*

$$\operatorname{dist}(x, K_1) = \operatorname{dist}(y, K_0) = \text{const} = H(K_0, K_1).$$

Observation 5. *If K_0, K_1 are minimal invariant sets then for any $\alpha \in [0,1]$ there exists a minimal invariant set K_α such that $H(K_0, K_\alpha) = \alpha H(K_0, K_1)$, $H(K_\alpha, K_1) = (1-\alpha)H(K_0, K_1)$.*

Proof. Let $r = H(K_0, K_1)$. Observe that for any $\varepsilon > 0$ the set

$$D_{\alpha,\varepsilon} = B(K_0, \alpha r + \varepsilon) \cap B(K_1, (1-\alpha)r + \varepsilon)$$

is nonempty and invariant. In view of weak compactness, the set

$$D_\alpha = \bigcap_{\varepsilon > 0} D_{\alpha,\varepsilon}$$

is nonempty and obviously invariant. Thus it contains a minimal invariant set K_α satisfying our requirements. □

We leave to the reader the justification of the fact that the limit of a convergent (with respect to the Hausdorff metric) sequence of minimal invariant sets is minimal invariant itself.

The above can be put in other form.

Observation 6. *The family of minimal T-invariant convex closed subsets of C is closed and metrically convex with respect to the Hausdorff metric.*

The above can be viewed as a counterpart of the following well known fact: if a nonexpansive mapping $T : C \to C$ has a fixed point in each T-invariant closed and convex subset of C then the set of fixed points of T is metrically convex.

The next observation concerns the class of strictly convex spaces. Let us recall that the space X is *strictly convex* if for any $x, y \in X$ the following implication holds

$$\left.\begin{array}{r} \|x\| \le 1 \\ \|y\| \le 1 \\ x \ne y \end{array}\right\} \Rightarrow \left\| \frac{x+y}{2} \right\| < 1.$$

The above condition means that the unit sphere in X does not contain any segment and it can also be equivalently rewritten as

$$\left.\begin{array}{c} \|x\| = r \\ \|y\| = r \\ \|x+y\| = 2r \end{array}\right\} \Rightarrow x = y.$$

It is not known whether strict convexity of the space X together with weak compactness of C implies the fixed point property of C. However, if not then the minimal invariant sets show a surprising property.

Observation 7. *Let X be a strictly convex space and let K_1, K_2 be two minimal invariant subsets of C. Then K_2 is a shifted copy of K_1, i.e. there exists $z \in X$ such that $K_2 = z + K_1$.*

Proof. Take any two points y_1, y_2 in K_2, $y_1 \neq y_2$, and let $x_1 = \mathrm{Proj}_{K_1} y_1$, $x_2 = \mathrm{Proj}_{K_1} y_2$ (since X is strictly convex, the metrical projection consists of one point). Let $v = (y_1 + y_2)/2$ and $u = (x_1 + x_2)/2$. We have

$$\begin{aligned} \|v - u\| &= \|(y_1 + y_2)/2 - (x_1 + x_2)/2\| \\ &\leq (\|y_1 - x_1\| + \|y_2 - x_2\|)/2 \\ &= (H(K_1, K_2) + H(K_1, K_2))/2 = H(K_1, K_2). \end{aligned}$$

But $\|u - v\|$ cannot be smaller than $H(K_1, K_2)$. Hence we have the implication (by strict convexity)

$$\left.\begin{aligned} \|y_1 - x_1\| &= H(K_1, K_2) \\ \|y_2 - x_2\| &= H(K_1, K_2) \\ \|(y_1 - x_1) + (y_2 - x_2)\| &= 2H(K_1, K_2) \end{aligned}\right\} \Rightarrow y_1 - x_1 = y_2 - x_2.$$

In other words the vector $y - \mathrm{Proj}_{K_1} y$ is constant on K_2 and denoting it by z we get the conclusion.

Not only all minimal invariant sets are identical but also the action of T on each set is the same.

Observation 8. *In the above setting, if $K_2 = z + K_1$ then, for any $y \in K_2$ and $x = \mathrm{Proj}_{K_1} y$, we have $Ty = z + Tx$.*

Proof. Indeed, $\|Ty - Tx\| \leq \|y - x\|$ but strict inequality does not hold. Thus $Tx = \mathrm{Proj}_{K_1} Ty$. □

Finally, let us present an observation concerning a kind of uniqueness fact. Recall that a mapping $T : C \to C$ is said to be *contractive* if for any $x, y \in C$, $x \neq y$, we have

$$\|Tx - Ty\| < \|x - y\|.$$

Contractive mapping can not have more than one fixed point. The counterpart of this is the following observation (valid in any space X).

Observation 9. *If $T : C \to C$ is contractive then C contains only one minimal invariant set.*

Proof. Suppose K_1, K_2 are two different minimal invariant sets. Obviously $K_1 \cap K_2 = \emptyset$. Take any $y \in K_2$ and let $x \in \mathrm{Proj}_{K_1} y$. Since

$$\mathrm{dist}(Ty, K_1) \leq \|Ty - Tx\| < \|x - y\| = \mathrm{dist}(y, K_1),$$

we have a contradiction with Observation 4. □

Let us end up with raising some problems which, in our opinion, open a new direction for further investigations.

Since the fixed point property (fpp) for a given set C depends only on its "internal geometry" and does not depend on "the size" of C, let us assume now that all the sets under concern are of the same diameter,

$$\operatorname{diam} C = 1.$$

Now for any $T : C \to C$ define the number

$$g(C,T) = \inf\{\operatorname{diam} K : K \subset C \text{is minimal invariant for } T\}.$$

Obviously,

$$0 \le g(C,T) \le 1$$

with $g(C,T) = 0$ if T has a fixed point and $g(C,T) = 1$ if C itself is minimal invariant for T. It leads to the first problem.

Question 1. *For weakly compact C, does $g(C,T) = 0$ imply that T has fixed point in C?*

The answer is unknown. Obviously the answer is affirmative for subsets of strictly convex spaces and also for T being contractive. Looking for an answer in general case S. Prus (private communication) produced an example of a bounded closed convex (but not weakly compact!) set C and a nonexpansive fixed point free mapping $T : C \to C$ having, for any $\varepsilon > 0$, a weakly compact T-invariant set K_ε satisfying $\operatorname{diam} K_\varepsilon < \varepsilon$.

The next step is to abstract of the mapping T. Put

$$g(C) = \sup\{g(C,T) : T : C \to C,\ T \text{ is nonexpansive}\}.$$

Again,

$$0 \le g(C) \le 1$$

with $g(C) = 0$ if C has the fixed point property (fpp) and $g(C) = 1$ if C is minimal invariant for at least one T. Here is the next question.

Question 2. *For weakly compact C, is the condition $g(C) = 0$ equivalent to* fpp (*does it imply* fpp)*?*

Again the answer is "yes" for subsets of strictly convex spaces. Regardless to the answer in general case, it seems to be interesting to ask

Question 3. *Given a real number $0 \le a \le 1$, what kind of "geometrical conditions" can imply $g(C) \le a$?*

The indicator $g(C)$ can be viewed as a kind of a tool to measure the "distortion" from the fixed point property. That's why Brailey Sims in private discussion jokingly proposed to call it "the measure of non-fpp-ness". Following him we ask: is it a good term?

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Manuscript received December 18, 1998

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