Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 12, 1998, 293–307

MULTIPLE PERIODIC SOLUTIONS FOR PROBLEMS AT RESONANCE WITH ARBITRARY EIGENVALUES

Patrick Habets — Enrico Serra — Massimo Tarallo

1. Introduction

Consider the Dirichlet problem

$$\begin{cases} \Delta u + \lambda_n u + g(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $N \geq 1$, g is a bounded (Carathéodory) function and λ_n is the n–th eigenvalue of the Laplacian with Dirichlet boundary conditions.

This and related problems (obtained by changing the boundary conditions), are called resonant and have been the object of much attention and study, as testified by the vast literature concerning the subject. The general aim of papers dealing with problem (1) is the understanding of the conditions on the function g or on the potential $G(x,u) = \int_0^u g(x,s)\,ds$ which ensure existence of one or more solutions to the problem.

In this framework, it has long been recognized that the behavior of

$$\int_\Omega G(x,u_0(x))\,dx, \tag{2}$$

1991 *Mathematics Subject Classification.* 35J20, 34B15.
Key words and phrases. Relative category, deformations, critical points.
Supported by EEC, CI1*–CT93–0323.

when u_0 varies in the kernel of the linear part of the equations, plays a fundamental role. Many conditions on the asymptotic behavior of (2) when u_0 "tends to infinity" in an appropriate sense have been discovered to be sufficient to prove existence results.

These conditions are of course most easily formulated if one assumes λ_n to be a simple eigenvalue. In this case indeed the corresponding eigenspace is one-dimensional, say $\mathbb{R}\varphi_n$, and the analysis of the asymptotic behavior of (2) reflects in the study of the function $\psi : \mathbb{R} \to \mathbb{R}$ defined as

$$\psi(r) = \int_\Omega G(x, r\varphi_n(x))\, dx.$$

It is well known, for instance, that if

$$\lim_{r\to\pm\infty} \psi(r) = -\infty, \tag{3}$$

then the corresponding problem has at least one solution (which can be obtained variationally as a minimum for the associated action functional, see [5]), while, if

$$\lim_{r\to\pm\infty} \psi(r) = +\infty, \tag{4}$$

then the saddle point geometry of the action functional again allows one to prove existence of a solution (see [1], [6]).

In view of these considerations, in [2] the authors analyzed, confining themselves to the *first* eigenvalue, an intermediate condition between the two above, namely the case in which ψ exhibits large oscillations, as for example when

$$\liminf_{r\to\pm\infty} \psi(r) = -\infty \quad \text{and} \quad \limsup_{r\to\pm\infty} \psi(r) = +\infty. \tag{5}$$

The authors found in this case two sequences of solutions, made up of local minima and mountain pass points, respectively. This result was extended to arbitrary simple eigenvalues in [3], where essentially the same condition as in [2] was again responsible for the existence of two sequences of solutions (intuitively having Morse index $n-1$ and n respectively, n being the dimension of the negative eigenspace).

In these papers resonance occurs, as we have described, at a *simple* eigenvalue. A natural question at this point is to ask what happens at multiple eigenvalues. This question is particularly relevant for the periodic problem associated to an ordinary differential equation, in which all eigenvalues (except λ_1) are double. This being the main motivation of this paper, we consider from now on only the periodic problem, which we write to be more consistent with the literature as

$$(6)\qquad \begin{cases} \ddot u + n^2 u + g(t,u) = 0 \quad \text{in } (0,2\pi), \\ u(0) = u(2\pi), \\ \dot u(0) = \dot u(2\pi). \end{cases}$$

Here g is a bounded Carathéodory function, 2π-periodic in t. Solutions to (6) arise naturally as critical points of the functional

$$f(u) = \frac{1}{2}\int_0^{2\pi} \dot u^2\,dt - \frac{n^2}{2}\int_0^{2\pi} u^2\,dt - \int_0^{2\pi} G(t,u)\,dt$$

in the space $H_{2\pi}$ of 2π-periodic H^1-functions.

Since existence results hold under conditions like (3) and (4), appropriately reformulated, we will turn our attention towards oscillation conditions in the spirit of (5).

Let $H^0 = \operatorname{span}\{\cos nt, \sin nt\} \simeq \mathbb{R}^2$ be the eigenspace associated to n^2, and define on it the function

$$\varphi(a,b) = \int_0^{2\pi} G(t, a\cos nt + b\sin nt)\,dt.$$

When $G(t,u)$ does not depend on time (we write $G(u)$ in this case), the function φ is radially symmetric since, as it is easy to see,

$$\varphi(a,b) = \psi(\sqrt{a^2+b^2}),$$

where of course

$$\psi(r) = \int_0^{2\pi} G(r\sin nt)\,dt.$$

The natural way to extend (5) is then to ask that

$$(7)\qquad \liminf_{r\to\infty} \psi(r) = -\infty \quad \text{and} \quad \limsup_{r\to\infty} \psi(r) = +\infty,$$

giving rise to a sequence of concentric circles in H^0 where the values of

$$\int_0^{2\pi} G(u_0(t))\,dt$$

are alternately very high and very low, in an uniform way.

When $G(t,u)$ depends on time the radial symmetry on H^0 is broken. However, we expect the functional to behave in a similar way on H^0 if we require some uniformity in the oscillations, namely

$$(8)\qquad \liminf_{r\to\infty} \psi^+(r) = -\infty \quad \text{and} \quad \limsup_{r\to\infty} \psi^-(r) = +\infty,$$

where, denoting B_r^0 the ball of radius r centered at zero in H_0, we have set

$$\psi^-(r) = \inf_{u_0\in\partial B_r^0} \int_0^{2\pi} G(t,u_0(t))\,dt$$

and

$$\psi^+(r) = \sup_{u_0 \in \partial B_r^0} \int_0^{2\pi} G(t, u_0(t))\, dt.$$

As we will see in Section 2, condition (8) is satisfied under the same hypotheses as in [2] and [3]. Moreover, in spite of the fact that the arguments of these papers do not apply to our situation (λ_n is not simple), the breaking of the radial symmetry described above suggests even stronger multiplicity results.

The main difficulty and, we think, the main point of interest of the present paper, consists in trying to embody these heuristic ideas in concrete variational arguments.

The result we obtain is the existence of four infinite families of solutions, in the following sense. We will explicitly construct two sequences of pairwise disjoint bounded set, say X_k and X'_k for every integer k, and two numbers $c_* < d_*$ (independent of k) such that

1. each set X_k contains at least two different solutions which have, in some intuitive way, Morse indices $2n-3$ (the dimension of the negative eigenspace) and $2n-2$, and whose level is low, i.e. smaller than c_*,
2. each set X'_k contains at least two different solutions which have Morse indices $2n-2$ and $2n-1$, and whose level is high, i.e. greater than d_*.

These couples of solutions come from dimensionally different variational principles, one of which is somewhat unusual. We briefly sketch the main ideas we followed to construct it.

The underlying "radial" symmetry of the problem suggests the use of Lusternik–Schnirelman category in order to obtain multiplicity results and, due to the indefiniteness of the action functional, the most convenient approach consists in working with the relative category (see [4], [7]). The main problem here is that, while the set X'_k is invariant with respect to a standard deformation flow (see Section 4), the set X_k is not, since, as we will see, the supposedly critical levels are attained also on its boundary.

This is the reason why the methods in [7] do not apply directly; to overcome this problem we introduce an unusual minimax argument in connection with a nonstandard deformation flow, tailored to deal with the geometrical properties of the functional.

The paper is organized as follows: in Section 2 we state the precise assumptions and the main result, and we exhibit some classes of potentials satisfying the hypotheses. Section 3 contains the main arguments and the proof of the existence of the first families of solutions. Finally, in Section 4 we construct the second families of solutions.

2. The main result

Consider the periodic problem

$$\begin{cases} \ddot{u} + n^2 u + g(t,u) = 0, \\ u(0) = u(2\pi), \\ \dot{u}(0) = \dot{u}(2\pi), \end{cases} \tag{9}$$

where $n \in \mathbb{N}$ and $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

(H1) g is a Carathéodory function, i.e.
$g(\,\cdot\,, u)$ is measurable for all $u \in \mathbb{R}$ and
$g(t, \,\cdot\,)$ is continuous for almost all $t \in [0, 2\pi]$;

(H2) there exists $k \in L^2$ such that for almost all t and all u, $|g(t,u)| \le k(t)$.

Consider next the Hilbert space $H_{2\pi} = \{u \in H^1(0,2\pi) \mid u(0) = u(2\pi)\}$ with scalar product $(u,v) = \int_0^{2\pi} (\dot{u}\dot{v} + uv)\,dt$ and norm $||u|| = (|\dot{u}|_2^2 + |u|_2^2)^{1/2}$. Here $|u|_2$ is the usual norm in L^2. It is well-known that the solutions of (9) correspond to critical points of the functional $f : H_{2\pi} \to \mathbb{R}$ defined by

$$f(u) = \frac{1}{2}\int_0^{2\pi} \dot{u}^2\,dt - \frac{n^2}{2}\int_0^{2\pi} u^2\,dt - \int_0^{2\pi} G(t,u)\,dt,$$

where $G(t,u) = \int_0^u g(t,s)\,ds$.

Let us decompose the space $H_{2\pi}$ into orthogonal complements

$$H_{2\pi} = H^- \oplus H^0 \oplus H^+,$$

where

$$\begin{aligned} H^- &= \operatorname{span}\{1, \cos t, \sin t, \dots, \cos(n-1)t, \sin(n-1)t\}, \\ H^0 &= \operatorname{span}\{\cos nt, \sin nt\}, \\ H^+ &= \overline{\operatorname{span}}\{\cos(n+1)t, \sin(n+1)t, \dots\}. \end{aligned}$$

In the following, we write $u \in H_{2\pi}$ as $u = u_- + u_0 + u_+$, where $u_- \in H^-$, $u_0 \in H^0$ and $u_+ \in H^+$.

The functional f has the following geometric features.

PROPOSITION 1. *Assume that $n \in \mathbb{N}$ and that g satisfies* (H1) *and* (H2). *Then there exists $\sigma > 0$ such that, for every $u \in H_{2\pi}$,*

(i) $||u_-|| \ge \sigma$ *implies* $\nabla f(u)u_- < 0$,
(ii) $||u_+|| \ge \sigma$ *implies* $\nabla f(u)u_+ > 0$.

PROOF. Claim (i) follows from the fact that for some $\alpha > 0$

$$\begin{aligned} \nabla f(u)u_- &= \int_0^{2\pi} (\dot{u}\dot{u}_- - n^2 u u_-)\,dt - \int_0^{2\pi} g(t,u)u_-\,dt \\ &\le -\alpha(|\dot{u}_-|_2^2 + |u_-|_2^2) + |k|_2|u_-|_2 \le -\alpha||u_-||^2 + |k|_2||u_-||. \end{aligned}$$

Claim (ii) follows in a similar way. □

A central role will be played by the set

$$\{u \in H_{2\pi} \mid ||u_-|| \le \sigma,\ ||u_+|| \le \sigma\}$$

where $\sigma > 0$ is given as in the previous proposition. Indeed, no critical point of f can lie outside this set, and it has some special invariance properties (with respect to deformation flows) which will be relevant when looking for critical points of f. Of course, suitable level estimates in that region are necessary in order to use minimax arguments. These estimates will be provided in the following propositions.

Given $0 < r_1 < r_2$, we define

$$C^0_{r_1 r_2} := \{u_0 \in H^0 \mid r_1 \le ||u_0|| \le r_2\}.$$

We also write

$$B^-_r, \quad B^0_r, \quad B^+_r$$

for closed balls, centered at the origin, of radius r, respectively in the spaces H^-, H^0 and H^+, and

$$\partial B^-_r, \quad \partial B^0_r, \quad \partial B^+_r$$

for the corresponding spheres.

Proposition 2. *Let $0 < r_1 < r_2$, $\sigma > 0$, $c_* \in \mathbb{R}$, $n \in \mathbb{N}$ and assume g satisfies assumptions* (H1) *and* (H2). *Then, there exists $\tau > 0$ such that*

$$\sup_{\partial B^-_\tau \times C^0_{r_1 r_2} \times B^+_\sigma} f \le c_*.$$

Proof. For some $\beta > 0$ and every $u \in \partial B^-_\tau \times C^0_{r_1 r_2} \times B^+_\sigma$, we compute

$$\begin{aligned} f(u) &= \frac{1}{2}\int_0^{2\pi} (\dot u_-^2 - n^2 u_-^2)\,dt + \frac{1}{2}\int_0^{2\pi} (\dot u_+^2 - n^2 u_+^2)\,dt \\ &\quad - \int_0^{2\pi} (G(t, u_- + u_0 + u_+) - G(t, u_0))\,dt - \int_0^{2\pi} G(t, u_0)\,dt \\ &\le -\beta ||u_-||^2 + \sup_{B^+_\sigma} \frac{1}{2}\int_0^{2\pi} (\dot u_+^2 - n^2 u_+^2)\,dt + |k|_2(|u_-|_2 + |u_+|_2) \\ &\quad + 2\pi \max_{[0,2\pi]} [\max_{C^0_{r_1 r_2}} |G(\,\cdot\,, u_0)|\,] \le c_*, \end{aligned} \tag{10}$$

which holds true if $\tau = ||u_-||$ is large enough. □

To describe the oscillating behavior of the action functional over H^0, we define the two functions

$$\psi^-(r) = \inf_{u_0 \in \partial B^0_r} \int_0^{2\pi} G(t, u_0(t))\,dt$$

and

$$\psi^+(r) = \sup_{u_0 \in \partial B_r^0} \int_0^{2\pi} G(t, u_0(t))\, dt.$$

Our main assumption is the following.

(H3) The functions $\psi^\pm$ satisfy

$$\liminf_{r\to\infty} \psi^+(r) = -\infty \quad \text{and} \quad \limsup_{r\to\infty} \psi^-(r) > -\infty.$$

Remark 1. Assumption (H3) can be changed into

(H3*) The functions $\psi^\pm$ satisfy

$$\liminf_{r\to\infty} \psi^+(r) < +\infty \quad \text{and} \quad \limsup_{r\to\infty} \psi^-(r) = +\infty.$$

As we will see in the proofs, the main condition in assumption (H3) is that the oscillations are large enough. Therefore (H3) can be relaxed into (H3*).

The following proposition gives sufficient conditions on the nonlinearity in order that (H3) be satisfied. These are modeled on analogous ones from [2] and [3].

Proposition 3. *Assume $g(t,u) = p(t)h(u) + e(t)$, where $p \in L^\infty$, h is continuous and $e \in L^2$ are such that for some α and β, m and M in $\mathbb{R}$,*

1. $0 < \alpha \le p(t) \le \beta$,
2. $e \in H^- \oplus H^+$,

(1) $-m \le \liminf_{|u|\to\infty} \dfrac{H(u)}{u} \le \limsup_{|u|\to\infty} \dfrac{H(u)}{u} \le M$, *where* $H(u) = \int_0^u h(s)\, ds$.

Define, for r, $s > 0$, the sets

$$A_s^+(r) := \{u \in \mathbb{R} \mid |u| \le r,\ H(u) \ge s|u|\},$$
$$A_s^-(r) := \{u \in \mathbb{R} \mid |u| \le r,\ H(u) \le -s|u|\},$$

and assume there is $s > 0$ such that

$$\limsup_{r\to\infty} \frac{\operatorname{meas}(A_s^+(r))}{2r} > \sqrt{1 - \left(\frac{s\alpha}{s\alpha + m\beta}\right)^2},$$
$$\limsup_{r\to\infty} \frac{\operatorname{meas}(A_s^-(r))}{2r} > \sqrt{1 - \left(\frac{s\alpha}{s\alpha + M\beta}\right)^2}.$$

Then assumptions (H3) (*and* (H3*)) *are satisfied.*

Proof. A function $u_0 \in \partial B_r^0$ reads

$$u_0(t) = \sqrt{2\pi} r \sin(nt + \varphi).$$

Extending p by periodicity, we compute

$$\begin{aligned}\int_0^{2\pi} G(t,u_0(t))\,dt &= \int_0^{2\pi} p(t)H(\sqrt{2\pi}r\sin(nt+\varphi))\,dt \\ &= \int_0^{2\pi} p(t-\frac{\varphi}{n})H(\sqrt{2\pi}r\sin nt)\,dt.\end{aligned}$$

It follows that

$$\sup_{\partial B_r^0}\int_0^{2\pi} G(t,u_0(t))\,dt \le \int_0^{2\pi} \widetilde{p}(t,r)H(\sqrt{2\pi}r\sin nt)\,dt,$$

where

$$\widetilde{p}(t,r) = \begin{cases} \alpha & \text{if } (t,r) \text{ is such that } H(\sqrt{2\pi}r\sin nt) < 0, \\ \beta & \text{if } (t,r) \text{ is such that } H(\sqrt{2\pi}r\sin nt) \ge 0. \end{cases}$$

Repeating the proof of Proposition 1 in [2] we obtain then

$$\liminf_{r\to\infty}\sup_{u_0\in\partial B_r^0}\int_0^{2\pi} G(t,u_0(t))\,dt = \liminf_{r\to\infty}\int_0^{2\pi}\widetilde{p}(t,r)H(\sqrt{2\pi}r\sin nt)\,dt = -\infty.$$

In a similar way it can be proved that

$$\limsup_{r\to\infty}\inf_{u_0\in\partial B_r^0}\int_0^{2\pi} G(t,u_0(t))\,dt = +\infty. \qquad \square$$

We now describe the basic geometrical framework that we will use in the main proof.

Proposition 4. *Let $\mu > 0$, $\nu > 0$, $n \in \mathbb{N}$ and assume that g satisfies assumptions* (H1), (H2) *and* (H3). *Then there exist two ordered sequences $(R_k)_k$ and $(r_k)_k$, going to infinity such that*

$$\ldots < R_k < r_k < R_{k+1} < \ldots$$

and that for some $c_* < d_*$ and all $k \in \mathbb{N}$,

$$\sup_{H^-\times\partial B_{r_k}^0\times B_\mu^+} f \le c_* < d_* \le \inf_{B_\nu^-\times\partial B_{R_k}^0\times H^+} f.$$

Proof. From computations similar to (10), we deduce that there exist positive constants a and A such that

$$f(u) \ge \frac{1}{2}\int_0^{2\pi}(\dot{u}_-^2 - nu_-^2)\,dt + a||u_+||^2 - |k|_2(||u_-|| + ||u_+||) - \int_0^{2\pi} G(t,u_0(t))\,dt$$

and

$$\inf_{B_\nu^-\times\partial B_R^0\times H^+} f \ge -A - \sup_{\partial B_R^0}\int_0^{2\pi} G(t,u_0(t))\,dt.$$

It follows then from (H3) that

$$\limsup_{R\to\infty}[\inf_{B_\nu^-\times\partial B_R^0\times H^+} f] \ge -A - \liminf_{R\to\infty}\psi^+(R) = +\infty. \tag{11}$$

In a similar way, there exists a constant $B > 0$ such that

$$\liminf_{r\to\infty}[\sup_{H^-\times\partial B^0_r\times B^+_\mu} f] \le B - \limsup_{r\to\infty}\psi^-(r) < +\infty. \tag{12}$$

The claim follows now from (11) and (12). □

We now state our main result.

Theorem 5. *Let $n \in \mathbb{N}$, $n > 0$ and assume $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ satisfies* (H1)–(H3). *Then there exist $\sigma > 0$, two numbers $c_* < d_*$ and two increasing sequences $(R_k)_k$, $(r_k)_k$ going to infinity, such that*

1. *each set $B^-_\sigma \times C^0_{R_kR_{k+1}} \times B^+_\sigma$ contains two different solutions, u_k and v_k of* (9) *such that*
$$f(u_k), f(v_k) \le c_*,$$
2. *each set $B^-_\sigma \times C^0_{r_kr_{k+1}} \times B^+_\sigma$ contains two different solutions, x_k and y_k of* (9) *such that*
$$f(x_k), f(y_k) \ge d_*.$$

Remark 2. The claims of Theorem 5 imply that as $k \to \infty$, the solutions u_k, v_k, x_k and y_k "look" like eigenfunctions $A \sin nt + B \cos nt$. See the conclusion of the proof in the next section and [2], [3] for similar results.

3. Existence of the solutions u_k and v_k

3.1. Construction of boxes X. We first choose $\sigma > 0$ from Proposition 1. Next we take sequences $(R_k)_k$ and $(r_k)_k$, with $\ldots < R_k < r_k < R_{k+1} < \ldots$, and two numbers $c_* < d_*$ according to Proposition 4 where $\mu = \sigma$ and $\nu = 2\sigma$. We fix one value of k and to simplify the notations we write $R = R_k$, $r = r_k$ and $S = R_{k+1}$. Summing up, we know that

$$\sup_{H^-\times\partial B^0_r\times B^+_\sigma} f \le c_* < d_* \le \inf_{B^-_{2\sigma}\times\partial C^0_{RS}\times H^+} f. \tag{13}$$

We will show that, due to this inequality and to the behavior of ∇f over $B^-_\sigma \times H^0 \times H^+$ and $H^- \times H^0 \times B^+_\sigma$, the set

$$X := B^-_{2\sigma} \times C^0_{RS} \times B^+_\sigma$$

contains two different solutions u and v. To this aim category arguments will be used in connection with a non trivial choice of a gradient flow. The non triviality is essentially due to the fact that ∂X contains points where the functional f is low (for instance $\partial B^-_{2\sigma}\times\partial B^0_r\times\{0\}$), as well as points where it is high (for instance $\{0\}\times\partial C^0_{RS}\times H^+$). Therefore we cannot expect to separate any interesting level from that ones f attains on ∂X, and the methods of [7] do not apply.

3.2. Definition of minimax classes. Define

$$D := B^-_{2\sigma} \times \partial B^0_r \times \{0\} \subset X,$$

its relative boundary

$$Y := \partial D = \partial B^-_{2\sigma} \times \partial B^0_r \times \{0\}$$

and

$$Z := B^-_\sigma \times H^0 \times H^+,$$

and consider the minimax classes

$$\Gamma_1 := \{A \subset X \mid A = \overline{A},\ Y \subset A,\ \operatorname{cat}_{X,Y}(A) \geq 1\},$$
$$\Gamma_2 := \{A \subset X \mid A = \overline{A},\ Y \subset A,\ \operatorname{cat}_{X,Y}(A) \geq 2\}.$$

By $\operatorname{cat}_{X,Y}(A)$ we denote the relative category of A in X with respect to Y; see for instance [4] for its definition and main properties.

We claim that the class $\Gamma_2 \subset \Gamma_1$ is not empty. To see this we show that $D \in \Gamma_2$, i.e. $\operatorname{cat}_{X,Y}(D) \geq 2$. To this aim, note that

$$h(u) = u_- + \frac{r}{||u_0||} u_0, \quad u \in X$$

retracts X on D. Then the convex combination $\Phi(u,\lambda) = (1-\lambda)u + \lambda h(u)$ deforms X into D keeping Y fixed, proving that $\operatorname{cat}_{X,Y}(D) = \operatorname{cat}_{D,Y}(D)$. Further, D is topologically equivalent to a torus T and Y to its boundary ∂T. Hence, $\operatorname{cat}_{D,Y}(D) = \operatorname{cat}_{T,\partial T}(T) = 2$ (see [4], [7]).

3.3. Critical levels and critical sets. Define the levels

$$c_i = \inf_{\Gamma_i \cap Z} \sup f, \quad i = 1, 2,$$

where $\Gamma_i \cap Z = \{A \cap Z \mid A \in \Gamma_i\}$, and the corresponding sets $K_i \subset X$ of critical points in X at level c_i. In what follows we write $K = K_1 \cup K_2$.

Remark 3. Note that $\Gamma_i \cap Z$ is *not* a minimax class in the usual sense, since it is not invariant along any gradient flow. As we will see, in spite of that, due to the invariance properties of Z the points whose level is close to c_i and whose gradient is close to zero cannot all leave Z. This is what we really need to construct Palais–Smale sequences; in fact, in the proof we will adopt a slightly different point of view, which is more convenient in order to obtain multiplicity results.

By construction, we clearly have $c_2 \geq c_1$; our aim is to show that in both cases $c_2 > c_1$ and $c_2 = c_1$ the set K contains at least two points.

First of all we show that the definition of the levels makes sense, namely that $c_1 > -\infty$. Since it is easy to recognize that $\inf_X f > -\infty$, all we have to do is to prove that $A \cap Z \neq \emptyset$ for every $A \in \Gamma_1$.

Claim. *Each* $A \in \Gamma_1$ *intersects* $W := \{0\} \times C^0_{RS} \times B^+_\sigma \subset Z$.

Assume for contradiction that $A \cap W = \emptyset$, and note that in this case A can be retracted on Y by means of

$$h(u) = \frac{2\sigma}{||u_-||} u_- + \frac{r}{||u_0||} u_0, \quad u \in X \setminus W.$$

Thus the convex combination $\Phi(u, \lambda) = (1-\lambda)u + \lambda h(u)$ deforms A into Y, keeping Y fixed, and well-known properties of the relative category give $\mathrm{cat}_{X,Y}(A) = \mathrm{cat}_{Y,Y}(Y) = 0$, which violates the definition of A.

Claim. $\mathrm{dist}(K, \partial X) > 0$.

To see this note that since Since K is compact (because f satisfies the Palais–Smale condition on bounded sets), it is enough to prove that $K \cap \partial X = \emptyset$.

Now there is no critical point on $\partial B^-_{2\sigma} \times C^0_{RS} \times B^+_\sigma$, as we chose σ such that on this set $\nabla f(u)u^- < 0$. In a similar way, there is no critical point on $B^-_{2\sigma} \times C^0_{RS} \times \partial B^+_\sigma$, since for such points $\nabla f(u)u^+ > 0$. Finally, due to (13) and to the obvious inequality

$$c_1 \le c_2 \le \sup_{D \cap Z} f \le \sup_{H^- \times \partial B^0_r \times B^+_\sigma} \le c_*, \tag{14}$$

the set $B^-_{2\sigma} \times \partial C^0_{RS} \times B^+_\sigma$ cannot intersect K because there the level is too high (recall (13)).

3.4. The deformation flow. Let c denote either c_1 or c_2. Define

$$N_\varrho = \{u \mid \mathrm{dist}(u, K) \le \varrho\}$$

(the empty set if $K = \emptyset$), and choose $\varrho > 0$ so small that $\varrho < \sigma$, $N_{2\varrho} \subset X$ and

$$\mathrm{cat}_{X,Y}(N_{2\varrho}) = \mathrm{cat}_{X,Y}(K).$$

Since the Palais-Smale condition holds on bounded sets, we can take a positive $\delta < (d_* - c_*)/2$ such that for every $u \in X \setminus N_\varrho$,

$$|f(u) - c| + ||\nabla f(u)|| \ge 2\delta. \tag{15}$$

Choose now two C^∞ cut-off functions α and $\beta : [0, \infty[\to [0, 1]$ such that

$$\alpha(s) = \begin{cases} 1 & \text{if } s \le \delta, \\ 0 & \text{if } s \ge 2\delta, \end{cases} \quad \text{and} \quad \beta(s) = \begin{cases} 1 & \text{if } s \le \sigma + \varrho, \\ 0 & \text{if } s \ge 2\sigma, \end{cases}$$

and consider the flow $\eta_t : H \to H$ defined as the value at time t of the solution of the Cauchy problem

$$\begin{cases} y' = -\alpha(|f(y) - c|)\beta(||y_-||) \dfrac{\nabla f(y)}{1 + ||\nabla f(y)||}, \\ y(0) = u. \end{cases}$$

One can easily check global existence, uniqueness and continuity of the solutions of this system.

Note that for all $t \geq 0$, η_t is the identity on Y; this is obvious since on Y, $||u_-|| = 2\sigma$ and therefore $\beta(||u_-||) = 0$, so that $\eta_t(u) = u$.

Claim. *For all $t \geq 0$, $\eta_t(X) \subset X$.*

Notice first that flow lines with initial conditions in X cannot cross $\partial B^-_{2\sigma} \times C^0_{RS} \times B^+_\sigma$, as $\beta(||u_-||) = 0$ if $||u_-|| \geq 2\sigma$.

In a similar way, these flow lines cannot cross $B^-_{2\sigma} \times C^0_{RS} \times \partial B^+_\sigma$, since there $\nabla f(u)u^+ > 0$, so that the vector field points inward.

Finally, they cannot reach $B^-_{2\sigma} \times \partial C^0_{RS} \times B^+_\sigma$, since there the level is too high. Indeed combining (13) and (14) we obtain that $|f(u) - c| \geq d_* - c_* \geq 2\delta$ for every $u \in B^-_{2\sigma} \times \partial C^0_{RS} \times H^+$; thus $\alpha(|f(u) - c|) = 0$ and $\eta_t(u) = u$.

3.5. If $c_1 = c_2$, $\mathrm{cat}_X(K) \geq 2$, i.e. the set K is infinite. Assume for contradiction that

$$\mathrm{cat}_X(K) < 2.$$

Firstly choose $\varepsilon > 0$ such that $\varepsilon < \delta$, $\varepsilon < \varrho\delta^2/2(1+\delta)$, and take some $A \in \Gamma_2$ such that

$$\sup_{A\cap Z} \leq c + \varepsilon.$$

By the properties of the category, we have

$$2 \leq \mathrm{cat}_{X,Y}(A) \leq \mathrm{cat}_{X,Y}(A \setminus N_{2\varrho}) + \mathrm{cat}_X(N_{2\varrho}) \leq \mathrm{cat}_{X,Y}(A \setminus N_{2\varrho}) + 1.$$

Therefore $\mathrm{cat}_{X,Y}(A \setminus N_{2\varrho}) \geq 1$, which implies that for every $t \geq 0$

$$\mathrm{cat}_{X,Y}(\eta_t(A \setminus N_{2\varrho})) \geq 1,$$

so that $\eta_t(A \setminus N_{2\varrho}) \in \Gamma_1$.

Let $u \in \eta_\varrho(A \setminus N_{2\varrho}) \cap Z$ be such that $f(u) > c - \varepsilon$. Such a point exists since $\sup\{f(u) \mid u \in \eta_\varrho(A \setminus N_{2\varrho}) \cap Z\} \geq c$. Next, pick $x \in A \setminus N_{2\varrho}$ such that $u = \eta_\varrho(x)$. Notice that, from Proposition 1, Z is negatively invariant for η_t, so that $u \in Z$ implies $x \in Z$.

Since f decreases along the flow, for every $t \in [0, \varrho]$ we have $f(\eta_t(x)) \in [f(u), f(x)] \subset [c - \varepsilon, c + \varepsilon]$, that is, $|f(\eta_t(x)) - c| \leq \varepsilon < \delta$ and therefore

$$\alpha(|f(\eta_t(x)) - c|) = 1$$

for all $t \in [0, \varrho]$. From (15) we also deduce that

$$||\nabla f(\eta_t(x))|| \geq \delta.$$

Notice now that $\eta_t(x)$ is one-Lipschitz in t so that, for every $t \in [0, \varrho]$ there results $||\eta_t(x) - x|| \leq t \leq \varrho$. Since $x \in Z = B_\sigma^- \times H^0 \times H^+$, we also have, for such t,

$$\beta(||(\eta_t(x))_-||) = 1.$$

Using the above inequalities, we compute

$$\begin{aligned} c - \varepsilon < f(u) &= f(x) + \int_0^\varrho \frac{d}{dt} f(\eta_t(x))\, dt \\ &= f(x) - \int_0^\varrho \frac{||\nabla f(\eta_t(x))||^2}{1 + ||\nabla f(\eta_t(x))||}\, dt \leq c + \varepsilon - \varrho \frac{\delta^2}{1+\delta}, \end{aligned}$$

which contradicts the choice of ε.

3.6. If $c_1 \neq c_2$, the set K_i are not empty. Assume for contradiction that $K_i = \emptyset$ and repeat the very same arguments of Part 5 after choosing $N_\varrho = N_{2\varrho} = \emptyset$.

4. Existence of the solutions x_k and y_k

The proof of existence of the solutions x_k and y_k is very similar to the proof in the last section. We only indicate the necessary changes.

4.1. Construction of boxes X. Take all constants (σ, μ, ν, R_k, r_k, c_*, d_*) exactly as in the previous section; then fix one value of k and, to simplify the notations, write $r = r_{k-1}$, $s = r_k$ and $R = R_k$. Then by construction we know that

$$\sup_{H^- \times \partial C_{rs}^0 \times B_\sigma^+} f \leq c_* < d_* \leq \inf_{B_{2\sigma}^+ \times \partial B_R^0 \times H^+} f. \tag{16}$$

Next choose $\tau > 0$ according to Proposition 2 such that

$$\sup_{\partial B_\tau^- \times C_{rs}^0 \times B_\sigma^+} f \leq c_*. \tag{17}$$

We will show that, due to these inequalities and to the behaviour of ∇f on $H^- \times H^0 \times \partial B_\sigma^+$, the set $X := B_\tau^- \times C_{rs}^0 \times B_\sigma^+$ contains two different solutions x and y.

4.2. Definition of minimax classes. Define the sets

$$\begin{aligned} D &:= B_\tau^- \times C_{rs}^0 \times \{0\}, \\ Y &:= \partial D = \partial B_\tau^- \times C_{rs}^0 \times \{0\} \cup B_\tau^- \times \partial C_{rs}^0 \times \{0\} \subset X \end{aligned}$$

and consider the minimax classes

$$\begin{aligned} \Gamma_1 &:= \{A \subset X \mid A = \overline{A},\, Y \subset A,\, \mathrm{cat}_{X,Y}(A) \geq 1\}, \\ \Gamma_2 &:= \{A \subset X \mid A = \overline{A},\, Y \subset A,\, \mathrm{cat}_{X,Y}(A) \geq 2\}. \end{aligned}$$

Once more, one can prove that the the classes are not empty; more precisely, $D \in \Gamma_2 \subset \Gamma_1$.

4.3. Critical levels and critical sets. Define the levels

$$c_i = \inf_{\Gamma_i} \sup f, \quad i = 1, 2,$$

and the corresponding sets $K_i \subset X$ of critical points in X, at level c_i. We write $K = K_1 \cup K_2$.

The sets K_i are easily shown to be compact, and it can be proved that

$$K \cap \partial X = \emptyset \quad \text{and} \quad c_2 \geq c_1 \geq d_*,$$

as a consequence of the estimates (16) and (17) in connection with the following intersection property.

Claim. *Each $A \in \Gamma_1$ intersects $W := \{0\} \times \partial B_R^0 \times B_\sigma^+$.*

First of all, for every $u \in X \setminus W$ define $h(u) \in Y$ to be the intersection with Y of the half line

$$\frac{R}{||u_0||} u_0 + \mu \left(u_- + u_0 - \frac{R}{||u_0||} u_0 \right), \quad \mu \geq 0.$$

Now if we had $A \cap W = \emptyset$, then the convex combination $\Phi(u, \lambda) = (1-\lambda)u + \lambda h(u)$ would deform A on Y, keeping Y fixed. Then we would have $\operatorname{cat}_{X,Y}(A) = 0$, contradicting the definition of A.

4.4. The deformation flow. Let $c = c_1$ or c_2 and define

$$N_\varrho = \{u \mid \operatorname{dist}(u, K) \leq \varrho\}.$$

Choose $\varrho > 0$ and $\delta > 0$ as in 3.4, pick a C^∞ cut-off function $\alpha : [0, \infty[\to [0, 1]$ such that

$$\alpha(s) = \begin{cases} 1 & \text{if } s \leq \delta, \\ 0 & \text{if } s \geq 2\delta, \end{cases}$$

and consider the flow $\eta_t : H \to H$ defined from the Cauchy problem

$$\begin{cases} y' = -\alpha(|f(y) - c|) \dfrac{\nabla f(y)}{1 + ||\nabla f(y)||}, \\ y(0) = u. \end{cases}$$

At this point all the ingredients to repeat the proof of the last section are available; we omit the details.

Conclusion. The above arguments show, as in the previous section, the existence of (a sequence of) couples of critical points having bounded H^- and H^+ components and unbounded H^0 components; this explains Remark 2 (see also [3]).

Acknowledgments. This work was partly carried out during the stay of the two last authors at Université Catholique de Louvain. These authors wish to express their gratitude to this institution for the invitation.

REFERENCES

[1] S. AHMAD, A. C. LAZER AND J. L. PAUL, *Elementary critical point theory and perturbations of elliptic boundary value problems at resonance*, Indiana Univ. Math. J. **25** (1976), 933–944.

[2] P. HABETS, R. MANASEVICH AND F. ZANOLIN, *A nonlinear boundary value problem with potential oscillating around the first eigenvalue*, J. Differential Equations **117** (1995), 428–445.

[3] P. HABETS, E. SERRA AND M. TARALLO, *Multiplicity results for boundary value problems with potentials oscillating around resonance*, J. Differential Equations **138** (1997), 133–156.

[4] D. LUPO, *Patchwork on Lusternik–Schnirelman category, relative category and limit relative category*, preprint Politecnico di Milano, N. 10/R, October 1994.

[5] J. MAWHIN AND M. WILLEM, *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag, New York, 1989.

[6] P. H. RABINOWITZ, *Some minimax theorems and applications to nonlinear partial differential equations*, Nonlinear Analysis (Cesari, Kannan and Weinberge, eds.), Academic Press, New York, 1978, pp. 161–177.

[7] M. WILLEM, *Minimax Theorems*, Progr. Nonlinear Differential Equations Appl., vol. 24, Birkhäuser, Boston, Basel, Berlin, 1996.

Manuscript received November 16, 1998

PATRICK HABETS
Institut de Mathématiques Pures et Appliquées
Chemin du Cyclotron 2
B-1348 Louvain-la-Neuve, BELGIUM

ENRICO SERRA
Dipartimento di Matematica
Politecnico di Torino
Corso Duca degli Abruzzi 24
I–10129, Torino, ITALY

E-mail address: serra@polito.it

MASSIMO TARALLO
Dipartimento di Matematica
Università degli Studi di Milano
Via Saldini 50
I–20133, Milano, ITALY