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ON A THEOREM OF TVERBERG

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1. Introduction

Let Δ^n denote the n-dimensional simplex. Any face of Δ^n is assumed to be closed. The well-known theorem of Radon (see [6]) can be formulated as follows

Theorem (Radon). *For any linear map $f : \Delta^{n+1} \to \mathbb{R}^n$ there exist two disjoint faces σ_1, σ_2 of Δ^{n+1} such that $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$.*

In 1966 the Radon theorem was generalized by Tverberg in the following way (see [20]):

Theorem (Tverberg). *For any linear map $f : \Delta^N \to \mathbb{R}^n$, where $N = (p-1)(n+1)$, there exist p pairwise disjoint faces $\sigma_1, \ldots, \sigma_p \subset \Delta^N$ such that*

$$\bigcap_{i=1}^{p} f(\sigma_i) \neq \emptyset.$$

There is a natural question whether the linearity condition for f can be replaced by continuity. The first positive answer was given by Bajmóczy and Bárány in [1] for $p = 2$. Next Bárány, Shlosman and Szücs in [3] proved the theorem for p being a prime number. In 1992 Volovikov obtained the positive answer for any number which is a prime power (see [21]). In all papers mentioned above various generalizations of the classical Borsuk–Ulam antipodal theorem

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was used in an essential way. Recently Sarkaria [16] gave a proof for an arbitrary natural p.

The aim of this paper is to prove Volovikov theorem (see Theorem 1 in [21]) for multivalued maps (cf. [13] for $p = 2$). In our considerations we will need an appropriate version of Bourgin–Yang theorem (cf. [10]) which generalizes Borsuk–Ulam theorem (see Theorem 3). We consider also the case when Δ^N is replaced by an arbitrary N-dimensional compact and convex polytope in $\mathbb{R}^N$ (see Theorem 9).

2. G-spaces and the G-index

We are going to use cohomology of the Čech type. The Čech cohomology theory has a continuity property which says that if a cohomology class vanishes on a closed set, then it vanishes on a neighbourhood of this set as well. Throughout the paper the group $\mathbb{Z}_p$ of integers mod p, p prime, will be used as a coefficient group in cohomology.

Let G be the Cartesian product of n copies of the group $\mathbb{Z}_p$. We assume that G acts freely on a paracompact space X. We call X a *G-space*. Any such G-space admits an equivariant map $h : X \to EG$ into a classifying space EG; any two such maps are equivariantly homotopic (see [8, Theorems 8.12 and 6.14]). The map h induces a map $\widehat{h} : X/G \to BG := EG/G$ on the orbit spaces. Consequently one has a uniquely determined homomorphism

$$\widehat{h}^* : H^*(BG, \mathbb{Z}_p) \to H^*(X/G, \mathbb{Z}_p).$$

Let us recall the definition of the G-index $\mathrm{ind}_G X$, for a G-space X (see [21]).

Definition 1. We say that the *G-index of X* is not less than k, if the homomorphism $\widehat{h}^k : H^k(BG, \mathbb{Z}_p) \to H^k(X/G, \mathbb{Z}_p)$ is a monomorphism.

Most of the properties of the G-index are immediate consequences of the definition. In particular, *monotonicity* says:

$$\textit{if } G \textit{ acts freely on } X \textit{ and } Y, \textit{ and } f : X \to Y \textit{ is an equivariant map, then} \quad \mathrm{ind}_G Y \geq \mathrm{ind}_G X.$$

The *dimension property*:

$$\textit{if } \dim X < m, \ \textit{then } \mathrm{ind}_G X < m,$$

where dim denotes the covering dimension.

An important special case of the above says:

$$\textit{if } \mathrm{ind}_G X = 0, \ \textit{then } X \neq \emptyset.$$

As the consequence of the continuity property for the Čech cohomology we obtain the following *continuity property* for G-index:

let G act freely on X, $A \subset X$ is a compact G-space. Then there is an open neighbourhood U of A in X which is a G-space such that $\mathrm{ind}_G U = \mathrm{ind}_G A$.

The concept of the G-index was introduced by Yang [24] for $G = \mathbb{Z}_2$ and next extended to other more general settings by several authors, notably to actions of compact Lie groups by Fadell and Husseini [9].

3. Multivalued maps

Let X, Y be two spaces. We say that $\varphi : X \to Y$ is *a multivalued map* if for every point $x \in X$ a nonempty subset $\varphi(x)$ of Y is given. We associate with φ *the graph* to be the set

$$\Gamma_\varphi := \{(x, y) \in X \times Y \mid y \in \varphi(x)\}.$$

The *image* of a subset $A \subset X$ is the set $\varphi(A) := \bigcup_{x \in A} \varphi(x)$. For a subset $B \subset Y$ we can define two types of *a counterimage*:

$$\varphi^{-1}(B) := \{x \in X \mid \varphi(x) \subset B\}, \qquad \varphi_+^{-1}(B) := \{x \in X \mid \varphi(x) \cap B \neq \emptyset\}.$$

They both coincide if φ is a singlevalued map.

One defines a composition of $\varphi : X \to Y$ and $\psi : Y \to Z$ as a map $\gamma : X \to Z$ given by $\gamma(x) = \psi(\varphi(x))$.

A multivalued map $\varphi : X \to Y$ is *upper semicontinuous* (*u.s.c.*) provided

(i) for each $x \in X$ $\varphi(x) \subset Y$ is compact,
(ii) for every open subset $V \subset Y$ the set $\varphi^{-1}(V)$ is open in X.

Let us recall some basic properties of u.s.c. maps:

(1) The image of a compact set is a compact set.
(2) The graph Γ_φ is a closed subset of $X \times Y$.
(3) The composition of two u.s.c. maps is an u.s.c. map, too.

We would like to remind a class of admissible multivalued maps considered by Górniewicz [11].

We say that a space X is *acyclic* if $H^*(X) = H^*$ (*point*).

A continuous map $p : X \to Y$ is a *Vietoris map* if:

(i) $p(X) = Y$,
(ii) p is proper (i.e. $p^{-1}(A)$ is compact whenever $A \subset Y$ is compact),
(iii) for every $y \in Y$ the set $p^{-1}(y)$ is acyclic.

An important feature of Vietoris maps is the famous Vietoris–Begle Mapping Theorem (see [18]) which says that if X, Y are paracompact spaces and $p : X \to Y$ is a Vietoris map, then it induces an isomorphism on cohomology.

Definition 2. An u.s.c. map $\varphi : X \to Y$ is *admissible* provided there exists a space Γ, and two continuous maps $p : \Gamma \to X$, $q : \Gamma \to Y$ such that

(i) p is a Vietoris map,
(ii) for every $x \in X$ $q(p^{-1}(x)) \subset \varphi(x)$.

We call every such pair (p, q) of maps *a selected pair* for φ.

The class of admissible maps is very broad. It includes all u.s.c. maps with acyclic values (see [11]), and in particular with convex values, if Y is a normed space. Moreover, a composition of two admissible maps is also admissible ([11]). Many results from topological fixed point theory of singlevalued maps carry onto this class of maps.

4. Borsuk–Ulam type theorems

Our first result is a multivalued version of a Bourgin–Yang type theorem for the group $G = (\mathbb{Z}_p)^n$. This is a generalization of a theorem due to Volovikov (see [21]).

Theorem 3. *Let G and X be as above and assume that* $\operatorname{ind}_G X \geq k$. *For each admissible map $\varphi : X \to \mathbb{R}^m$ the G-index of the set*

$$A_\varphi = \left\{ x \in X : \bigcap_{g \in G} \varphi(gx) \neq \emptyset \right\}$$

is not less than $k - (p^n - 1)m$.

Proof. Let's denote by d the order of the group G, $d = p^n$. Let $\varphi : X \to \mathbb{R}^m$ be an admissible map. We consider a selected pair $X \xleftarrow{p} \Gamma \xrightarrow{q} \mathbb{R}^m$ for φ.

We choose a linear order in the set of all elements of G, $g_1 \prec g_2 \prec \ldots \prec g_d$. In the Cartesian product of d copies of Γ the coordinates of each point will be indexed by elements of G. This allows us to define an action of G on Γ^d. For any $g \in G$ we let $g(\gamma_{g_1}, \ldots, \gamma_{g_d}) = (\gamma_{gg_1}, \ldots, \gamma_{gg_d})$ and define a subset $\widetilde{X} \subset \Gamma^d$:

$$(*) \qquad \widetilde{X} = \{(\gamma_{g_1}, \ldots, \gamma_{g_d}) \in \Gamma^d : \exists x \in X \ p(\gamma_{g_i}) = g_i x, \ i = 1, \ldots, d\}.$$

Notice that for each $(\gamma_{g_1}, \ldots, \gamma_{g_d}) \in \widetilde{X}$ there is only one $x \in X$ satisfying the conditions of $(*)$. It is clear that $\widetilde{X}$ is a G-subset of Γ^d, and G acts freely on it. Consider the following diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\pi"] \arrow[d, "s"'] & \Gamma \arrow[dl, "p"] \arrow[dr, "q"] & \\ X & & \mathbb{R}^m \end{tikzcd}$$

where $\pi : \widetilde{X} \to \Gamma$ is the projection $\pi(\gamma_{g_1}, \dots, \gamma_{g_d}) = \gamma_{g_1}$ and $s : \widetilde{X} \to X$ is the composition $s = p \circ \pi$. One can see that s is a G-equivariant map:

$$\begin{aligned} s(g(\gamma_{g_1}, \dots, \gamma_{g_d})) &= s(\gamma_{gg_1}, \dots, \gamma_{gg_d}) = p \circ \pi(\gamma_{gg_1}, \dots, \gamma_{gg_d}) \\ &= p(\gamma_{gg_1}) = gg_1x = gp(\gamma_{g_1}) = gs(\gamma_{g_1}, \dots, \gamma_{g_d}). \end{aligned}$$

For each subset $A \subset X$,

$$s^{-1}(A) = [p^{-1}(g_1g_1^{-1}A) \times p^{-1}(g_2g_1^{-1}A) \times \dots \times p^{-1}(g_dg_1^{-1}A)] \cap \widetilde{X}.$$

In particular, if $A \subset X$ is compact then $s^{-1}(A)$ is compact and therefore s is a proper map. On the other hand, for every $x \in X$,

$$s^{-1}(x) = p^{-1}(x) \times p^{-1}(g_2g_1^{-1}x) \times \dots \times p^{-1}(g_dg_1^{-1}x),$$

and thus $s^{-1}(x)$ is an acyclic set as it is a Cartesian product of acyclic sets. Consequently we have shown that $s : \widetilde{X} \to X$ is a Vietoris map.

Now, we consider the following commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{s} & X \\ \downarrow & & \downarrow \\ \widetilde{X}/G & \xrightarrow{\widehat{s}} & X/G \end{array}$$

where $\widehat{s}$ is a map induced by s on the orbit spaces and vertical arrows denote the natural projections. Since G is a finite group and s is a Vietoris map, it follows that $\widehat{s}$ is also a Vietoris map. Hence the homomorphism $\widehat{s}^* : H^*(X/G) \to H^*(\widetilde{X}/G)$ is an isomorphism. In the diagram

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{s} & X & \xrightarrow{h} & EG \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{X}/G & \xrightarrow{\widehat{s}} & X/G & \xrightarrow{\widehat{h}} & BG \end{array}$$

h is an arbitrary G-equivariant map. By our assumptions $\widehat{h}^k : H^k(BG, \mathbb{Z}_p) \to H^k(X/G, \mathbb{Z}_p)$ is a monomorphism. The composition $h \circ s : \widetilde{X} \to EG$ is an equivariant map and $(\widehat{h \circ s})^k = \widehat{s}^k \circ \widehat{h}^k : H^k(BG, \mathbb{Z}_p) \to H^k(\widetilde{X}/G, \mathbb{Z}_p)$ is a monomorphism, thus $\mathrm{ind}_G\widetilde{X} \geq k$.

Now, applying Volovikov Theorem (see [21]) to the map $f = q \circ \pi : \widetilde{X} \to \mathbb{R}^m$, we find that the index of A_f, $\mathrm{ind}_G A_f \geq k - (d-1)m$. Thus by monotonicity property of the index we obtain

$$\mathrm{ind}_G s(A_f) \geq k - (d-1)m.$$

We check that $s(A_f)$ is a G-subset of A_φ. Let's take $x \in s(A_f)$. There is a point $(\gamma_{g_1}, \dots, \gamma_{g_d}) \in \widetilde{X}$ such that $x = s(\gamma_{g_1}, \dots, \gamma_{g_d}) = p(\gamma_{g_1})$ and $q(\gamma_{g_1}) = q(\gamma_g)$ for every $g \in G$. On the other hand we have $q(\gamma_{gg_1}) \in q(p^{-1}(gx)) \subset \varphi(gx)$. Hence

$$q(\gamma_{g_1}) \in \bigcap_{g\in G} \varphi(gx)$$

and therefore $x \in A_\varphi$. Again by the monotonicity property of the index we obtain the inequality $\mathrm{ind}_G A_\varphi \geq k-(d-1)m$ which completes the proof. $\square$

Our next problem is devoted to a topological generalization of Tverberg theorem (see [3] and [22]). We are going to generalize Theorem 1 in [22] to multivalued maps. We begin with the following:

PROPOSITION 4. *Let X be a paracompact G-space such that $X^G = \emptyset$. We assume that X is l-acyclic, i.e. $\widetilde{H}^k(X,\mathbb{Z}_p) = 0$ for $k = 0,\dots,l$. Then* $\mathrm{ind}_G X \geq l+1$.

For the proof see e.g. [22].

Denote by d the order of a group $G = \mathbb{Z}_p^n$, $d = p^n$. For $N = (d-1)(m+1)$, we denote by Δ^N the N-dimensional simplex and by $\partial\Delta^N$ its boundary.

THEOREM 5. *Let $\varphi : \partial\Delta^N \to \mathbb{R}^m$ be an admissible mapping into an m-dimensional Euclidean space. Then there are d mutually disjoint closed faces of $\Delta^N, \sigma_1,\dots,\sigma_d$ such that*

$$\bigcap_{i=1}^{d} \varphi(\sigma_i) \neq \emptyset.$$

PROOF. In [3] the following CW-complex was considered: in the Cartesian product $(\Delta^N)^d$ of d copies of the simplex Δ^N we choose the set $Y_{N,d}$ of all points $(y_1,\dots,y_d)$, $y_i \in \partial\Delta^N$ that have mutually disjoint carriers. It was shown in [3] that for all natural numbers N and d, $N > d$, the CW-complex $Y_{N,d}$ is $(N-d)$-connected. One can easily define a free action of the group G on $Y_{N,d}$ as follows:

Let $\alpha : G \to S_d$ be an arbitrary monomorphism of G into the permutation group S_d of d elements. Since S_d acts freely on $Y_{N,d}$ (by permutation of coordinates), the action of G induced by α is also free.

By Hurewicz theorem [18] $Y_{N,d}$ is $(N-d)$-acyclic as it is $(N-d)$-connected. From Proposition 4 we obtain that $\mathrm{ind}_G Y_{N,d} \geq N-d+1$.

Let us define an admissible map $\widetilde{\varphi} : Y_{N,d} \to \mathbb{R}^m, \widetilde{\varphi}(y_1,\dots,y_d) = \varphi(y_1)$. By Theorem 2 $\mathrm{ind}_G A_{\widetilde{\varphi}} \geq N-d+1-(d-1)m = 0$ which means in particular that $A_{\widetilde{\varphi}} \neq \emptyset$. Thus there is a point $(y_1,\dots,y_d) \in Y_{N,d}$ such that

$$\bigcap_{g\in G} \widetilde{\varphi}(g(y_1,\dots,y_d)) = \bigcap_{i=1}^{d} \varphi(y_i) \neq \emptyset.$$

By definition of $Y_{N,d}$ there are mutually disjoint faces of $\Delta^N, \sigma_1, \dots, \sigma_d$, with $y_i \in \sigma_i$ for all $i = 1, \dots, d$. Therefore $\bigcap_{i=1}^d \varphi(\sigma_i) \neq \emptyset$ which completes the proof. □

In order to formulate a generalization of Theorem 5 we introduce some notation. Given a convex compact set $C \subset \mathbb{R}^n$ with nonempty interior and a vector $v \in \mathbb{R}^n, v \neq 0$, we write

$$C(v) = \{x \in C : \langle v, x\rangle = \max\{\langle v, y\rangle,\ y \in C\}\}.$$

$C(v)$ is called a *proper face* of C. Clearly, it may happen that two different nonzero vectors define the same face of C. If $C(v)$ consists of one point x, we call it *a vertex* of C. For a vertex $x \in C$ we define *the star* of x, denoted by $\mathrm{st}(x)$, to be the union of all proper faces of C containing x. If $x \in C$ then *the carrier* of x is the minimal face containing x.

Definition 6. Let C be a compact and convex subset of $\mathbb{R}^n$ with nonempty interior. We say that C *has a property* (Δ), if there exists a homeomorphism $f : \Delta^n \to C$ such that the image of every face of Δ^n by f is a sum of faces in C.

The following theorem is an easy consequence of Theorem 4.

Theorem 7. *Let C be a compact convex subset of $\mathbb{R}^N$ with nonempty interior, where $N = (d-1)(m+1)$, $d = p^n$ for some prime p, and n, m are natural numbers. We assume that C has property (Δ). Then for every admissible map $\varphi : \partial C \to \mathbb{R}^m$ there are pairwise disjoint faces $A_1, \dots, A_d$ of C such that*

$$\bigcap_{i=1}^{d} \varphi(A_i) \neq \emptyset.$$

Proof. By our assumptions there is a homeomorphism $f : \Delta^N \to C$ sending each face of Δ^N onto a sum of faces in C. Define $h : \partial\Delta^N \to \partial C$, $h(x) = f(x)$. We consider the composition $\psi := \varphi \circ h : \partial\Delta^N \to \mathbb{R}^m$, which is an admissible map. In view of Theorem 5 there exist points $x_1, \dots, x_d$ in $\partial\Delta^N$ with mutually disjoint carriers such that

$$\bigcap_{i=1}^{d} \psi(x_i) = \bigcap_{i=1}^{d} \varphi(h(x_i)) \neq \emptyset.$$

Since the carriers of the points $h(x_i)$, $i = 1, \dots, d$ are mutually disjoint, the proof is complete. □

Theorem 8. *Every compact and convex polytope $C \subset \mathbb{R}^n$ with nonempty interior has the property (Δ).*

Proof. We proceed by induction with respect to the dimension of C. If $n = 0$, then our theorem is obvious.

We assume that the theorem holds for $n = k$ and consider a compact and convex polytope $C \subset \mathbb{R}^{k+1}$ with nonempty interior. We choose any vertex $x_0 \in C$. Since C has nonempty interior, there is a vector $v \in \mathbb{R}^{k+1}$ at x_0 such that $-v$ is directed inward C and $\{x_0\} = C(v)$. Let H be a hyperplane through x_0 in $\mathbb{R}^{k+1}$ which is orthogonal to v. Then the orthogonal projection π of $\mathrm{st}(x_0)$ into H is a homeomorphism onto its image. Moreover, the set $\pi(\mathrm{st}(x_0))$ is star-shaped with the center at x_0.

We say that $B \subset \pi(\mathrm{st}(x_0))$ is *a π-face* in $\pi(\mathrm{st}(x_0))$ if it is an image of a face of C (contained in $\mathrm{st}(x_0)$).

Let $p_1, \ldots, p_m$ be the π-vertices in $\pi(\mathrm{st}(x_0))$ which are joined with x_0 by a 1-dimensional π-face and let $D = \mathrm{conv}\{p_1, \ldots, p_m\}$. Thus $D \subset H$ is a polytope of dimension k with nonempty interior (in H). The projection along rays (starting at x_0) from ∂D onto $\partial\pi(\mathrm{st}(x_0))$ can be extended radially to a homeomorphism $f_1 : D \to \pi(\mathrm{st}(x_0))$ such that the image of every face of D by f_1 is a sum of π-faces in $\pi(\mathrm{st}(x_0))$. In particular $f_1(x_0) = x_0$.

By induction there is a homeomorphism $f_0 : \Delta^k \to D$ such that the image of every face in Δ^k by f_0 is a sum of faces in D.

Let $\Delta^{k+1} = \mathrm{conv}\{w_0, \ldots, w_{k+1}\}$. Simplex Δ^k is considered as a face of Δ^{k+1}, $\Delta^k = \mathrm{conv}\{w_0, \ldots, w_k\}$. Notice that $\partial\Delta^k$ is equal to the boundary of $\mathrm{st}(w_{k+1})$ which allows us to define a map $\widetilde{f}_2 : \partial\mathrm{st}(w_{k+1}) \cup \{w_{k+1}\} \to D$ putting

$$\widetilde{f}_2(x) = \begin{cases} f_0(x) & \text{if } x \in \partial\mathrm{st}(w_{k+1}) = \partial\Delta^k, \\ x_0 & \text{if } x = w_{k+1}. \end{cases}$$

Now we extend it radially to a homeomorphism $f_2 : \mathrm{st}(w_{k+1}) \to D$. One can see that the composition

$$h_1 = \pi^{-1} \circ f_1 \circ f_2 : \mathrm{st}(w_{k+1}) \to \mathrm{st}(x_0),$$

is a homeomorphism such that the image of every simplex in $\mathrm{st}(w_{k+1})$ by h_1 is a sum of faces in $\mathrm{st}(x_0)$.

Let us also notice that the sets $\overline{\partial\Delta^{k+1} \setminus \mathrm{st}(w_{k+1})} = \Delta^k$ and $\overline{\partial C \setminus \mathrm{st}(x_0)}$ are homeomorphic to each other since they are both homeomorphic to a closed disc of dimension k. In our case h_1 defines a homeomorphism $h_2 : \Delta^k \to \overline{\partial C \setminus \mathrm{st}(x_0)}$. Hence we have a homeomorphism $h : \partial\Delta^{k+1} \to \partial C$ defined by

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in \mathrm{st}(w_{k+1}), \\ h_2(x) & \text{otherwise,} \end{cases}$$

which obviously maps every simplex in $\partial\Delta^{k+1}$ onto a sum of faces in ∂C.

Finally, h can be extended to a homeomorphism $f : \Delta^{k+1} \to C$ and therefore C has the property (Δ) . □

As a direct consequence of Theorems 7 and 8 we obtain

THEOREM 9. *Let C be a compact and convex polytope in $\mathbb{R}^N$ with nonempty interior, where $N = (d-1)(m+1)$ and $d = p^n$ for some prime p and $n, m \in \mathbb{N}$. Then for any admissible map $\varphi : \partial C \to \mathbb{R}^m$ there are pairwise disjoint faces $A_1, ..., A_d$ of C such that*

$$\bigcap_{i=1}^{d} \varphi(A_i) \neq \emptyset.$$

It is obvious that compact and convex polytopes are not the only sets with property (Δ). For example the following is true.

PROPOSITION 10. *Let C be a compact, convex subset of $\mathbb{R}^n$ with nonempty interior. If we assume that for some vertex $x_0 \in \partial C$ there is an open neighbourhood U of x_0 in ∂C consisting of vertices only then C has the property (Δ).*

PROOF. Let $\Delta^n = \operatorname{conv}(w_0, \ldots, w_n)$. We consider an arbitrary injection $h_1 : \operatorname{st}(w_n) \to U$. It is a homeomorphism onto its image and the image of every simplex in $\operatorname{st}(w_n)$ by h_1 is a sum of faces (0-dimensional faces) in C.

Now, any extension of h_1 to a homeomorphism $h : \Delta^n \to C$ maps each face of Δ^n onto a sum of faces in C, which completes the proof. □

The analysis of many concrete examples suggests the following

Conjecture. Every compact and convex subset of $\mathbb{R}^n$ with nonempty interior has the property (Δ).

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