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GLOBAL BIFURCATION OF OSCILLATORY PERIODIC SOLUTIONS OF DELAY DIFFERENTIAL EQUATIONS

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1. Introduction

In this work we prove a global bifurcation result for periodic solutions of a general class of delay differential equations (allowing for state dependent delays). More precisely, assume that $M > 0$ is any real number and we are given a continuous function $f : C([-M, 0]; \mathbb{R}) \to \mathbb{R}$. Consider the singularly perturbed equation

$$\varepsilon \cdot x'(t) = f(x_t), \quad t > 0,$$

or equivalently (with $\lambda = 1/\varepsilon$)

$$(\mathcal{E}_\lambda) \qquad x'(t) = \lambda \cdot f(x_t), \quad t > 0,$$

where $\lambda > 0$ is a parameter. We will be interested in large values of this parameter. Note that, for every function $y : \mathbb{R} \to \mathbb{R}$ the "translates" $y_t : [-M, 0] \to \mathbb{R}$ are defined by $y_t(\theta) \doteq y(t + \theta)$, where $-M \le \theta \le 0$. Equation $(\mathcal{E}_\lambda)$ includes equations with constant delays (e.g. $f(\phi) = g(\phi(-M))$, $\phi \in C([-M, 0]; \mathbb{R})$, for some $g : \mathbb{R} \to \mathbb{R}$) or state dependent delays (e.g. $f(\phi) = g(\phi(-r(\phi(0))))$, for some $r : \mathbb{R} \to [0, \infty)$).

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Oscillatory periodic solutions of $(\mathcal{E}_\lambda)$ are very important in the study of the long term dynamics of this equation. For example, such solutions provide a lower bound on the complexity (or "fractal dimension") of the attractor of $(\mathcal{E}_\lambda)$ (for details see Mallet–Paret [5]). Here we prove that, under "natural" assumptions on f (*negative feedback*, instability of the *steady-state* solution), equation $(\mathcal{E}_\lambda)$ admits a continuum of periodic solutions that oscillate "slowly" round the steady-state solution. Such solutions are called *slowly oscillating periodic solutions* (S.O.Ps for short). For the proof of this result we follow the standard approach: we define a (global) Poincaré map $\Pi(\lambda,\,\cdot\,) : K \to K$ (some set K), so that fixed points of $\Pi(\lambda,\,\cdot\,)$ correspond to S.O.P. solutions of $(\mathcal{E}_\lambda)$. Then one wants to prove a *continuation* result for fixed points of $\Pi(\lambda,\,\cdot\,)$ (in the sense of the global bifurcation result of Rabinowitz [14]). A major problem though arises if we allow for state-dependent delays in $(\mathcal{E}_\lambda)$: the function $\Pi(\lambda,\,\cdot\,)$ is *discontinuous* on K. Obviously, discontinuity does not mesh well with bifurcation considerations.

It is the purpose of this work to show that by exploiting the properties of the Poincaré map $\Pi(\lambda,\,\cdot\,)$ (specifically the structure of its set of discontinuity), one can prove a continuation result for its fixed points. The set-up for our bifurcation theorem (Section 2) is such that our main hypotheses can be readily checked for a large class of delay equations. We also give an example of how the existing literature on the subject can be used to facilitate this checking (Section 3).

We need to mention that the approach of ours, described above, is pertaining to the seminal work of Rabinowitz [14] and subsequent modifications, applicable in the study of Functional Differential Equations (see Nussbaum [10]–[12]).

We now give some notation which will be used throughout this work.

Notation 1.1. Let (X, d_X) be a metric space. The (usual) distance between two nonempty subsets A, B of X is denoted by $\delta_X(A,B)$, i.e.

$$\delta_X(A,B) \doteq \inf\{d_X(x,y) \,:\, x \in A,\ y \in B\}. \tag{1.1}$$

Further, if A is any nonempty subset of X and J is any subinterval of $[0,\infty)$ we put

$$A_J \doteq \{x \in X \,:\, \delta_X(x,A) \in J\}. \tag{1.2}$$

Notation 1.2. Let (X_1, d_1), (X_2, d_2) be any two metric spaces. The product space $X \doteq X_1 \times X_2$ is topologized by the corresponding product metric

$$d_X((x_1,x_2),(y_1,y_2)) \doteq d_1(x_1,y_1) + d_2(x_2,y_2), \quad x_1, y_1 \in X_1,\ x_2, y_2 \in X_2. \tag{1.3}$$

If S is any nonempty subset of $X_1 \times X_2$ we put

$$S\langle x_1\rangle \doteq \{x_2 \in X_2 \,:\, (x_1,x_2) \in S\}. \tag{1.4}$$

Notation 1.3. If (X_1, d_1), (X_2, d_2) are any two metric spaces and $f : X_1 \to X_2$ is any function we put

$$\mathrm{lip}(f) \doteq \sup\{d_2(f(x), f(y))/d_1(x, y) \,:\, x, y \in X_1,\ x \neq y\}. \tag{1.5}$$

Assume now that X_1 and X_2 are Banach spaces and n is any positive integer. Denote by $C^n(X_1; X_2)$ the space of functions $f : X_1 \to X_2$ which have continuous mth order Fréchet derivatives $D^m f$, $1 \leq m \leq n$, i.e.

$$D^m f \,:\, X_1 \to \mathcal{L}^m(X_1; X_2),$$

and $\mathcal{L}^m(X_1; X_2)$ is the space of m-linear operators from X_1 to X_2. Then we define the space $C^{n,1}(X_1; X_2)$ as follows:

$$C^{n,1}(X_1; X_2) \doteq \{f \in C^n(X_1; X_2) \text{ with } \mathrm{lip}(D^n f) < \infty\}. \tag{1.6}$$

The latter is a Banach space with respect to the norm

$$\|f\|_{n,1} \doteq \sum_{k=1}^{n} \|D^k f\|_{\mathcal{L}^k} + \mathrm{lip}(D^n f). \tag{1.7}$$

2. A global bifurcation result

In this section we give a list of our hypotheses and prove the related bifurcation result. We start with a brief discussion on the underlying ideas from point set topology. The following lemma is essentially due to Kuratowski (see e.g. [4]; the version for metric spaces is attributed to Whyburn [15]). For a proof see also Deimling [2].

Lemma 2.1. *Let (M, d_M) be a compact metric space, $A \subset M$ be a component and $B \subset M$ be a closed set such that $A \cap B = \emptyset$. Then there exist disjoint compact sets M_A, M_B so that $M = M_A \cup M_B$ and $A \subset M_A$, $B \subset M_B$.*

The following application of Lemma 2.1 provides a separation which is useful for the bifurcation problem in mind.

Lemma 2.2. *Let (X, d_X) be a locally compact metric space and A be a compact component of X. Then there exists a number $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0]$, we can find closed, disjoint subsets C_1^ε, C_2^ε of X such that $X = C_1^\varepsilon \cup C_2^\varepsilon$ and $A \subset C_1^\varepsilon \subset A_{[0,\varepsilon]}$.*

Proof. Since A is compact and X is locally compact, there exists a number $\varepsilon_0 > 0$ such that the set $A_{[0,\varepsilon_0]}$ is compact. If $\varepsilon \in (0, \varepsilon_0]$ is any number, we put $B \doteq A_{\{\varepsilon\}}$ and apply Lemma 2.1 with $M \doteq A_{[0,\varepsilon]}$. If M_A, M_B are as in that lemma we define $C_1^\varepsilon \doteq M_A$ and $C_2^\varepsilon \doteq M_B \cup A_{[\varepsilon,\infty)}$. It is easily proved that the sets C_1^ε, C_2^ε have the required properties. $\square$

It is well known that degree theory is useful in bifurcation problems as a way of checking whether bifurcation actually occurs. It is the so-called fixed-point index which best fits the space settings met in bifurcation problems that arise from the study of functional differential equations. More precisely, assume that Y is a compact, absolute neighbourhood retract (e.g. a compact, convex subset of a Banach space), and $f : Y \to Y$ is a map. If U is any open subset of Y such that f is continuous in $\overline{U}$ and has no fixed points in ∂U, then, an integer denoted by $i_Y(f;U)$ can be defined. This integer is called the *fixed-point index* of f in U, and can be thought of as an algebraic count of the fixed points of f in U. For a list of the properties of the fixed-point index which axiomatically define it, one may consult Brown [1] and Nussbaum [8], [11], [12]. In applications one wants to actually compute the number $i_Y(f;U)$ for specific f and U. This is not allways easy but in certain cases is possible; such a situation arises when the fixed points of f enjoy the so-called *attractivity/ejectivity* property (see e.g. Nussbaum [9], [12]).

Notation 2.3. For the rest of this section we assume that (Y, d_Y) is a given metric space. We also put $X \doteq \mathbb{R}_+ \times Y$, where $\mathbb{R}_+ \equiv (0,\infty)$, and assume that X is topologized with the corresponding product metric d_X (see (1.3)). Further, assume that $F : X \to Y$ is a given function.

We are interested in elements $x = (\lambda, y) \in X$ for which $F(\lambda, y) = y$. It turns out that we can give a somewhat detailed description of such elements, provided the following is satisfied.

Standing Hypothesis 2.4.

(H1) There exists a closed set $E \subset Y$ and an element y_0 of Y such that the following are true:
- (a) $y_0 \in E$,
- (b) $F(\mathbb{R}_+ \times E) = \{y_0\}$,
- (c) F is continuous on $\mathbb{R}_+ \times (Y \setminus E)$.

(H2) There exists an increasing family $\{K_\lambda\}_{\lambda \geq 0}$ of compact absolute neighbourhood retracts, with $K_0 \equiv \{y_0\}$, such that, for all $\mu \geq 0$ the following are true:
- (a) $K_\mu \subset Y$,
- (b) $F((0,\mu] \times K_\mu) \subset K_\mu$,
- (c) If $(\lambda, y) \in (0,\mu] \times Y$ and $F(\lambda, y) = y$ then $y \in K_\mu$.

(H3) There exists a number λ_0 such that if the sequence $\{(\lambda_m, y_m)\}_{m\geq 1} \subset X$ satisfies:
- (a) $\lim_{m\to\infty} d_X((\lambda_m, y_m), (\lambda_\infty, y_\infty)) = 0$, for some $(\lambda_\infty, y_\infty) \in X$,
- (b) $F(\lambda_m, y_m) = y_m \neq y_0$, for all $m \geq 1$,
- (c) $\lim_{m\to\infty} \delta_Y(y_m, E) = 0$,

then $(\lambda_\infty, y_\infty) = (\lambda_0, y_0)$.

(H4) If the function $\rho(\lambda)$, $\lambda > 0$ is given by

$$\rho(\lambda) \doteq \inf\{d_Y(y_1, y_2) : y_1 \in E,\ F(\lambda, y_2) = y_2 \neq y_0\}$$

(we agree that $\inf\emptyset = 1$), then for all numbers λ, Λ, r with $\Lambda > \lambda_0$, $0 < \lambda \leq \Lambda$, $\lambda \neq \lambda_0$, $0 < r < \rho(\lambda)$ we have

$$i_{K_\Lambda}(F(\lambda, \cdot); K_\Lambda \cap E_{(r,\infty)}) = \begin{cases} 0 & \text{if } 0 < \lambda < \lambda_0, \\ 1 & \text{if } \lambda > \lambda_0. \end{cases}$$

A brief discussion on the list of Standing Hypotheses is necessary.

2.5 Remark on the Standing Hypothesis. (H1) It is an obvious consequence that the set $\mathbb{R}_+ \times \{y_0\}$ is the trivial solution to equation $F(\lambda, y) = y$. Note also that we do not require that F be continuous on its domain of definition. We thus allow for discontinuities; E contains all discontinuities of F. Indeed, state dependent delay equations give rise to Poincaré maps which are, in general, discontinuous.

(H2) Note that the set Y may not itself be an absolute neighbourhood retract. Therefore we cannot in general define the fixed point index for the pair (F, Y). Although more general index theories can accommodate (F, Z), with $Z \doteq \bigcup_{\lambda \geq 0} K_\lambda$, (see for instance Nussbaum [8]), we choose not to bother the reader with technical details related to such index generalizations. Instead, we require that properties (a) and (b) be satisfied; for the restricted pairs $(F(\lambda, \cdot), K_\mu)$, with $0 < \lambda < \mu$, the fixed point indices are well defined. Note also that in order that index generalizations accommodate (F, Z), analogs of properties (a) through (c) will have to hold true.

(H3) Here we put some structure on the set of nontrivial solutions of $F(\lambda, y) = y$, i.e. on the set

$$\sigma \doteq \{(\lambda, y) \in X \,:\, F(\lambda, y) = y \neq y_0\} \tag{2.1}$$

by requiring that σ can approach the set of discontinuity $\mathbb{R}_+ \times E$ only at the point (λ_0, y_0). A fortiori (H3) requires that the set σ can approach the trivial solution $\mathbb{R}_+ \times \{y_0\}$ only at the point (λ_0, y_0). In the terminology of bifurcation theory we say that (λ_0, y_0) proports to be a *bifurcation point*.

(H4) Here we essentially require that (λ_0, y_0) is *indeed* a bifurcation point. Although the assumption on the values of the fixed point index seems overly restrictive, it covers all the family of delay equations under consideration. The case is, this family is homotopic to a simple delay equation for which the actual calculations have been carried out and resulted to (H4) (see Mallet–Paret and Nussbaum [6], Mallet–Paret and Nussbaum and Paraskevopoulos [7]). We choose therefore to forgo listing the properties that F has to satisfy so that index calculations can be carried out. The interested reader should look up the notions of attractivity and ejectivity in, say, Nussbaum [9], [11] and the references cited therein. The fact that $\rho(\lambda) > 0$ for all $\lambda \neq \lambda_0$ follows from (H2) and (H3); this is proved in Lemma 2.6. Further, to see that the fixed point index $i_{K_\Lambda}(F(\lambda, \cdot);\ K_\Lambda \cap E_{(r,\infty)})$ is well defined for $\lambda \neq \lambda_0$ it is enough to note that

$$F(\lambda, \cdot):\ K_\Lambda \to K_\Lambda, \quad \lambda \leq \Lambda$$

as (H2) gives and that, for all r with $0 < r < \rho(\lambda)$, the boundary of the open set $K_\Lambda \cap E_{(r,\infty)}$ contains neither fixed points nor points of discontinuity of $F(\lambda, \cdot)$. Finally, using definitions (1.4) and (2.1) we can rewrite the function ρ in a more instructive manner, namely as $\rho(\lambda) = \delta_Y(\sigma\langle\lambda\rangle, E)$.

Lemma 2.6. *Assume that the Hypotheses* (H1)–(H3) *hold true. Then the function ρ defined in* (H4) *satisfies $\rho(\lambda) > 0$, for all $\lambda \neq \lambda_0$.*

Proof. Fix $\lambda > 0$, $\lambda \neq \lambda_0$. If the set $\sigma\langle\lambda\rangle$ is empty then $\rho(\lambda) = \inf\emptyset = 1$ and we are done. So, assume that there exists at least one $y \in \sigma\langle\lambda\rangle$. Also recall that $\rho(\lambda) = \delta_Y(\sigma\langle\lambda\rangle, E)$. By way of contradiction assume that $\rho(\lambda) = 0$. Then we can find a sequence $\{y_n\}_{n\geq 1} \subset Y$ so that $(\lambda, y_n) \in \sigma$, for all $n \geq 1$ and $\lim_{n\to\infty} \delta_Y(y_n, E) = 0$. Further, note that part (c) of (H2) implies that $\{y \in Y : (\lambda, y) \in \sigma\} \subset K_\lambda$. Since K_λ is compact there exists an element y_∞ of K_λ so that, passing to a subsequence if necessary, $\lim_{n\to\infty}((\lambda, y_n), (\lambda, y_\infty)) = 0$. Now (H3) implies that $\lambda = \lambda_0$ which is the sought-for contradiction. $\square$

The following lemma provides additional information about the set of nontrivial solutions of $F(\lambda, y) = y$.

Lemma 2.7. *Assume that the Standing Hypothesis holds and define a set $\Sigma \subset X$ as follows*

$$\Sigma \doteq \{(\lambda, y) \in X\ :\ F(\lambda, y) = y \neq y_0\} \cup \{(\lambda_0, y_0)\} = \sigma \cup \{(\lambda_0, y_0)\}. \tag{2.2}$$

Then the set Σ is closed in X and the metric space (Σ, d_X) is locally compact.

Proof. Using the definition of the set Σ and parts (a) and (b) of Hypothesis (H1) one can easily see that, for every $(\lambda, y) \in \Sigma$ the following implications are true:

$$y \notin E \Rightarrow F(\lambda, y) = y \neq y_0, \tag{2.3a}$$

$$y \in E \Rightarrow (\lambda, y) = (\lambda_0, y_0). \tag{2.3b}$$

We now prove that Σ is closed in X. To this end assume that the sequence $\{(\lambda_m, y_m)\}_{m\geq 1} \subset \Sigma$ is such that $\lim_{m\to\infty} d_X((\lambda_m, y_m), (\lambda_\infty, y_\infty)) = 0$, for some $(\lambda_\infty, y_\infty) \in X$. Obviously, it is enough to prove that $(\lambda_\infty, y_\infty) \in \Sigma$. Next we put

$b_m \doteq \delta_Y(y_m, E)$, $m \geq 1$ and note that $\lim_{m\to\infty} b_m = \delta_Y(y_\infty, E)$. The following three cases are possible:

(a) There exists an integer $m_1 \geq 1$ such that $b_m = 0$, for all $m \geq m_1$. Then (2.3b) gives that, for all $m \geq m_1$, we have $(\lambda_m, y_m) = (\lambda_0, y_0)$ and the claim follows.

(b) $\delta_Y(y_\infty, E) = 0$, but we can assume, passing to a subsequence if necessary, that $b_m > 0$, for all $m \geq 1$. Then (2.3a) implies that, for all $m \geq 1$, $F(\lambda_m, y_m) = y_m \neq y_0$ and Hypothesis (H3) gives that $(\lambda_\infty, y_\infty) = (\lambda_0, y_0) \in \Sigma$. The claim follows.

(c) $\delta_Y(y_\infty, E) > 0$. Then, for some $m_1 \geq 1$ we have $b_m > 0$, for all $m \geq m_1$ and from (2.3a) we find that

$$F(\lambda_m, y_m) = y_m \neq y_0 \quad \forall m \geq m_1.$$

Since F is continuous on the set $\mathbb{R}_+ \times (Y \setminus E)$ we find passing to the limit $m \to \infty$ that $F(\lambda_\infty, y_\infty) = y_\infty \neq y_0$ and the claim follows. Therefore Σ is closed.

We next prove that the space (Σ, d_X) is locally compact. To this end let $(\lambda, y) \in \Sigma$. If $0 < \varepsilon < \lambda$ then the set $([\lambda-\varepsilon, \lambda+\varepsilon] \times \{y_1 \in Y : d_Y(y, y_1) \leq \varepsilon\}) \cap \Sigma$ is a compact neighbourhood of (λ, y) in Σ. Indeed, it is enough to note that part (c) of Hypothesis (H2) gives that

$$([\lambda-\varepsilon, \lambda+\varepsilon] \times \{y_1 \in Y : d_Y(y, y_1) \leq \varepsilon\}) \cap \Sigma \subset ([\lambda-\varepsilon, \lambda+\varepsilon] \times K_{\lambda+\varepsilon})$$

The proof is complete. □

Remark 2.8. It is an obvious consequence of the definition of Σ and Hypothesis (H1) that $\Sigma \cap (\mathbb{R}_+ \times E) = \{(\lambda_0, y_0)\}$. We are now in position to state our main result; namely that the branch of non-trivial solutions of $F(\lambda, y) = y$ that bifurcates from (λ_0, y_0) (as Hypothesis (H4) guarantees) extends globally.

Theorem 2.9. *Assume that the Standing Hypothesis holds and let Σ_0 be the component of Σ which contains the point $\{(\lambda_0, y_0)\}$. Then Σ_0 is not compact in X.*

Proof. By way of contradiction assume that Σ_0 is compact in X. Then there exist numbers λ_L, λ_R with $0 < \lambda_L \leq \lambda_0 \leq \lambda_R < \infty$ satisfying

$$[\lambda_L, \lambda_R] = \{\lambda \in \mathbb{R}_+ \,:\, \Sigma_0\langle\lambda\rangle \neq \emptyset\}.$$

Since (Σ, d_X) is locally compact (Lemma 2.7), we can apply Lemma 2.2 with $A = \Sigma_0$ and $0 < \varepsilon < \min\{\varepsilon_0, \lambda_L\}$. Let Σ_1^ε, Σ_2^ε be as in the conclusion of that lemma; i.e. $\Sigma_1^\varepsilon \cap \Sigma_2^\varepsilon = \emptyset$, $\Sigma_1^\varepsilon \cup \Sigma_2^\varepsilon = \Sigma$ and $\Sigma_0 \subset \Sigma_1^\varepsilon$. Since Σ is closed in X (see Lemma 2.7) it follows that Σ_1^ε and Σ_2^ε are also closed in X. Now put $\widehat{\Sigma}_1^\varepsilon \doteq \Sigma_1^\varepsilon \cup (\mathbb{R}_+ \times E)$ and note that $\widehat{\Sigma}_1^\varepsilon$ is closed in X. Further, use Remark 2.8

to verify that $\widehat{\Sigma}_1^\varepsilon \cap \Sigma_2^\varepsilon = \emptyset$. Next define a continuous function $\rho_\varepsilon : X \to [0,\infty)$ and an open subset Ω_ε of X as follows

$$\rho_\varepsilon(x) \doteq \begin{cases} \varepsilon & \text{if } \Sigma_2^\varepsilon = \emptyset, \\ \delta_X(x, \Sigma_2^\varepsilon) & \text{if } \Sigma_2^\varepsilon \neq \emptyset, \end{cases}$$
$$\Omega_\varepsilon \doteq \{x \in X \,:\, \delta_X(x, \widehat{\Sigma}_1^\varepsilon) > \rho_\varepsilon(x)\}.$$

Then it is easily seen that $\Sigma_2^\varepsilon \subset \Omega_\varepsilon$. In view of the above definitions, it is an elementary exercise in point-set topology to verify that the following are true:

$$\partial\Omega_\varepsilon \cap \Sigma = \overline{\Omega}_\varepsilon \cap (\mathbb{R}_+ \times E) = \emptyset, \tag{2.4a}$$

$$\Sigma\langle\lambda\rangle \subset \Omega_\varepsilon\langle\lambda\rangle, \quad \text{whenever } 0 < \lambda < \lambda_L - \varepsilon \text{ or } \lambda > \lambda_R + \varepsilon. \tag{2.4b}$$

Now fix numbers Λ_L and Λ_R with $0 < \Lambda_L < \lambda_L - \varepsilon$ and $\Lambda_R > \lambda_R + \varepsilon$. For notational convenience we put $K \doteq K_{\Lambda_R}$. In view of (2.4a) the set $\partial\Omega_\varepsilon$ contains neither fixed points nor points of discontinuity of $F(\lambda, \cdot\,)$ and the fixed point index

$$i_K(F(\lambda, \cdot\,); K \cap \Omega_\varepsilon\langle\lambda\rangle)$$

is well defined for all $\lambda \in [\Lambda_L, \Lambda_R]$. Further, the homotopy property gives (use (2.4a), (2.4b))

$$i_K(F(\lambda, \cdot\,); K \cap \Omega_\varepsilon\langle\lambda\rangle) = n_0, \quad \forall \lambda \in [\Lambda_L, \Lambda_R] \tag{2.5}$$

for some integer n_0. Next, inclusion (2.4b) and the additivity property of the fixed point index give

$$i_K(F(\Lambda_L, \cdot\,); K \cap \Omega_\varepsilon\langle\Lambda_L\rangle) = i_K(F(\Lambda_L, \cdot\,); K \cap E_{(r,\infty)}), \quad 0 < r < \rho(\Lambda_L),$$

$$i_K(F(\Lambda_R, \cdot\,); K \cap \Omega_\varepsilon\langle\Lambda_R\rangle) = i_K(F(\Lambda_R, \cdot\,); K \cap E_{(r,\infty)}), \quad 0 < r < \rho(\Lambda_R).$$

Using Hypothesis (H4) this gives

$$i_K(F(\lambda, \cdot\,); K \cap \Omega_\varepsilon\langle\lambda\rangle) = \begin{cases} 0 & \text{if } \lambda = \Lambda_L, \\ 1 & \text{if } \lambda = \Lambda_R. \end{cases} \tag{2.6}$$

Equations (2.5) and (2.6) provide the required contradiction. Thus Σ_0 is not compact. $\square$

For certain applications, more information about the structure of Σ is known; in turn further information about Σ_0 can be obtained, as the following corollary indicates.

Corollary 2.10. *Assume that the Standing Hypothesis holds and in addition $\Sigma \subset [\lambda_*, \infty) \times Y$ for some $\lambda_* > 0$. Then the set Σ_0 is unbounded.*

Proof. We claim that if the inclusion $\Sigma \subset [\lambda_*, \infty) \times Y$ holds then the following implication is true

$$\Sigma_0 \text{ is bounded } \Rightarrow \Sigma_0 \text{ is compact.} \tag{2.7}$$

Indeed, part (a) of Hypothesis (H3) gives that $\Sigma_0 \cap [(0, \lambda] \times Y] \subset (0, \lambda] \times K_\lambda$. Therefore $\Sigma_0 \cap [(0, \lambda] \times Y] \subset [\lambda_*, \lambda] \times K_\lambda$; since $[\lambda_*, \lambda] \times K_\lambda$ is compact, (2.7) holds. In view of (2.7) the assertion is an immediate consequence of Theorem 2.9. $\square$

Remark 2.11. It is an obvious consequence of Hypothesis (H2) and Corollary 2.10 that the continuum Σ_0 of nontrivial solutions of $F(\lambda, y) = y$ persists for all $\lambda > \lambda_0$. This observation justifies our claim that the branch of bifurcating solutions can be continued globally.

3. Applications to functional differential equations

Here we show how the bifurcation result we established in the previous section can be applied to give global continua of periodic solutions of differential equations of the form $(\mathcal{E}_\lambda)$. We first formulate the initial value problem for equation $(\mathcal{E}_\lambda)$.

Definition 3.1. Assume that we are given real numbers $\lambda > 0$, $M > 0$ and continuous functions $f : C([-M, 0]; \mathbb{R}) \to \mathbb{R}$ and $\phi : [-M, 0] \to \mathbb{R}$. We say that the function $x \in C([-M, \infty); \mathbb{R}) \cap C^1((0, \infty); \mathbb{R})$ solves the initial value problem $\mathcal{P}(\lambda, \phi)$ if and only if it satisfies:

$$\mathcal{P}(\lambda; \phi) \qquad \begin{cases} x'(t) = \lambda \cdot f(x_t), & t > 0, \\ x_0 = \phi. \end{cases}$$

In the sequel the solution to the i.v.p. $\mathcal{P}(\lambda, \phi)$ will be denoted by $x(\,\cdot\,; \lambda; \phi)$

We now list a few regularity properties which the nonlinearity f will be required to satisfy.

Definition 3.2. For notational convenience put

$$\mathcal{L}_R \doteq \{\phi \in C^{0,1}([-M, 0]; \mathbb{R}) \text{ with } \mathrm{lip}(\phi) < R\}$$

(for the notation see (1.5) in the Introduction). We say that the function $f : C([-M, 0]; \mathbb{R}) \to \mathbb{R}$ is *σ-Lipschitzian* if and only if its restriction on each of the sets $\mathcal{L}_R$ is Lipschitz continuous with respect to the supremum norm of $C([-M, 0]; \mathbb{R})$, i.e. if and only if

$$\sup\{|f(\phi) - f(\psi)|/\|\phi - \psi\| \,:\, \phi, \psi \in \mathcal{L}_R,\ \phi \neq \psi\} < \infty$$

where, of course, $\|\phi\| = \sup\{|\phi(t)| : t \in [-M, 0]\}$.

Remark 3.3. The notion of a σ-Lipschitzian map is essentially a weakening of that of a Lipschitz continuous map. This weakening is necessary if the family $(\mathcal{E}_\lambda)$ is to include equations with state-dependent delays. As an example the reader may easily verify that the nonlinearity $f(\phi) \doteq -\phi(0) + g(\phi(-r(\phi(0))))$, with smooth functions $g : \mathbb{R} \to \mathbb{R}$, $r : \mathbb{R} \to [0,\infty)$, gives rise to a state-dependent delay equation. Furthermore, f *fails* to be locally Lipschitzian but it is σ-Lipschitzian.

The notion of σ Lipschitzian functions is equivalent to that of *almost Lipschitz continuous* functions defined in Mallet–Paret, Nussbaum and Paraskevopoulos [7]. The interested reader may consult Definition 1.1 in that paper.

From the same paper we also reproduce the following definitions:

Definition 3.4. A function $f : C([-M,0];\mathbb{R}) \to \mathbb{R}$ satisfies a *negative feedback condition* if and only if the following are true:

(a) For every function $\phi \in C([-M,0];\mathbb{R})$ with $\phi(t) \geq 0$ (respectively, $\phi(t) > 0$, $\phi(t) \leq 0$, $\phi(t) < 0$), for all $t \in [-M,0]$, we have $f(\phi) \leq 0$ (respectively, $f(\phi) < 0$, $f(\phi) \geq 0$, $f(\phi) > 0$).

(b) There exists a locally Lipschitz function $g : \mathbb{R} \to \mathbb{R}$, with $x \cdot g(x) < 0$ when $x \neq 0$, and a number $\tau_0 \in (0,M]$ such that $f(\phi) = g\big(\phi(-\tau_0)\big)$ for all $\phi \in C\big([-M,0];\mathbb{R}\big)$ with $\phi(0) = 0$.

Definition 3.5. A function $f : C([-M,0];\mathbb{R}) \to \mathbb{R}$ is *σ-Fréchet differentiable at* 0 if and only if the restriction of f on the space $C^{0,1}([-M,0];\mathbb{R})$ is Fréchet differentiable at 0, i.e. if and only if the linear functional $Df(0) \in (C^{0,1}([-M,0];\mathbb{R}))'$ satisfies $f(\phi) = f(0) + [Df(0)](\phi) + o(\|\phi\|_{0,1})$, for all $\phi \in C^{0,1}([-M,0];\mathbb{R})$.

In the sequel we will study equations $(\mathcal{E}_\lambda)$ which admit global solutions that take values in bounded intervals. For such equations we have

Definition 3.6. Let I be any compact interval of reals. The nonlinearity $f : C([-M,0];\mathbb{R}) \to \mathbb{R}$ will be called *I-proper* if and only if:

(a) $\sup\{|f(\phi)| : \phi \in C([-M,0];J)\} < \infty$, for all compact intervals $J \subset \mathbb{R}$.

(b) For every $\phi \in C([-M,0];I)$ with $\phi(0) = \max I$ (respectively, $\phi(0) = \min I$) we have $f(\phi) \leq 0$ (respectively, $f(\phi) \geq 0$).

Remark 3.7. There is a fairly general class of nonlinearities which are I-proper for some interval I. An extensive list may be found in [7]. If f is I-proper and satisfies a negative feedback condition then the obvious observation $f(0) = 0$ implies that $\min I < 0 < \max I$.

We now list the Standing Hypotheses for this section.

Standing Hypothesis. Let $f : C([-M,0];\mathbb{R}) \to \mathbb{R}$ be a given function.

(F1) f is continuous, σ-Lipschitzian and I-proper for some interval I.

(F2) f satisfies a negative feedback condition.

(F3) f is σ-Fréchet differentiable at 0 with Fréchet derivative $[Df(0)](\phi) = -\alpha\phi(0) - \beta\phi(-\tau_0)$, for all $\phi \in C^{0,1}([-M,0];\mathbb{R})$, where $\beta > \alpha > 0$.

The following results are standard. For a proof the reader is referred to [7].

Proposition 3.8. *Assume that $f : C([-M,0];\mathbb{R}) \to \mathbb{R}$ satisfies Hypothesis* (F1) *and let $\lambda > 0$ be a given number and $\phi \in C^{0,1}([-M,0];I)$ be a given function. Then the initial value problem $\mathcal{P}(\lambda;\phi)$ admits exactly one solution $x(\,\cdot\,;\lambda;\phi) : [-M,\infty) \to I$. Further, this solution depends continuously on λ and ϕ, i.e. if the numbers $\lambda_n > 0$ satisfy $\lim_{n\to\infty} \lambda_n = \lambda$ and the functions $\phi_n \in C^{0,1}([-M,0];\mathbb{R})$ satisfy $\lim_{n\to\infty} \|\phi_n - \phi\| = 0$, then $\lim_{n\to\infty} x(\,\cdot\,;\lambda_n;\phi_n) = x(\,\cdot\,;\lambda;\phi)$ uniformly on compact subsets of $[-M,\infty)$.*

The next result illustrates how the negative feedback condition forces solutions of $(\mathcal{E}_\lambda)$ to oscillate around the trivial (zero) solution; it also provides a means of locating zeros of nontrivial solutions.

Proposition 3.9. *Assume that $f : C([-M,0];\mathbb{R}) \to \mathbb{R}$ satisfies Hypotheses* (F1) *and* (F2) *and define convex sets*

$$Y \doteq \{\phi \in C^{0,1}([-M,0];I) : \phi(0) = 0,\ \phi([-\tau_0,0]) \subset [0,\infty)\}, \tag{3.1}$$

$$E \doteq \{\phi \in Y : \phi(t) = 0 \text{ for all } t \in [-\tau_0,0]\}. \tag{3.2}$$

Let $\lambda > 0$, $\phi \in Y$ and $x(\,\cdot\,;\lambda;\phi)$ be the solution of the corresponding initial value problem $\mathcal{P}(\lambda;\phi)$. Then there exists a nondecreasing sequence of numbers $\{q_n(\lambda;\phi)\}_{n\ge 0} \subset [0,\infty]$ with the following properties:

(a) *$x(t;\lambda;\phi) = 0$, for all $t \in [0,q_0]$*

(b) *if $q_N(\lambda;\phi) < \infty$ for some integer $N \ge 1$ then*

$$q_n(\lambda;\phi) - q_{n-1}(\lambda;\phi) > \tau_0, \quad n = 1,\dots,N, \tag{3.3}$$

$$(-1)^n \cdot x(t;\lambda;\phi) > 0, \quad t \in (q_{n-1},q_n),\ n = 1,\dots,N. \tag{3.4}$$

Therefore the sequence $\{q_n(\lambda;\phi)\}_{n\ge 0}$ contains all the zeros of $x(\,\cdot\,;\lambda;\phi)$ in the interval $[0,\infty]$. Furthermore, if $\phi \in E$ then $q_0(\lambda;\phi) = \infty$ for all $\lambda > 0$.

Any solution for which (3.3) and (3.4) are satisfied will be called *slowly oscillating*. A very interesting class of solutions of $(\mathcal{E}_\lambda)$ are those who are slowly oscillating, periodic and satisfy $x(t+q_2;\lambda;\phi) = x(t;\lambda;\phi)$, for all $t \in \mathbb{R}$, with $q_2(\lambda;\phi) < \infty$. These are called *slowly oscillating periodic solutions*, or S.O.P.(2) solutions for short. The following theorem provides an indispensable device for the search for S.O.P.(2) solutions of functional differential equations.

Theorem 3.10. *Assume that* $f : C([-M,0];\mathbb{R}) \to \mathbb{R}$ *satisfies Hypotheses* (F1) *and* (F2) *and define a function* $\Pi : \mathbb{R}_+ \times Y \to Y$ *as follows:*

$$\Pi(\lambda;\phi) \doteq \begin{cases} x_{q_2} & \text{if } q_2 = q_2(\lambda;\phi) < \infty, \\ 0 & \text{otherwise}. \end{cases} \tag{3.5}$$

Then Π *is continuous on* $\mathbb{R}_+ \times [Y \setminus E]$ (Y *is equipped with the supremum norm* $\|\cdot\|$). *Further, if* $\Pi(\lambda;\phi) = \phi \neq 0$, *for some* $\phi \in Y$, *then the solution* $x(\,\cdot\,;\lambda;\phi)$ *is a* S.O.P.(2) *solution of* $\mathcal{P}(\lambda;\phi)$.

Remark 3.11. The use of a Poincaré map (such as $\Pi(\lambda;\phi)$) as a tool for searching for periodic solutions is commonplace in the study of differential equations. The reader should nevertheless note that, in our case, the map $\Pi(\lambda;\phi)$ is *not continuous*. In fact this map is, in general, discontinuous on the set E (see (3.2)) even in the case of a single, state-dependent delay (see e.g. the nonlinearity used in Remark 3.3). It is our bifurcation result of the previous section that allows us to carry out the program of proving the existence of global continua of S.O.P.(2) solutions of $(\mathcal{E}_\lambda)$. The following lemma provides a necessary link between this work and the standard literature on equations with state-dependent delays.

Lemma 3.12. *Assume that the sets* Y *and* E *are defined as in equations* (3.1) *and* (3.2), *respectively,. For every* $\phi \in Y$ *we have* $\delta_Y(\phi, E) = \max\{\phi(t) : -\tau_0 \le t \le 0\}$.

Proof. We first note that for every $\phi \in Y$ we have that

$$\delta_Y(\phi, E) \ge \max\{\phi(t) \,:\, -\tau_0 \le t \le 0\}.$$

Next we fix an arbitrary $\phi \in Y$ and define the functions

$$\begin{aligned} \phi_*(t) &\doteq -\mathrm{lip}(\phi)\cdot\min\{0, t+\tau_0\}, && -M \le t \le 0, \\ \widehat{\phi}(t) &\doteq \min\{\phi(t), \phi_*(t)\}, && -M \le t \le 0. \end{aligned}$$

Then we have that $\mathrm{lip}(\phi_*) = \mathrm{lip}(\phi) \ge \mathrm{lip}(\widehat{\phi})$ and therefore $\widehat{\phi} \in E$. Furthermore,

$$\phi(t) - \widehat{\phi}(t) = \max\{0, \phi(t) - \phi_*(t)\}, \quad t \in [-M, 0]. \tag{3.6}$$

The Lipschitz continuity of ϕ gives that $|\phi(t) - \phi(-\tau_0)| \le -\mathrm{lip}(\phi)\cdot(t+\tau_0)$ for all $t \in [-M, -\tau_0]$, which can be rewritten as

$$\phi(-\tau_0) - 2\phi_*(t) \le \phi(t) - \phi_*(t) \le \phi(-\tau_0), \quad t \in [-M, -\tau_0]. \tag{3.7}$$

Using (3.6) and (3.7) we find that

$$\max\{|\phi(t) - \widehat{\phi}(t)| \,:\, -M \le t \le 0\} = \max\{\phi(t) \,:\, -\tau_0 \le t \le 0\}.$$

Since $\widehat{\phi} \in E$ this implies that $\delta_Y(\phi, E) \le \max\{\phi(t) \,:\, -\tau_0 \le t \le 0\}$ and the lemma follows. □

Next we state and prove the main result of this section.

THEOREM 3.13. *Assume that the function* $f : C([-M,0];\mathbb{R}) \to \mathbb{R}$ *satisfies Hypotheses* (F1)–(F3). *Then there exists a global continuum of* S.O.P.(2) *solutions of equation* ($\mathcal{E}_\lambda$) *which bifurcates from the trivial* (*zero*) *solution at the point* $\lambda_0 = \nu_0/\sqrt{\beta^2-\alpha^2}$. *Here* ν_0 *is the unique solution of the equation* $\alpha + \beta\cdot\cos(\nu\cdot\tau_0) = 0$ *in the open interval* $(\pi/2\tau_0, \pi/\tau_0)$.

PROOF. We are going to apply Theorem 2.9, with Π in place of F. To this end we only need to check that the Standing Hypothesis in Section 1 is satisfied. Assume that the sets Y and E defined in (3.1) and (3.2), respectively, are equipped with the supremum norm $\|\cdot\|$ and let $y_0 \equiv 0$. With this set-up it is immediate that Hypothesis (H1) holds. Next we put:

$$\kappa \doteq \sup\{|f(\phi)| \,:\, \phi \in C([-M,0];I)\},$$

and define a family K_λ, $\lambda \geq 0$ of compact, convex subsets of Y as

$$K_\lambda \doteq \{\phi \in Y \,:\, \mathrm{lip}(\phi) \leq \lambda\cdot\kappa\}.$$

The fact that $K_0 = \{0\}$ is obvious (recall that $\phi(0) = 0$ for all $\phi \in Y$). Also note that

$$\begin{aligned}\mathrm{lip}(\Pi(\lambda;\phi)) &\leq \mathrm{lip}(x(\,\cdot\,;\lambda;\phi)) = \max\{\mathrm{lip}(\phi), \sup_{t\geq 0}|x'(t;\lambda;\phi)|\} \\ &\leq \max\{\mathrm{lip}(\phi), \lambda\cdot\kappa\}.\end{aligned} \tag{3.8}$$

Therefore

$$\mathrm{lip}(\Pi(\lambda;\phi)) \leq \mu\cdot\kappa, \quad \text{for all } 0 \leq \lambda \leq \mu,\ \phi \in K_\mu.$$

Further, if $q_2(\lambda;\phi) < \infty$ then $q_2(\lambda;\phi) - q_1(\lambda;\phi) > \tau_0$ and thus $\Pi(\lambda;\phi)(t) = x_{q_2}(t) = x(t+q_2;\lambda;\phi) \geq 0$ for all $t \in [-\tau_0, 0]$. Therefore $\Pi((0,\mu]\times K_\mu) \subset K_\mu$ and part (b) of (H2) holds. Finally, if $\Pi(\lambda;\phi) = \phi$ for some $\lambda \in (0,\mu]$ then part (c) of (H2) holds when $\phi = 0$. If $\phi \neq 0$ let $n_0 \in \mathbb{N}$ be such that $n_0\cdot\tau_0 > M$. Then $q_{2n_0}(\lambda;\phi) + t \geq 0$ for all $t \in [-M,0]$ and, iterating $\Pi(\lambda;\,\cdot\,)$ we find that $\phi = \Pi^n(\lambda;\phi) = x_{q_{2n_0}}(\cdot;\lambda;\phi)$, which gives that $\mathrm{lip}(\phi) \leq \sup_{t\geq 0}|x'(t;\lambda;\phi)| \leq \lambda\cdot\kappa$. Thus $\phi \in K_\mu$ and Hypothesis (H2) is satisfied. The fact that Hypotheses (H3) and (H4) hold true is proved (among other things) in Theorem (3.1) of [7]. The interested reader should consult the proof of that Theorem; the use of Lemma 3.12 is crucial for the verification of (H3). We have therefore shown that Hypotheses (H1) through (H4) hold; the proof is now complete. $\square$

We next prove that if the Standing Hypothesis holds then the following inclusion is satisfied:

$$\{(\lambda,\phi) \in \mathbb{R}_+ \times Y \,:\, \Pi(\lambda;\phi) = \phi \neq 0\} \subset [\lambda_*, \infty) \times Y \tag{3.9}$$

for some $\lambda_* > 0$. We start with a preliminary Lemma; the obvious proof is left to the reader.

Lemma 3.14. *Assume that f satisfies the Standing Hypothesis and, for every real number x define the (constant) function $T[x] \in C^{0,1}([-M,0];\mathbb{R})$ by $T[x](\theta) \doteq x$, for all $\theta \in [-M,0]$. Further, for every $\phi \in C^{0,1}([-M,0];\mathbb{R})$, every $\xi \in \mathbb{R}$ and every $r \in \mathbb{R}_+$ put*

$$F(\xi,\phi) \doteq \begin{cases} f(\xi\phi)/\xi & if\ \xi \neq 0, \\ -\alpha\phi(0) - \beta\phi(-\tau_0) & if\ \xi = 0, \end{cases}$$

$$\mathcal{B}_r(J) \doteq \sup\{|F(\xi,\phi)| \,:\, |\xi| \leq \max_{\eta\in I} |\eta|, \quad \phi \in \mathcal{L}_r,\ \phi([-M,0]) \subset J\}.$$

Then the restriction of F on each of the sets $\mathbb{R} \times \mathcal{L}_r$, $r \geq 0$ is continuous ($\mathcal{L}_r$ is equipped with the sup-norm) and $\mathcal{B}_r(J) < \infty$. If we put

$$G(\xi,\zeta) \doteq F(\xi, T[\zeta])$$

then G is continuous and $\zeta \cdot G(\xi,\zeta) < 0$ for all $\zeta \neq 0$.

Theorem 3.15. *Assume that the Standing Hypothesis holds true. Then* (3.9) *also holds.*

Proof. We argue by way of contradiction and assume that there exist sequences $\{\phi_n\}_{n\geq 1} \subset Y$, $\{\lambda_n\}_{n\geq 1} \subset \mathbb{R}_+$, such that $\Pi(\lambda_n,\phi_n) = \phi_n \neq 0$ for every $n \geq 1$ and $\lim_{n\to\infty} \lambda_n = 0$. Put, for convenience, $x^n \doteq x(\,\cdot\,;\lambda_n;\phi_n)$, $n \geq 1$. Then, for every $n \geq 1$, x^n is a periodic solution of $(\mathcal{E}_\lambda)$ and we can assume, without loss of generality, that $\|x^n\| = \sup_{t\in\mathbb{R}} |x^n(t)| = |x^n(0)|$. Now put $y^n(t) \doteq x^n(t/\lambda)/\|x^n\|$ and note that

$$\|y^n\| = |y^n(0)| = 1, \quad \text{for all } n \geq 1. \tag{3.10}$$

It is immediate that, for each $n \geq 1$ the function y^n is a (global) solution of the following equation

$$\frac{dy^n(t)}{dt} = F(\|x^n\|, y^n(t + \lambda_n\,\cdot\,)), \quad t \in \mathbb{R},\ n \geq 1, \tag{3.11}$$

where F is defined in Lemma 3.14 and the function $y^n(t+\lambda_n\,\cdot\,) \in C^{0,1}([-M,0];\mathbb{R})$ is given by

$$y^n(t + \lambda_n\,\cdot\,)(\theta) \doteq y^n(t + \lambda_n\theta), \quad \theta \in [-M,0],\ n \geq 1.$$

Since $\|y^n(t + \lambda_n\,\cdot\,)\| \leq 1$ and $\|x^n\| \leq \max_{\eta\in I} |\eta|$ for all $n \geq 1$, equation (3.11) gives

$$\sup_{t\in\mathbb{R}} \left|\frac{dy^n(t)}{dt}\right| \leq \mathcal{B}_r([-1,1]), \quad n \geq 1. \tag{3.12}$$

In view of the estimates (3.10) and (3.12), the Ascoli–Arzéla theorem implies that the set $\{y^n\}_{n\ge 1}$ is a compact subset of $C(\mathbb{R};\mathbb{R})$ with respect to the topology of uniform convergence on compact subsets of $\mathbb{R}$. Thus there exist a number ξ_0 with $0 \le \xi_0 \le \max_{\eta\in I}|\eta|$ and a function $y_\infty \in C(\mathbb{R};\mathbb{R})$ such that, passing to a subsequence, $\lim_{n\to\infty}\|x^n\| = \xi_0$ and $\lim_{n\to\infty} y^n = y_\infty$, uniformly on compacta. Elementary arguments using (3.11) imply that y_∞ is also a global solution of the scalar, autonomous O.D.E.

$$\frac{dw(t)}{dt} = G(\xi_0, w(t)), \quad t \in \mathbb{R}. \tag{3.13}$$

In view of (3.10), y_∞ also satisfies

$$\|y_\infty\| = |y_\infty(0)| = 1. \tag{3.14}$$

Since $\zeta\cdot G(\xi_0,\zeta) < 0$ for all $\zeta \ne 0$ (see Lemma 3.14), we see that equation (3.13) does not admit global solutions which satisfy (3.14). This provides the required contradiction. The proof is now complete. □

Corollary 3.16. *Assume that the Standing Hypothesis holds. Then the global continuum of* S.O.P.(2) *solutions of* $(\mathcal{E}_\lambda)$ *which bifurcates from the point* $(\lambda_0, 0)$ *persists for all* $\lambda > \lambda_*$.

Proof. Immediate consequence of Theorem 3.15 and Corollary 2.10. □

Remark 3.17. The framework described above includes a variety of state dependent delay equations; a number of examples is presented in Chapter 4 of [7]. We should mention that the author has studied (see Paraskevopoulos [13]) equations of the form

$$\dot{x}(t) = -\lambda\cdot x(t) + \lambda\cdot F(x(t-r_1),\ldots,x(t-r_N)), \quad r_i = r_i(x(t)),\ 1\le i\le N, \tag{3.15}$$

where $F : \mathbb{R}^N \to \mathbb{R}$ and $r_i : \mathbb{R} \to \mathbb{R}_+$, $1 \le i \le N$ are given functions. In that work a slightly different approach was followed, leading to the definition of a *continuous* Poincaré map Π. Under additional assumptions (similar in nature to (F1)–(F3) given above) on the functions F and r_i, $i = 1,\ldots,N$, it is proved there that Π undergoes a bifurcation which gives rise to *unbounded* continua of S.O.P.(2) solutions of (3.15).

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