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THE EFFECT OF THE GRAPH TOPOLOGY ON A SEMILINEAR ELLIPTIC EQUATION WITH CRITICAL EXPONENT

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1. Introduction

The aim of this paper is to study the effect of the topological structure of the graph of the coefficient $Q(y)$ on the number of the positive solutions of the following elliptic problem:

$$\begin{cases} -\Delta u = Q(y)u^{2^*-1} + \varepsilon u & y \text{ in } \Omega, \\ u > 0 & y \text{ in } \Omega, \\ u = 0 & y \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where ε is a small nonnegative number, $2^* = 2N/(N-2)$, $N \geq 4$, Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$ and $Q(y)$ is a smooth positive function in $\overline{\Omega}$.

Problem (1.1) stems from differential geometry and has attracted a lot of attention. In the case $\varepsilon > 0$, the existence of at least one solution for (1.1) was established by Brézis and Nirenberg [9] in the case $Q =$ Const. and by Escobar [12] for a continuous function $Q(y)$ satisfying some additional assumptions. In the case $\varepsilon = 0$, it follows from the Pohozaev identity that problem (1.1) has no solution if Ω is star shaped and $\langle DQ(y), y\rangle \leq 0$. Thus we expect that a solution of problem (1.1) will concentrate at some point as $\varepsilon \to 0+$. So it is

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interesting to know where the concentration point is and to estimate the number of the solutions if there are such points. In the case $Q = \text{Const.}$, Rey [19], [20] studied the role of the Green function in problem (1.1) and used the category of the domain to estimate the number of the solutions of (1.1) for $\varepsilon > 0$ small. For general $Q(y)$, Cao and Noussair [10] proved that (1.1) has at least as many solutions as the number of degenerate isolated global maximum points of $Q(y)$ if ε is small.

In the case $\varepsilon = 0$ and $Q(y) = 1$, Bahri and Coron [2] investigated the effect of the domain topology on the existence of a solution for (1.1). Thus another problem to consider is the effect of the graph topology of $Q(y)$ on the existence result for (1.1) in the case $\varepsilon = 0$ and the domain Ω is contractible.

The aim of this paper is two-fold. First, we construct a solution for (1.1) which concentrates at an interior or a boundary local maximum point of $Q(y)$ as $\varepsilon \to 0$. We also estimate the number of such solutions using the category of the set on which $Q(y)$ attains its local maximum. Second, we study the effect of the graph topology of $Q(y)$ on the existence of a solution for (1.1) in the case $\varepsilon = 0$. Actually, we will construct a solution for (1.1) for $\varepsilon > 0$ small, such that this sequence of solution converges strongly in $H^1(\Omega)$ as $\varepsilon \to 0$.

Before we introduce our main results, we give some notation. Let

$$U_{x,\lambda}(y) = [N(N-2)]^{(N-2)/4} \frac{\lambda^{(N-2)/2}}{(1 + \lambda^2 |y - x|^2)^{(N-2)/2}}.$$

It is well known that $U_{x,\lambda}$ satisfies

$$\Delta U_{x,\lambda} = U_{x,\lambda}^{2^*-1}, \quad y \in \mathbb{R}^N.$$

Let P denote the projection from $H^1(\Omega)$ into $H^1_\circ(\Omega)$; that is, if $w \in H^1(\Omega)$, then Pw is a unique solution of the following Dirichlet problem

$$\begin{cases} \Delta u = \Delta w & y \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let

$$\langle u, v \rangle = \int_\Omega DuDv, \qquad u, v \in \mathrm{H}^1_\circ(\Omega), \tag{1.2}$$

$$\|u\| = \left(\int_\Omega |Du|^2 \right)^{1/2}, \quad u \in \mathrm{H}^1_\circ(\Omega), \tag{1.3}$$

$$E_{x,\lambda} = \bigg\{ v : v \in \mathrm{H}^1_\circ(\Omega), \\ \langle v, PU_{x,\lambda} \rangle = \left\langle v, \frac{PU_{x,\lambda}}{\partial \lambda} \right\rangle = \left\langle v, \frac{PU_{x,\lambda}}{\partial x_j} \right\rangle = 0,\ j = 1, \ldots, N \bigg\}. \tag{1.4}$$

We now state the main results of this paper.

Theorem 1.1. *Let M and M_* be two connected closed sets compactly contained in Ω satisfying $M \subset M_*$, $\max_{x\in M} Q(x) > \max_{x\in\partial M_*} Q(x)$ and*

$$\begin{cases} Q(x) =: Q_M = \text{Const.} & \text{for all } x \in M, \\ M_* \cap \{x : Q(x) = Q_M\} = M, \\ M_* \cap \{x : Q(x) > Q_M\} = \emptyset. \end{cases} \tag{1.5}$$

Suppose that $N \geq 4$ and that $D^2_{ij}Q(x) = 0$, $i, j = 1, \dots, N$, for all $x \in M$ if $N \geq 5$. Then there is an $\varepsilon_\circ > 0$, such that for each $\varepsilon \in (0, \varepsilon_\circ]$ problem (1.1) *has at least* $\mathrm{Cat}_{M_*}(M)$ *solutions of the form*

$$u_\varepsilon = \alpha_\varepsilon P U_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon, \tag{1.6}$$

where $v_\varepsilon \in E_{x_\varepsilon,\lambda_\varepsilon}$, and as $\varepsilon \to 0$,

$$\alpha_\varepsilon \to Q_M^{-1/(2^*-2)}, \tag{1.7}$$

$$\|v_\varepsilon\| \to 0, \tag{1.8}$$

$$x_\varepsilon \to x_\circ \in M, \tag{1.9}$$

$$\lambda_\varepsilon \to \infty, \tag{1.10}$$

Theorem 1.2. *Suppose that $N \geq 5$. Let M be a connected closed set in $\partial\Omega$, satisfying*

$$\begin{cases} Q(x) =: Q_M = \text{Const.} & \text{for all } x \in M, \\ Q(x) \leq Q_M - a(d(x, M))^k & \text{for all } d(x, M) \leq \delta, \\ |D^i Q(x)| = O(d(x, M)^{k-i}) & \text{for all } d(x, M) \leq \delta,\ i = 1, \dots, [k], \end{cases} \tag{1.11}$$

where a is some positive constant, k is some constant satisfying $k > 4/(N-4)+2$. Then there is an $\varepsilon_\circ > 0$, such that for each $\varepsilon \in (0, \varepsilon_\circ]$, problem (1.1) *has at least* $\mathrm{Cat}_M(M)$ *solutions of the form*

$$u_\varepsilon = \alpha_\varepsilon P U_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon, \tag{1.12}$$

where $v_\varepsilon \in E_{x_\varepsilon,\lambda_\varepsilon}$, and as $\varepsilon \to 0$,

$$\alpha_\varepsilon \to Q_M^{-1/(2^*-2)}, \tag{1.13}$$

$$\|v_\varepsilon\| \to 0, \tag{1.14}$$

$$x_\varepsilon \to x_\circ \in M, \tag{1.15}$$

$$\lambda_\varepsilon \to \infty, \quad \lambda_\varepsilon d(x_\varepsilon, \partial\Omega) \to \infty. \tag{1.16}$$

Theorem 1.3. *Let $M = \{x : Q(x) = Q_{\max}\}$, where*

$$Q_{\max} = \max_{x\in\Omega} Q(x).$$

Suppose that $N \geq 5$ and that the following conditions hold:

(i) *M is not contractible in a small neighbourhood of itself, but M is contractible in $\{x : Q(x) \geq t\}$ for some constant t belonging to*

$$(2^{-2/(N-2)}Q_{\max}, Q_{\max})$$

and such that $\max_{\partial\Omega} Q(x) < t$ and

(ii) *for each $x \in \Omega$ satisfying $DQ(x) = 0$ and $Q_{\max} > Q(x) \geq t$, we have*

$$\Delta Q(x) > 0.$$

Then for each $\varepsilon \in [0, \varepsilon_\circ]$, (1.1) has a solution u_ε such that u_ε converges (up to a subsequence) strongly in $H^1(\Omega)$ as $\varepsilon \to 0$.

Remark 1.4. From the proof of Theorem 1.3, we see that in the case $N = 4$ and $\varepsilon = 0$, if (i) and (ii) hold and for each $x \in \Omega$ satisfying $DQ(x) = 0$ and $Q_{max} > Q(x) \geq t$, we have

$$\Delta Q(x) > \frac{N^2 H(x,x) \int_{\mathbb{R}^N} U_{0,1}^{2^*-1}}{\int_{\mathbb{R}^N} |y|^2 U_{0,1}^{2^*}},$$

where $H(y, x)$ is the regular part of the Green's function, then (1.1) has a solution.

In order to obtain the existence of one solution for (1.1), conditions similar to (1.5) or (1.11) were imposed on a global maximum point in [12]. The degeneracy condition on the maximum point is necessary to get a solution concentrating at that point. In fact, in the next section we will prove that there is no solution of the form (1.6) which concentrates at an interior critical point $x_\circ$ with $\Delta Q(x_\circ) \neq 0$.

In the last several years, a number of results have been obtained concerning the effect of the domain topology, the domain shape and the shape of the graph of the coefficient on the number of the positive solutions for nonlinear elliptic problem with nearly critical and critical exponent, see for example [3], [10], [11], [18]–[20], [22]–24]. As far as the authors know, the first paper dealing with the effect of the topological structure of the coefficient on the number of solutions is due to Musina [18]. However, the method in [18], similar to that in [4], [5], cannot be used to construct a solution concentrating at a local maximum point of $Q(y)$.

It is not difficult to prove that if (1.1) has a solution of the form (1.6) with $x_\varepsilon \to x_\circ \in \Omega$, then $x_\circ$ must be a critical point of $Q(y)$. Thus it is interesting to know what kind of critical points of $Q(y)$ can generate a solution of the form (1.6) for (1.1). Using a similar method developed in [11], [23], we can prove that if $Q(y)$ is flat enough around a minimum point $x_\circ \in \Omega$, that is, $|D^j Q(x)| \le C|x-x_\circ|^{L-j}$, $j = 1, \dots, N-2$, $|Q(x) - Q(x_\circ)| \ge C_0|x-x_\circ|^L$, for all $x \in B_\delta(x_\circ)$, then $x_\circ$ will generate a solution of the form (1.6) for (1.1). On the other hand, if Ω is convex and $x_\circ \in \partial\Omega$ is a minimum point of $Q(y)$ such that $Q(y)$ is nondecreasing in the direction n in a neighbourhood of $x_\circ$, where n is the inward unit normal of $\partial\Omega$ at $x_\circ$, then using the moving plane method of Gidas, Ni and Nirenberg [13], we see that the distance between the maximum point of any positive solution of (1.1) and $x_\circ$ has a positive lower bound. As a result, there is no solution concentrating at $x_\circ$. So the problem of what kind of boundary point can generate a solution is far from well understood.

Our main results here show that the topological structure of the global maximum set can not only affect the number of the single peak solution, but also create a new kind of solution, that is, solution which does not concentrate at certain points. It is easy to check that the energy of the solution for (1.1) in the case $\varepsilon = 0$ is at least $(1/N)S^{N/2}/Q_{\max}^{(N-2)/2}$, where S is the best Sobolev constant for the embedding $H^1(\mathbb{R}^N) \to L^{2^*}(\mathbb{R}^N)$, but above this energy level, the corresponding functional does not satisfies the PS condition. To overcome this difficulty, we first perturbe the original problem suitably and construct a solution for this perturbed problem, whose energy is strictly greater than $(1/N)S^{N/2}/Q_{\max}^{(N-2)/2}$. Then we prove that the solution for the perturbed problem converges strongly in $H^1(\Omega)$ to a solution of the original problem.

There are papers on the existence of solutions for (1.1) in the case $\varepsilon = 0$ and $\Omega = \mathbb{R}^N$ under some symmetry assumptions on the coefficient $Q(x)$, see for example [7], [8], [14]–[16]. In [6], Bianchi considered (1.1) on $\mathbb{R}^N$ with the general coefficient $Q(x)$. Among other things, he assumed that $Q(x)$ has only a finite number of critical points and $Q(x)$ possesses at least two isolated global maximum points (so the maximum set of $Q(x)$ is not contractible in a small neighbourhood of itself). Thus his result does not apply to the case where the maximum set of $Q(x)$ is a sphere.

Let $K : H^1(\Omega) - \{0\} \to \mathbb{R}$ be a functional defined by

$$K(u) = \frac{\int_\Omega |Du|^2 - \varepsilon \int_\Omega u^2}{(\int_\Omega Q(y)|u|^{2^*})^{2/2^*}}. \tag{1.17}$$

Let $\mathcal{M} =: \{x \in \Omega,\ \lambda \ge \lambda_\circ,\ v \in E_{x,\lambda}\}$, where $\lambda_\circ$ is a large positive number. For $(x, \lambda, v) \in \mathcal{M}$ we set

$$J(x, \lambda, v) = K(PU_{x,\lambda} + v). \tag{1.18}$$

It is well known that if $\|v\|$ is small enough, $PU_{x,\lambda} + v$ is a critical point of $K(u)$ if and only if $(x, \lambda, v) \in \mathcal{M}$ is a critical point of $J(x, \lambda, v)$ on $\mathcal{M}$, see for example

[6], [2], [19]. Moreover, if $\|v\|$ is small enough, then the critical point $PU_{x,\lambda}+v$ of K is positive. On the other hand, $(x,\lambda,v)\in\mathcal{M}$ is a critical point of $J(x,\lambda,v)$ on $\mathcal{M}$ if and only if there are $A\in\mathbb{R}$, $B\in\mathbb{R}$ and $G_j\in\mathbb{R}$, such that

$$\frac{\partial J}{\partial x_i}=B\Big\langle\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda\partial x_i},v\Big\rangle+\sum_{j=1}^{N}G_j\Big\langle\frac{\partial^2 PU_{x,\lambda}}{\partial x_j\partial x_i},v\Big\rangle,\quad i=1,\dots,N,\tag{1.19}$$

$$\frac{\partial J}{\partial\lambda}=B\Big\langle\frac{\partial^2 PU_{x,\lambda}}{\partial\lambda^2},v\Big\rangle+\sum_{j=1}^{N}G_j\Big\langle\frac{\partial^2 PU_{x,\lambda}}{\partial x_j\partial\lambda},v\Big\rangle,\tag{1.20}$$

$$\frac{\partial J}{\partial v}=APU_{x,\lambda}+B\frac{\partial PU_{x,\lambda}}{\partial\lambda}+\sum_{j=1}^{N}G_j\frac{\partial PU_{x,\lambda}}{\partial x_j}.\tag{1.21}$$

The paper is organized as follows. In Section 2 we study the interior case. Section 3 is devoted to the study of boundary case, and the proof of Theorem 1.3 is given in Section 4. Some technical estimates needed in the proofs of our main results are given in the Appendices.

2. Proof of Theorem 1.1

We commence with the following result which enables us to reduce the original problem into a finite dimensional problem.

Proposition 2.1. *There exist an $\varepsilon_\circ>0$ and $\lambda_\circ>0$, such that for each $\varepsilon\in(0,\varepsilon_\circ]$, there is a C^1-map*

$$v_\varepsilon=v_\varepsilon(x,\lambda):\Omega\times\{\lambda\ge\lambda_\circ\}\to E_{x,\lambda}$$

such that (1.21) *is satisfied. Moreover,*

$$\|v_\varepsilon\|=O\bigg(\sum_{j=1}^{k}\frac{|D^jQ(x)|}{\lambda^j}+\frac{1}{(\lambda d)^{(N-2)/2+\sigma}}+\varepsilon\lambda^{-1}\bigg),\tag{2.1}$$

where $\sigma>0$ is a constant.

Proof. The proof of Proposition 2.1 is standard and we refer to the paper [19] (see the proof of Proposition 4 there). Estimate (2.1) follows from Lemmas A.3 and A.4. □

Without loss of generality we may assume $Q_M=1$. To prove Theorem 1.1 we need the following estimate.

Lemma 2.2. *Let $x\in M_*$ and let v_ε be the map from Proposition* 2.1. *Then*

$$\begin{aligned}J(x,\lambda,v_\varepsilon)\ge{}&A^{1-2/2^*}\bigg(1+\frac{K_1H(x,x)}{\lambda^{N-2}}-K_2\varepsilon\lambda^{-2}\bigg)\\&+O\bigg(\frac{1}{\lambda^{N-2+\sigma}}+\varepsilon^2\lambda^{-2}\bigg),\quad \textit{if } N\ge5,\end{aligned}\tag{2.2}$$

$$J(x,\lambda,v_\varepsilon) \ge A^{1-2/2^*}\bigg(1+\frac{K_1H(x,x)}{\lambda^2}-\varepsilon\lambda^{-2}(K_3+o(1))\ln\lambda\bigg) + O\bigg(\frac{1}{\lambda^{2+\sigma}}+\varepsilon^2\lambda^{-2}\bigg), \quad \text{if } N=4, \tag{2.3}$$

where σ is a small positive constant, A, K_1, K_2 and K_3 are constants from Lemmas A.1 *and* A.2 *and $H(x,y)$ denotes the regular part of the Green function.*

PROOF. Let $v=v_\varepsilon$. First, in view of (A.4) from the proof of Lemma A.1, we have

$$\begin{aligned}\int_\Omega Q(y)|PU_{x,\lambda}+v|^{2^*} &= \int_{B_{\delta(x_\circ)}} Q(y)|PU_{x,\lambda}+v|^{2^*}+O(\|v\|^{2^*}+\lambda^{-N}) \\ &\le \int_{B_{\delta(x_\circ)}} |PU_{x,\lambda}+v|^{2^*}+O(\|v\|^{2^*}+\lambda^{-N}) \\ &= \int_\Omega |PU_{x,\lambda}+v|^{2^*}+O(\|v\|^{2^*}+\lambda^{-N}) \\ &\le \bigg(\int_\Omega |PU_{x,\lambda}|^{2^*}+\frac{2^*(2^*-1)}{2}\int_\Omega |PU_{x,\lambda}|^{2^*-2}v^2\bigg) \\ &\quad + O\bigg(\frac{1}{\lambda^{\theta+(N-2)/2}}\bigg)\|v\|+O(\|v\|^{2^*}+\lambda^{-N}) \\ &= \bigg(A-2^*\frac{B_1H(x,x)}{\lambda^{N-2}}+\frac{2^*(2^*-1)}{2}\int_\Omega |PU_{x,\lambda}|^{2^*-2}v^2\bigg) \\ &\quad + O\bigg(\frac{1}{\lambda^{\theta+(N-2)/2}}\bigg)\|v\|+O(\|v\|^{2^*}+\lambda^{-N}).\end{aligned} \tag{2.4}$$

We also have (see (A.2))

$$\int_\Omega |D(PU_{x,\lambda}+v)|^2 = A-\frac{B_1H(x,x)}{\lambda^{N-2}}+\|v\|^2+O(\lambda^{-N}) \tag{2.5}$$

and

$$\begin{aligned}\int_\Omega |PU_{x,\lambda}+v|^2 &= \int_\Omega |PU_{x,\lambda}|^2+O(\|v\|^2+\lambda^{-1}\|v\|) \\ &\ge \int_\Omega |PU_{x,\lambda}|^2+(\varepsilon-\tau)\|v\|^2+O(\lambda^{-2}),\end{aligned} \tag{2.6}$$

where $\tau>0$ is a small constant.

Combining (2.4)–(2.6), using (D.1) in [19], we obtain

$$J(x,\lambda,v_\varepsilon) \ge A^{1-2/2^*}\bigg(1+\frac{K_1H(x,x)}{\lambda^{N-2}}-\int_\Omega |PU_{x,\lambda}|^2\bigg)+\rho\|v\|^2 + O\bigg(\frac{1}{\lambda^{N-2+\sigma}}+\varepsilon^2\lambda^{-2}\bigg), \tag{2.7}$$

where $\rho>0$ and the result readily follows. □

We now consider the case $N \geq 5$. We define

$$c_\varepsilon =: A^{1-2/2^*}\left(1 - \frac{1}{2}K_1\varepsilon^5\right), \tag{2.8}$$

where K_1 is the constant in Lemma A.1.

Let $D = \{(x,\lambda) : x \in M_*,\ \lambda \in [\varepsilon^{-l}, \varepsilon^{-L}]\}$, where $L > l > 0$ are to be determined later. For $(x,\lambda) \in D$ we set

$$F(x,\lambda) =: J(x,\lambda, v_\varepsilon(x,\lambda)).$$

In order to use the Lusternik–Schnirelman theory of critical points to prove Theorem 1.1, we need to check that the following flow will not leave D:

$$\begin{cases} \dfrac{dY(t)}{dt} = -\mathrm{grad} F(Y(t)) \quad textfort \geq 0, \\ Y(0) = Y_\circ \in F^{c_\varepsilon}, \end{cases}$$

where $Y = (x,\lambda)$ and $F^c = \{(x,\lambda) : (x,\lambda) \in D,\ F(x,\lambda) \leq c\}$. Since along the flow $Y(t)$, $F(Y(t))$ decreases, we see that if $F(x,\lambda) > c_\varepsilon$, for all $(x,\lambda) \in \partial D$, then the flow will not touch ∂D.

Lemma 2.3. *Suppose that $N \geq 5$. Then $F(x,\lambda) > c_\varepsilon$, for all $(x,\lambda) \in \partial D$.*

Proof. Let $(x,\lambda) \in \partial D$.

Case 1. Suppose that $\lambda = \varepsilon^{-l}$. In this case, Lemma 2.2 yields

$$\begin{aligned} F(x,\lambda) \geq\, & A^{1-2/2^*}(1 + K_1 H(x,x)\varepsilon^{l(N-2)} - K_2\varepsilon^{1+2l}) \\ & + O(\varepsilon^{l(N-2+\sigma)} + \varepsilon^{2+2l}) > c_\varepsilon, \end{aligned} \tag{2.9}$$

if $\varepsilon > 0$ and $l > 0$ are small enough.

Case 2. Suppose that $\lambda = \varepsilon^{-L}$. It follows from Lemma 2.1 that

$$\begin{aligned} F(x,\lambda) \geq\, & A^{1-2/2^*}(1 + K_1 H(x,x)\varepsilon^{L(N-2)} - K_2\varepsilon^{1+2L}) \\ & + O(\varepsilon^{L(N-2+\sigma)} + \varepsilon^{2+2L}) > c_\varepsilon, \end{aligned} \tag{2.10}$$

if $L > 0$ is large enough.

Case 3. Suppose that $x \in \partial M_*$. According to our assumption there is a positive γ such that

$$Q(x) \leq 1 - \gamma, \quad x \in \partial M_*.$$

Consequently, as in Lemma A.1, we have

$$F(x,\lambda) \geq \frac{A^{1-2/2^*}}{Q(x)^{2/2^*}}(1 + o(1)) \geq \frac{A^{1-2/2^*}}{(1-\gamma)^{2/2^*}}(1 + o(1)) > c_\varepsilon \tag{2.11}$$

and the result follows. □

Proof of Theorem 1.1. *Case $N \geq 5$.* It follows from Lemma 2.3 that

$$\#\{(x,\lambda) : (x,\lambda) \in F^{c_\varepsilon},\ DF(x,\lambda) = 0\} \geq \mathrm{Cat}_D(F^{c_\varepsilon}).$$

Next, we claim that

$$M \times \{\lambda = \varepsilon^{-2}\} \subset F^{c_\varepsilon}. \tag{2.12}$$

In fact, for any $(x, \lambda) \in M \times \{\lambda = \varepsilon^{-2}\}$, we have $|D^j Q(x)| = 0$, $j = 1, 2$. From Lemma A.1, we get

$$\begin{aligned} F(x,\lambda) &= J(x,\lambda,0) + O(\lambda^{-3} + \varepsilon^2\lambda^{-2}) \\ &= A^{1-2/2^*}(1 - K_1\varepsilon\lambda^{-2}) + O(\lambda^{-3} + \varepsilon^2\lambda^{-2}) \\ &= A^{1-2/2^*}(1 - K_1\varepsilon^5) + O(\varepsilon^6) < c_\varepsilon. \end{aligned} \tag{2.13}$$

Consequently,

$$\begin{aligned} \#\{(x,\lambda) : (x,\lambda) \in F^{c_\varepsilon},\ DF(x,\lambda) = 0\} &\geq \mathrm{Cat}_D(M \times \{\lambda = \varepsilon^{-2}\}) \\ &= \mathrm{Cat}_{M_* \times \{\lambda = \varepsilon^{-2}\}}(M \times \{\lambda = \varepsilon^{-2}\}) = \mathrm{Cat}_{M_*}(M) \end{aligned} \tag{2.14}$$

and this completes the proof of Theorem 1.1 in the case $N \geq 5$.

Case $N = 4$. We define

$$\begin{aligned} c'_\varepsilon &= A^{1-2/2^*}(1 - e^{-2L_1/\varepsilon}), \\ D_1 &=: \{(x,\lambda) : x \in M_*,\ \lambda \in [\varepsilon^{-l}, e^{L_2/\varepsilon}]\}, \end{aligned}$$

where $L_2 > L_1 > 0$ are to be determined later. Then as in Lemma 2.3, we have $F(x,\lambda) > c'_\varepsilon$ if $\lambda = \varepsilon^{-l}$ or $x \in \partial M_*$. Moreover, if $\lambda = e^{L_2/\varepsilon}$, then

$$F(x,\lambda) = A^{1-2/2^*}(1 - e^{-2L_2/\varepsilon}(K_3 L_2 + O(1))) > c'_\varepsilon.$$

Hence $F(x,\lambda) > c'_\varepsilon$, for all $(x,\lambda) \in \partial D_1$. On the other hand for any $(x,\lambda) \in M \times \{\lambda = e^{L_1/\varepsilon}\}$, we have

$$\begin{aligned} F(x,\lambda) &= J(x,\lambda,0) + O(\lambda^{-2}) \\ &= A^{1-2/2^*}(1 - (K_3 + o(1))\varepsilon\lambda^{-2}\ln\lambda) + O(\lambda^{-2}) \\ &= A^{1-2/2^*}(1 - (K_3 + o(1))(L_1 + O(1))e^{-2L_1/\varepsilon}) < c'_\varepsilon, \end{aligned} \tag{2.15}$$

if $L_1 > 0$ is large enough. Therefore

$$M \times \{\lambda = e^{L_1/\varepsilon}\} \subset F^{c'_\varepsilon}.$$

Consequently,

$$\begin{aligned} &\#\{(x,\lambda) : (x,\lambda) \in F^{c_\varepsilon},\ DF(x,\lambda) = 0\} \\ &\geq \mathrm{Cat}_D(M \times \{\lambda = e^{L_1/\varepsilon}\}) \\ &= \mathrm{Cat}_{M_* \times \{\lambda = e^{L_1/\varepsilon}\}}(M \times \{\lambda = e^{L_1/\varepsilon}\}) = \mathrm{Cat}_{M_*}(M) \end{aligned} \tag{2.16}$$

and the result follows. □

To close this section we give the following nonexistence result.

Theorem 2.4. *Suppose that $N \geq 5$ and $x_\circ \in \Omega$ is a critical point of $Q(y)$ satisfying $\Delta Q(x_\circ) \neq 0$. Then* (1.1) *has no solution of the form* (1.6) *satisfying* (1.13)–(1.16).

Proof. Suppose that (1.1) has a solution of the form

$$u_\varepsilon = \alpha_\varepsilon PU_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon,$$

satisfying (1.13)–(1.16). First, we estimate v_ε. Multiplying (1.1) by v_ε and integrating over Ω, we get

$$\begin{aligned}(2.17)\quad \int_\Omega |Dv_\varepsilon|^2 &= \int_\Omega Q(y)|\alpha_\varepsilon PU_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon|^{2^*-1}v_\varepsilon \quad + \varepsilon\int_\Omega(\alpha_\varepsilon PU_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon)v_\varepsilon \\ &= \bigg[\alpha_\varepsilon^{2^*-1}\int_\Omega Q(y)PU^{2^*-1}_{x_\varepsilon,\lambda_\varepsilon}v_\varepsilon \\ &\quad + (2^*-1)\alpha_\varepsilon^{2^*-2}\int_\Omega Q(y)PU^{2^*-2}_{x_\varepsilon,\lambda_\varepsilon}v_\varepsilon^2\bigg] \\ &\quad + \varepsilon\int_\Omega(PU_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon)v_\varepsilon + O(\|v_\varepsilon\|^{2+\theta_1}),\end{aligned}$$

where $\theta_1 > 0$ is a constant. It follows from Appendix D in [19] that there exists a $\rho > 0$, such that

$$(2.18)\quad \int_\Omega |Dv_\varepsilon|^2 - (2^*-1)\alpha_\varepsilon^{2^*-2}\int_\Omega Q(y)PU^{2^*-2}_{x_\varepsilon,\lambda_\varepsilon}v_\varepsilon^2 - \varepsilon\int_\Omega v_\varepsilon^2 \geq \rho\int_\Omega |Dv_\varepsilon|^2.$$

Combining (2.17) and (2.18) we get

$$\|v_\varepsilon\|^2 \leq O\bigg(\int_\Omega Q(y)PU^{2^*-1}_{x_\varepsilon,\lambda_\varepsilon}v_\varepsilon + \varepsilon\int_\Omega PU_{x_\varepsilon,\lambda_\varepsilon}v_\varepsilon\bigg) + O(\|v_\varepsilon\|^{2+\theta_1}).$$

From this, with the aid of Lemma A.3, we obtain

$$(2.19)\quad \|v_\varepsilon\| \leq O\bigg(\frac{|DQ(x_\varepsilon)|}{\lambda_\varepsilon} + \lambda_\varepsilon^{-2} + \varepsilon\lambda_\varepsilon^{-1-\sigma}\bigg).$$

Next, multiplying (1.1) by $\partial PU_{x_\varepsilon,\lambda_\varepsilon}/\partial\lambda$ and integrating over Ω, we get

$$\alpha_\varepsilon\bigg\langle PU_{x_\varepsilon,\lambda_\varepsilon}, \frac{\partial PU_{x_\varepsilon,\lambda_\varepsilon}}{\partial\lambda}\bigg\rangle = \int_\Omega Q(y)(\alpha_\varepsilon PU_{x_\varepsilon,\lambda_\varepsilon} + v_\varepsilon)^{2^*-1}\frac{\partial PU_{x_\varepsilon,\lambda_\varepsilon}}{\partial\lambda},$$

which, together with Lemma B.1 and

$$\bigg\langle PU_{x_\varepsilon,\lambda_\varepsilon}, \frac{\partial PU_{x_\varepsilon,\lambda_\varepsilon}}{\partial\lambda}\bigg\rangle = O\bigg(\frac{1}{\lambda^{N-1}}\bigg) = O\bigg(\frac{1}{\lambda^4}\bigg),$$

yields

$$\frac{2K_0\Delta Q(x_\varepsilon)}{\lambda_\varepsilon^3} + \frac{K_2\varepsilon}{\lambda_\varepsilon^3} + O\bigg(\frac{|DQ(x_\varepsilon)| + \varepsilon}{\lambda_\varepsilon^3} + \lambda_\varepsilon^{-4}\bigg) = 0.$$

Thus we get a contradiction since $\Delta Q(x_\varepsilon) \to \Delta Q(x_\circ) \neq 0$. □

3. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. First, we define

$$\lambda_{\varepsilon,x} = \left[\frac{H(x,x)}{\varepsilon}\right]^{1/(N-4)}.$$

Choose $l = 1/(k-2) + \tau$ with $\tau > 0$ small enough. Then we have

$$l(k-2) > 1, \tag{3.1}$$

$$\left(k - \frac{2(N-2)}{N-4}\right)l < 1 + \frac{2}{N-4}. \tag{3.2}$$

Let

$$\begin{aligned} N_\varepsilon &= \{x : d(x,M) \le \varepsilon^l\} \cap \{x : d(x,\partial\Omega) \ge \varepsilon^L\}, \\ D_\varepsilon &= \{ (x,\lambda) : x \in N_\varepsilon,\ \lambda \in [\eta\lambda_{\varepsilon,x}, \varepsilon^{-T}]\}, \end{aligned}$$

where η is small fixed constant, L and T are large constants.

In this section we also assume that $Q_M = 1$. Let

$$c_\varepsilon =: A^{1-2/2^*}(1 - \tau\varepsilon^{1+2[1+t(N-2)]/(N-4)}),$$

where $t < L$ is a large constant to be determined later and $\tau > 0$ is a fixed small constant. Define

$$F(x,\lambda) = J(x,\lambda,v_\varepsilon(x,\lambda)), \quad (x,\lambda) \in D_\varepsilon.$$

In order to keep the following flow inside D_ε:

$$\begin{cases} \dfrac{dY(t)}{dt} = -\mathrm{grad}F(Y(t)), \quad Y = (x,\lambda), \\ Y(0) = Y_\circ \in F^{c_\varepsilon}, \end{cases}$$

we need the following lemma.

Lemma 3.1. *If $(x,\lambda) \in \partial D_\varepsilon$, then $F(x,\lambda) > c_\varepsilon$.*

Proof. It follows from Lemmas A.1 and A.3 that

$$\begin{aligned} F(x,\lambda) = {} & \frac{A^{1-2/2^*}}{Q(x)^{2/2^*}}\left[1 + \frac{K_1 H(x,x)}{\lambda^{N-2}} - K_2\varepsilon\lambda^{-2}\right] \\ & + O\left(\frac{|DQ(x)|^2}{\lambda^2} + \sum_{j=2}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^j} + \frac{\varepsilon}{(\lambda d)^{N-2}} + \frac{1}{(\lambda d)^{N-1}} + \varepsilon^2\lambda^{-2}\right). \end{aligned} \tag{3.5}$$

For any $\lambda \in [\eta\lambda_{\varepsilon,x}, \varepsilon^{-T}]$ we have

$$\lambda d \ge \left[\frac{H(x,x)}{\varepsilon}\right]^{1/(N-4)} d\eta \ge c\varepsilon^{-1/(N-4)}d^{1-(N-2)/(N-4)}) \to \infty, \text{ as } \varepsilon \to 0. \tag{3.6}$$

On the other hand, noting (3.1), we see that for $j \geq 2$,

$$
\begin{aligned}
\frac{|D^{(j)}Q(x)|}{\lambda^j} &= O\bigg(\frac{d(x,M)^{k-j}}{\lambda^j}\bigg) \\
&= O\Big(\varepsilon^{l(k-j)}\varepsilon^{[1+(N-2)l](j-2)/(N-4)}\lambda^{-2}\Big) \\
&= O\Big(\varepsilon^{(j-2)/(N-4)+l[k-2+2(j-2)/(N-4)]}\lambda^{-2}\Big) \\
&= O(\varepsilon^{\gamma})\varepsilon\lambda^{-2},
\end{aligned} \tag{3.7}
$$

for some $\gamma > 0$. Similarly

$$
\frac{|DQ(x)|^2}{\lambda^2} = O\bigg(\frac{\varepsilon^{l(k-1)}}{\lambda^2}\bigg) = O(\varepsilon^{\gamma})\varepsilon\lambda^{-2}.
$$

Inserting the above estimates into (3.5) yields

$$
F(x,\lambda) = \frac{A^{1-2/2^*}}{Q(x)^{2/2^*}}\bigg[1 + \frac{K_1 H(x,x)}{\lambda^{N-2}} - K_2\varepsilon\lambda^{-2}\bigg] + O(\varepsilon^{\gamma})\varepsilon\lambda^{-2}. \tag{3.8}
$$

Case 1. Suppose that $d(x,\partial\Omega) = \varepsilon^L$. Then

$$
\begin{aligned}
\varepsilon\lambda^{-2} &= O(\varepsilon(\varepsilon d^{N-2})^{2/(N-4)}) = O(\varepsilon^{1+2(1+L(N-2))/(N-4)}), \\
\frac{K_1 H(x,x)}{\lambda^{N-2}} &= O(\varepsilon^{(N-2)/(N-4)} H(x,x)^{-2/(N-4)}) = O(\varepsilon^{1+2(1+L(N-2))/(N-4)}).
\end{aligned}
$$

Hence, since $L > t$,

$$
\begin{aligned}
F(x,\lambda) &= \frac{A^{1-2/2^*}}{Q(x)^{2/2^*}}[1 + O(\varepsilon^{1+2(1+L(N-2))/(N-4)})] \\
&\geq A^{1-2/2^*}[1 + O(\varepsilon^{1+2(1+L(N-2))/(N-4)})] > c_{\varepsilon}.
\end{aligned} \tag{3.9}
$$

Case 2. Suppose that $d(x,M) = \varepsilon^l$. Then

$$
\frac{1}{Q(x)^{2/2^*}} \geq \frac{1}{(1 - a d(x,M)^k)^{2/2^*}} = \frac{1}{(1 - a\varepsilon^{lk})^{2/2^*}} \geq 1 + a'\varepsilon^{lk}, \tag{3.10}
$$

for some $a' > 0$. On the other hand, by (3.2), we get

$$
\begin{aligned}
\frac{K_1 H(x,x)}{\lambda^{N-2}} - K_2\varepsilon\lambda^{-2} &= O(\varepsilon^{(N-2)/(N-4)} d^{2(N-2)/(N-4)}) \\
&= O(\varepsilon^{1+2[1+(N-2)l]/(N-4)}) = O(\varepsilon^{\gamma})\varepsilon^{lk}.
\end{aligned} \tag{3.11}
$$

Consequently, there is a $a'' > 0$, such that

$$
\begin{aligned}
F(x,\lambda) &\geq A^{1-2/2^*}(1 + a'\varepsilon^{lk})(1 + O(\varepsilon^{\gamma})\varepsilon^{lk}) \\
&\geq A^{1-2/2^*}(1 + a''\varepsilon^{lk}) > c_{\varepsilon}.
\end{aligned} \tag{3.12}
$$

Case 3. Suppose that $\lambda = \eta\lambda_{\varepsilon,x}$. Then

$$\frac{K_1 H(x,x)}{\lambda^{N-2}} - K_2\varepsilon\lambda^{-2} = \left(\frac{K_1}{\eta^{N-2}} - K_2\eta^{-2}\right)\varepsilon^{(N-2)/(N-4)}H(x,x)^{-2/(N-4)} > 0, \tag{3.13}$$

if $\eta > 0$ is small enough. As a result,

$$F(x,\lambda) \geq A^{1-2/2^*} > c_\varepsilon.$$

Case 4. Suppose that $\lambda = \varepsilon^{-T}$. Then, if $T > 0$ is large enough,

$$F(x,\lambda) = A^{1-2/2^*}(1 + O(\varepsilon^{T(N-2)-l(N-2)} + \varepsilon^{1+2T})) > c_\varepsilon. \tag{3.14}$$

So the result follows from Steps 1–4. □

Proof of Theorem 1.2. In view of Lemma 3.1 we conclude

$$\#\{\,(x,\lambda) : DF(x,\lambda) = 0,\, (x,\lambda) \in D_\varepsilon\,\} \geq \mathrm{Cat}_{D_\varepsilon}(F^{c_\varepsilon}).$$

Next, we claim that

$$D_\varepsilon^* =: \left\{(x,\lambda)\ :\ x \in \bigcup_{x_\circ \in M}\{x_\circ + \varepsilon^t n\},\ \lambda = \lambda_{\varepsilon,x,*}\right\} \subset F^{c_\varepsilon} \tag{3.15}$$

where n is the inward unit normal to $\partial\Omega$ at $x_\circ$,

$$\lambda_{\varepsilon,x,*} =: t_*\left[\frac{H(x,x)}{\varepsilon}\right]^{1/(N-4)} \quad \text{and} \quad t_* =: \left[\frac{(N-2)K_1}{2K_2}\right]^{1/(N-4)}.$$

In fact, suppose that $(x,\lambda) \in D_\varepsilon^*$. Then

$$F(x,\lambda) = \frac{A^{1-2/2^*}}{Q(x)^{2/2^*}}(1 - c_\circ\varepsilon\lambda_{\varepsilon,x,*}^{-2}) + O(\varepsilon^\gamma)\varepsilon\lambda_{\varepsilon,x,*}^{-2}, \tag{3.16}$$

where $c_\circ = K_2 t_*^{-2} - K_1/t_*^{N-2} > 0$. On the other hand we have

$$Q(x) = 1 + O(d(x,M)^k) = 1 + O(\varepsilon^{tk}). \tag{3.17}$$

$$\varepsilon\lambda_{\varepsilon,x,*}^{-2} \geq c'\varepsilon[\varepsilon d(x,\partial\Omega)^{N-2}]^{2/(N-4)} = c'\varepsilon\varepsilon^{[1+t(N-2)]\cdot 2/(N-4)}, \tag{3.18}$$

for some $c' > 0$. Since $k > 2(N-2)/(N-4)$, we see that we can choose a suitably large t, such that

$$\varepsilon^{tk} = O(\varepsilon^\gamma)\varepsilon\lambda_{\varepsilon,x,*}^{-2}.$$

Consequently,

$$\begin{aligned} F(x,\lambda) &= A^{1-2/2^*}(1 - c_\circ\varepsilon\lambda_{\varepsilon,x,*}^{-2}) + O(\varepsilon^\gamma)\varepsilon\lambda_{\varepsilon,x,*}^{-2} \\ &\leq A^{1-2/2^*}(1 - c_\circ\varepsilon\lambda_{\varepsilon,x,*}^{-2}/2) \\ &\leq A^{1-2/2^*}(1 - c_\circ\varepsilon\varepsilon^{2[1+t(N-2)]/2(N-4)}) \leq c_\varepsilon. \end{aligned} \tag{3.19}$$

It follows from (3.15) that

$$\#\{(x,\lambda) : DF(x,\lambda)=0,\ (x,\lambda)\in D_\varepsilon\}\geq \mathrm{Cat}_{D_\varepsilon}(D_\varepsilon^*)=\mathrm{Cat}_{N_\varepsilon}(N_\varepsilon^*),$$

where $N_\varepsilon^*=\bigcup_{x_\circ\in M}\{x_\circ+\varepsilon^t n\}$, and n is the inward unit normal of $\partial\Omega$ at $x_\circ$. On the other hand, we have

$$N_\varepsilon\subset\bigcup_{x_\circ\in M}\overline{\Omega\cap B_{\varepsilon^l}(x_\circ)}=:N_\varepsilon^{**}.$$

Thus $\mathrm{Cat}_{N_\varepsilon}(N_\varepsilon^*)\geq\mathrm{Cat}_{N_\varepsilon^{**}}(N_\varepsilon^*)$. Since N_ε^{**}, N_ε^* and M are homotopically equivalent, we see that

$$\mathrm{Cat}_{N_\varepsilon^{**}}(N_\varepsilon^*)=\mathrm{Cat}_M(M)$$

and the result follows. □

4. Proof of Theorem 1.3

Let $\tau\geq 0$ be a small constant. For each fixed small $\varepsilon>0$, consider the following problem:

$$\begin{cases} -\Delta u=Q(y)u^{2^*-1-\tau}+\varepsilon u & y \text{ in } \Omega,\\ u>0 & y \text{ in } \Omega,\\ u=0 & y \text{ on } \partial\Omega. \end{cases}\tag{4.1}$$

The corresponding functional of the above problem is

$$I_\tau(u)=\frac{1}{2}\int_\Omega(|Du|^2-\varepsilon u^2)-\frac{1}{2^*-\tau}\int_\Omega Q(x)|u|^{2^*-\tau},\quad u\in H_0^1(\Omega).$$

First, we follow the basic idea of [18] to construct a solution for (4.1), whose energy is strictly greater than $S^{N/2}/NQ_{\max}^{(N-2)/2}$.

Theorem 4.1. *Suppose that the global maximum set M is not contractible in a small neighbourhood of itself, but there is a t belonging to the interval $(2^{-2/(N-2)}Q_{\max},Q_{\max})$ such that M is contractible within $\{x:Q(x)\geq t\}$. Then there are a $\tau_\circ>0$ and $\varepsilon_\circ>0$, such that for each $\tau\in(0,\tau_\circ]$ and $\varepsilon\in[0,\varepsilon_\circ]$, (4.1) has a solution u_τ satisfying*

$$\frac{1}{N}\frac{S^{N/2}}{Q_{\max}^{(N-2)/2}}+\delta<I_\tau(u_\tau)<\frac{1}{N}\frac{S^{N/2}}{t^{(N-2)/2}}+\delta,$$

where $\delta>0$ is some small constant independent of τ and ε.

Proof. Denote

$$J(u)=\int_\Omega|Du|^2-\varepsilon u^2,\quad u\in V_\tau,$$

where $V_\tau = \{u : u \in H_0^1(\Omega), \int_\Omega Q(x)|u|^{2^*-\tau} = 1\}$. Then for each fixed $\tau > 0$, $J(u)$ satisfies PS condition. We claim that $J(u)$ has a critical point $u_\tau \in V$ with

$$\frac{S}{Q_{\max}^{2/2^*}} + \delta < J(u_\tau) < \frac{S}{t^{2/2^*}} + \delta.$$

We argue by contradiction. Suppose that $J(u)$ does not have critical point in $J^{c_2} \setminus J^{c_1}$, where $J^c = \{u : u \in V_\tau, J(u) \le c\}$, $c_2 = S/t^{2/2^*} + \delta$, $c_1 = S/Q_{\max}^{2/2^*} + \delta$. Then there is a continuous map $\alpha(u) : J^{c_2} \to J^{c_1}$ satisfying $\alpha(u) = u$ for $u \in J^{c_1}$. By assumption, there is a continuous map $h(x,s) : M \times [0,1] \to \{x : Q(x) \ge t\}$ satisfying $h(x,0) = x$, $h(x,1) = x_\circ$, for all $x \in M$. Define

$$f(x,s)(y) = \beta\biggl(\alpha\biggl(\frac{\eta(|y-x|)U_{h(x,s),\lambda}}{(\int_\Omega Q(y)|\eta(|y-h(x,s)|)U_{h(x,s),\lambda}|^{2^*-\tau}\,dy)^{1/(2^*-\tau)}}\biggr)\biggr),$$

where $\lambda = \tau^{-L}$, L is some large constant, $\eta(r)$ is a smooth function with $\eta(r) = 0$ outside a small neighbourhood of 0, and

$$\beta(u) = \frac{\int_\Omega y|u|^{2^*}\,dy}{\int_\Omega |u|^{2^*}\,dy}.$$

It is easy to check that

$$J(f(x,s)(\,\cdot\,)) = \frac{S}{Q(h(x,s))^{1/2^*}} + o(1),$$

where $o(1) \to 0$ as $\tau \to 0$ and $\varepsilon \to 0$. Since $h(x,s) \in \{x : Q(x) \ge t\}$, we see that

$$\frac{\eta(|\,\cdot\, - x|)U_{h(x,s),\lambda}}{(\int_\Omega Q(y)|\eta(|y-h(x,s)|)U_{h(x,s),\lambda}|^{2^*-\tau}\,dy)^{1/(2^*-\tau)}} \in J^{c_2}$$

if $\lambda > 0$ is large enough. On the other hand, it follows from concentration compactness principle [17], [21] that if $\delta > 0$ and $\varepsilon \ge 0$ are small enough, then for any $u \in J^{c_1}$, $\beta(u)$ is in a small neighbourhood of M. So we see that $f(x,s)$ is a point in a small neighbourhood of M. Since for $x \in M$, $\eta(|y-x|)U_{x,\lambda} \in J^{c_1}$ if $\lambda > 0$ is large, we have $f(x,0) = x$. But $f(x,1) = x'_\circ$. This means that M can be deformed to a point within a small neighbourhood of M, a contradiction to our assumption. □

The rest of this section is devoted to proving the solution u_τ for (4.1) with

$$\frac{1}{N}\frac{S^{N/2}}{Q_{\max}^{(N-2)/2}} + \delta < I_\tau(u_\tau) < \frac{1}{N}\frac{S^{N/2}}{t^{(N-2)/2}} + \delta,$$

converges strongly in $H^1(\Omega)$ to a function $u_\circ$ as $\tau \to 0$. Thus $u_\circ$ is a solution of (1.1).

Proposition 4.2. *Suppose that u_τ is a solution of* (4.1) *with*

$$\frac{1}{N}\frac{S^{N/2}}{t_1^{(N-2)/2}} < I_\tau(u_\tau) < \frac{1}{N}\frac{S^{N/2}}{t_2^{(N-2)/2}},$$

where $2^{-2/(N-2)}Q_{\max} < t_2 < t_1 < Q_{\max}$. Assume that $\max_{x\in\partial\Omega} Q(x) < t_2$ and for each $x \in \Omega$ with $t_2 \le Q(x) \le t_1$, $DQ(x) = 0$, we have $\Delta Q(x) > 0$. Then u_τ converges strongly in $H^1(\Omega)$ as $\tau \to 0$.

We divide the proof of Proposition 4.2 into two lemmas. In the following, we always assume that $Q_{\max} = 1$.

Lemma 4.3. *Suppose that u_τ is a solution of* (4.1) *with*

$$I_\tau(u_\tau) < \frac{S^{N/2}}{t^{(N-2)/2}},$$

where $t > 2^{-2/(N-2)}Q_{\max}$, then, there is an $\varepsilon_\circ > 0$, such that for each fixed $\varepsilon \in [0, \varepsilon_\circ]$, we have that as $\tau \to 0$, either u_τ converges strongly, or there are $z_\tau \in \Omega$ and $\mu_\tau \to \infty$, such that

$$\left\| u_\tau - \left(\frac{Q(x_\circ)}{\mu}\right)^{-1/(2^*-2)} U_{z_\tau,\mu_\tau} \right\| \to 0,$$

where $z_\tau \to x_\circ$ and $\mu = \lim_{\tau\to 0} \mu_\tau^\tau \ge 1$ is a constant.

Proof. It is easy to check that u_τ is bounded in $H^1(\Omega)$. We assume

$$u_\tau \rightharpoonup u_\circ \text{ weakly in } H^1(\Omega) \text{ as } \tau \to 0.$$

On the other hand, by the Sobolev inequality, we see that if $u_\circ \ne 0$, then

$$I(u_\circ) \ge \frac{1}{N}S^{N/2} + o(1), \tag{4.2}$$

where $o(1) \to 0$ as $\tau \to 0$ and $\varepsilon \to 0$ and $I = I_\circ$.

Denote $v_\tau = u_\tau - u_\circ$. Then

$$\begin{aligned} -\Delta v_\tau = {}& Q(y)|v_\tau|^{2^*-2-\tau}v_\tau + \varepsilon v_\tau \\ & + Q(y)[(v_\tau + u_\circ)^{2^*-1-\tau} - |v_\tau|^{2^*-2-\tau}v_\tau - u_\circ^{2^*-1-\tau}]. \end{aligned} \tag{4.3}$$

Assume $\|v_\tau\|^2 \to l \ge 0$ as $\tau \to 0$. It follows from (4.3) and the Sobolev inequality that

$$l \le \left(\frac{l}{S}\right)^{2^*/2} + o(1), \tag{4.4}$$

which implies

$$l \ge (1 + o(1))S^{N/2}, \quad \text{if } l \ne 0, \tag{4.5}$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Combining (4.2) and (4.5), we conclude that if $u_\circ \neq 0$ and $l \neq 0$, then

$$I_\tau(u_\tau) = I(u_\circ) + I_\tau(v_\tau) + o(1) \geq \frac{2}{N} S^{N/2} + o(1).$$

So, under our assumption, we see that if $\|u_\tau - u_\circ\| \to l > 0$, then $u_\circ = 0$.

Now we assume $u_\circ = 0$. Then we claim that $\max_{x\in\Omega} u_\tau \to \infty$ as $\tau \to 0$. Otherwise, the boundedness of L^∞-norm of u_τ would imply the boundedness of $C^{1,\alpha}$-norm of u_τ. So $\|u_\tau\| \to 0$. This is a contradiction.

Let $z_\tau \in \Omega$ be such that $u_\tau(z_\tau) = \max_{x\in\Omega} u_\tau =: \mu_\tau^{(N-2)/2}$. Denote

$$w_\tau(z) = \mu_\tau^{-(N-2)/2} u_\tau\left(\frac{1}{\mu_\tau} z + z_\tau\right).$$

Then w_τ is a bounded sequence in $H^1(\mathbb{R}^N)$ and satisfies

$$-\Delta w = Q\left(\frac{1}{\mu_\tau} z + z_\tau\right) \frac{1}{\mu_\tau^{\tau(N-2)/2}} w^{2^*-1-\tau} + \varepsilon \mu_\tau^{-2} w.$$

Let $z_\tau \to x_\circ$ and $\mu_\tau^{\tau(N-2)/2} \to \mu \geq 1$. We assume

$$\begin{aligned} &w_\tau \rightharpoonup w_\circ \text{ weakly in } H^1(D) \text{ as } \tau \to 0, \\ &w_\tau \to w_0 \text{ in } C^{1,\alpha}_{\mathrm{loc}}(D), \end{aligned}$$

where D is $\mathbb{R}^N$ or half space. Then $w_\circ$ satisfies

$$-\Delta w = \frac{Q(x_\circ)}{\mu_\circ} w^{2^*-1}. \tag{4.6}$$

Since $w_\circ(0) = 1$, we see that $\mu < \infty$. By Pohozaev identity, (4.6) does not have positive solution if D is half space. So we conclude that $D = \mathbb{R}^N$.

Let $\omega_\tau = w_\tau - w_\circ$. As before, we see that if $\|\omega_\tau\| \to l > 0$, then $l \geq S^{N/2}$. Thus

$$\begin{aligned} I(u_\tau) &= \frac{1}{2}\|w_\tau\|^2 - \frac{1}{2^*-\tau}\int_{\mathbb{R}^N} \frac{Q(\mu_\tau^{-1} z + z_\tau)}{\mu} |w_\tau|^{2^*-\tau} + o(1) \\ &= \frac{1}{2}\|w_\circ\|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} \frac{Q(x_\circ)}{\mu} |w_\circ|^{2^*} \\ &\quad + \frac{1}{2}\|\omega_\tau\|^2 - \frac{1}{2^*-\tau}\int_{\mathbb{R}^N} \frac{Q(\mu_\tau^{-1} z + z_\tau)}{\mu} |\omega_\tau|^{2^*-\tau} + o(1) \\ &\geq \frac{2}{N} S^{N/2} + o(1). \end{aligned}$$

So we conclude that $\|\omega_\tau\| \to 0$, and the result follows. □

By Lemma 4.3 and Proposition 7 in [2], we know that a solution u_τ of (4.1) with $I_\tau(u_\tau) < (1/N)2^{2/(N-2)} S^{N/2}$ can be written in the form

$$u_\tau = \alpha_\tau P U_{x_\tau,\lambda_\tau} + v_\tau, \tag{4.7}$$

where $\alpha_\tau \to \mu/Q(x_\circ))^{1/(2^*-2)}$, $\mu_\tau/\lambda_\tau+\lambda_\tau/\mu_\tau+\mu_\tau\lambda_\tau|x_\tau-z_\tau|^2 \le C$, $v_\tau \in E_{x_\tau,\lambda_\tau}$, $\|v_\tau\| \to 0$. As a result, $x_\tau \to x_\circ$, $\lambda_\tau \to \infty$, and $\mu = \lim_{\tau\to 0}\mu_\tau^\tau$.

Next, we give a necessary condition for the location of $x_\circ$ and prove $\mu = 1$ for any solution of (4.1) of the form (4.7).

Lemma 4.4. *Suppose that u_τ is a solution of the form* (4.7) *for (4.1), satisfying $I_\tau(u_\tau) < 2S^{N/2}/N$. Then $\mu = 1$ and $DQ(x_\circ) = 0$ and $\Delta Q(x_\circ) \le 0$.*

Proof. First, using Lemma B.1 and arguing as Theorem 2.1, we obtain

$$\|v_\tau\|^2 = O\bigg(\tau^2 + \frac{|DQ(x)|^2}{\lambda^2} + \frac{1}{\lambda^{2+\sigma}}\bigg). \tag{4.8}$$

We claim that $x_\circ \in \Omega$. In fact, if $x_\circ \in \partial\Omega$, then it follows from $Q(x_\circ) \le \max_{x\in\partial\Omega} Q(x) \le 2^{-2/(N-2)}$ that

$$I_\tau(u_\tau) = \frac{1}{N}\frac{S^{N/2}}{Q(x_\circ)^{2/(N-2)}} + o(1) \ge \frac{2}{N}S^{N/2}.$$

This is a contradiction. On the other hand, we have

$$\bigg\langle u_\tau, \frac{\partial PU_{x_\tau,\lambda_\tau}}{\partial\lambda}\bigg\rangle = \int_\Omega Q(y)u_\tau^{2^*-1-\tau}\frac{\partial PU_{x_\tau,\lambda_\tau}}{\partial\lambda}. \tag{4.9}$$

Using Lemma B.1, we get

$$\frac{\tau}{\lambda^{1+(N-2)\tau/2}} \le \frac{C}{\lambda^3},$$

which, in view of $\lambda^\tau \le C < \infty$, implies $\lambda \le C\tau^{-1/2}$. Hence, $\mu = \lim_{\tau\to 0}\lambda^\tau = 1$.

We also have

$$\bigg\langle u_\tau, \frac{\partial PU_{x_\tau,\lambda_\tau}}{\partial x_j}\bigg\rangle = \int_\Omega Q(y)u_\tau^{2^*-1-\tau}\frac{\partial PU_{x_\tau,\lambda_\tau}}{\partial x_j}, \tag{4.10}$$

which, in view of Lemma A.3, is equivalent to

$$\int_\Omega Q(y)PU_{x_\tau,\lambda_\tau}^{2^*-1-\tau}\frac{\partial PU_{x_\tau,\lambda_\tau}}{\partial x_j} = o(1). \tag{4.11}$$

But

$$\begin{aligned}\int_\Omega Q(y)&PU_{x_\tau,\lambda_\tau}^{2^*-1-\tau}\frac{\partial PU_{x_\tau,\lambda_\tau}}{\partial x_j} \\ &= \int_\Omega Q(y)U_{x_\tau,\lambda_\tau}^{2^*-1-\tau}\frac{\partial U_{x_\tau,\lambda_\tau}}{\partial x_j} + o(1) \\ &= \int_\Omega \langle DQ(x), y-x\rangle U_{x_\tau,\lambda_\tau}^{2^*-1-\tau}\frac{\partial U_{x_\tau,\lambda_\tau}}{\partial x_j} + o(1) \\ &= D_jQ(x)\int_\Omega (y_j-x_j)U_{x_\tau,\lambda_\tau}^{2^*-1-\tau}\frac{\partial U_{x_\tau,\lambda_\tau}}{\partial x_j} + o(1).\end{aligned} \tag{4.12}$$

Since $\int_\Omega (y_j - x_j)U_{x_\tau,\lambda_\tau}^{2^*-1-\tau}\partial U_{x_\tau,\lambda_\tau}/\partial x_j \le -c' < 0$, (4.11) and (4.12) imply $DQ(x_\circ) = 0$.

Using (4.9) and Lemma B.1, we have

$$\frac{N-2}{22^*}Q(x)A\frac{\tau}{\lambda}+\frac{\Delta Q(x_\circ)}{2^*N\lambda^3}\int_{\mathbb{R}^N}|y|^2U_{0,1}^{2^*}+\frac{2\varepsilon}{\lambda^3}\int_{\mathbb{R}^N}U_{0,1}^2=o\biggl(\frac{1}{\lambda^3}\biggr).$$

Thus we obtain $\Delta Q(x_\circ)\le -\varepsilon 22^*N\int_{\mathbb{R}^N}U_{0,1}^2\le 0$. □

Proof of Proposition 4.2. In view of Lemma 4.3, to prove Proposition 4.2, we only need to prove $u_\circ\ne 0$. Suppose that $u_\circ=0$. Then it follows from Lemma 4.4 that

$$u_\tau=\alpha_\tau PU_{x_\tau,\lambda_\tau}+v_\tau,$$

and as $\tau\to 0$, $\alpha_\tau\to 1/Q(x_\circ)^{1/(2^*-2)}$, $x_\tau\to x_\circ$, $DQ(x_\circ)=0$ and $\Delta Q(x_\circ)\le 0$. It is easy to see

$$I_\tau(u_\tau)=\frac{1}{N}\frac{S^{N/2}}{Q(x_\circ)^{1/(2^*-2)}}+o(1).$$

Thus we deduce

$$t_2\le Q(x_\circ)\le t_1.$$

According to our assumption, we have $\Delta Q(x_\circ)>0$. This is a contradiction. □

Proof of Theorem 1.3. The existence part is just a direct consequence of Theorem 4.1 and Proposition 4.2. To prove that u_τ (up to a subsequence) converges strongly in $H^1(\Omega)$ as $\tau\to 0$, we just need to repeat the proof of Proposition 4.2 and thus we omit the details. □

Appendix A

Let $d=d(x,\partial\Omega)$.

Lemma A.1. *Suppose that $N\ge 5$. We have*

$$\begin{aligned}K(PU_{x,\lambda})&=\frac{A^{1-2/2^*}}{Q(x)^{2/2^*}}\biggl[1+\frac{K_1H(x,x)}{\lambda^{N-2}}-K_2\varepsilon\lambda^{-2}\biggr]\\&\quad+O\biggl(\sum_{j=2}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^j}+\frac{\varepsilon}{(\lambda d)^{N-2}}+\frac{1}{(\lambda d)^{N-1}}\biggr),\end{aligned}\tag{A.1}$$

where

$$A=\int_{\mathbb{R}^N}U^{2^*},\quad K_1=\frac{1}{A}\int_{\mathbb{R}^N}U^{2^*-1},\quad K_2=\frac{1}{A}\int_{\mathbb{R}^N}U^2.$$

Proof. Let $\varphi_{x,\lambda}=U_{x,\lambda}-PU_{x,\lambda}$. Then it follows from Proposition 1 in [19] that

$$0\le\varphi_{x,\lambda}\le\frac{H(x,x)}{\lambda^{(N-2)/2}}.$$

As in [6] and [19], we have

$$\int_\Omega |DPU_{x,\lambda}|^2 = \int_\Omega U_{x,\lambda}^{2^*-1} PU_{x,\lambda} = A - \frac{B_1 H(x,x)}{\lambda^{N-2}} + O\bigg(\frac{1}{(\lambda d)^N}\bigg), \tag{A.2}$$

$$\int_\Omega |PU_{x,\lambda}|^2 = B_2\lambda^{-2} + O\bigg(\frac{1}{(\lambda d)^{N-2}}\bigg), \tag{A.3}$$

where $B_1 = \int_\Omega U_{x,\lambda}^{2^*-1}$, $B_2 = \int_{\mathbb{R}^N} U^2$. We also have (see [19])

$$\begin{aligned}\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*} &= \int_\Omega Q(y)|U_{x,\lambda} - \varphi_{x,\lambda}|^{2^*} \\ &= \int_\Omega Q(y)U_{x,\lambda}^{2^*} - 2^*\int_\Omega Q(y)U_{x,\lambda}^{2^*-1}\varphi_{x,\lambda} + O\bigg(\frac{1}{(\lambda d)^{N-1}}\bigg) \\ &= \int_\Omega Q(y)U_{x,\lambda}^{2^*} - 2^*\frac{B_1 Q(x)H(x,x)}{\lambda^{N-2}} + O\bigg(\frac{1}{(\lambda d)^{N-1}}\bigg).\end{aligned} \tag{A.4}$$

Using Taylor's expansion and the radial symmetry of U we write

$$\begin{aligned}\int_\Omega Q(y)U_{x,\lambda}^{2^*} = {}& Q(x)A + \int_\Omega \langle DQ(x), y-x\rangle U_{x,\lambda}^{2^*} \\ &+ O\bigg(\sum_{j=2}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^j}\bigg) + O\bigg(\frac{1}{\lambda^{N-1}}\bigg).\end{aligned} \tag{A.5}$$

Using the symmetry of U, we deduce easily

$$\int_\Omega \langle DQ(x), y-x\rangle U_{x,\lambda}^{2^*} = O\bigg(\frac{1}{\lambda}\frac{1}{(\lambda d)^{N-1}}\bigg). \tag{A.6}$$

Combining (A.4) and (A.5) we get

$$\begin{aligned}\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*} = {}& Q(x)A - 2^*\frac{B_1 Q(x)H(x,x)}{\lambda^{N-2}} \\ &+ O\bigg(\sum_{j=2}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^j}\bigg) + O\bigg(\frac{1}{(\lambda d)^{N-1}}\bigg).\end{aligned} \tag{A.7}$$

Clearly, Lemma A.1 follows from (A.2), (A.3) and (A.7). □

Lemma A.2. *Suppose that $N = 4$. We have*

$$\begin{aligned}K(PU_{x,\lambda}) = {}& \frac{A^{1-2/2^*}}{Q(x)^{2/2^*}}\bigg[1 + \frac{K_1 H(x,x)}{\lambda^{N-2}} - \varepsilon\lambda^{-2}\ln(\lambda d)\Big(K_3 + o(1)\Big)\bigg] \\ &+ O\bigg(\sum_{j=2}^{N-2}\frac{|D^{(j)}Q(x)|}{\lambda^j} + \frac{\varepsilon}{(\lambda d)^{N-2}} + \frac{1}{(\lambda d)^{N-1}}\bigg),\end{aligned} \tag{A.8}$$

where K_3 is some positive constant and $o(1) \to 0$ as $\lambda d \to \infty$.

PROOF. In order to prove Lemma A.2, we only need to note that

$$\frac{\int_\Omega U_{x,\lambda}^2}{\lambda^2 \ln(\lambda d)} \to K_3 > 0,$$

as $\lambda d \to \infty$. □

LEMMA A.3. *Let k be the biggest positive integer satisfying $k \le (N-2)/2$. Suppose that $\lambda^\tau \le C$. Then for any $v \in E_{x,\lambda}$ and $\tau \ge 0$, we have*

$$\begin{aligned}\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-1-\tau} v \\ = O\biggl(\tau + \sum_{j=1}^{k} \frac{|D^j Q(x)|}{\lambda^j} + \frac{1}{(\lambda d)^{\theta+(N-2)/2}}\biggr)\|v\|,\end{aligned} \tag{A.9}$$

$$\begin{aligned}\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-2-\tau} \frac{\partial PU_{x,\lambda}}{\partial \lambda} v \\ = O\biggl(\tau + \sum_{j=1}^{k} \frac{|D^j Q(x)|}{\lambda^j} + \frac{1}{(\lambda d)^{\theta+(N-2)/2}}\biggr)\lambda^{-1}\|v\|,\end{aligned} \tag{A.10}$$

$$\begin{aligned}\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-2-\tau} \frac{\partial PU_{x,\lambda}}{\partial x_j} v \\ = O\biggl(\tau + \sum_{j=1}^{k} \frac{|D^j Q(x)|}{\lambda^j} + \frac{1}{(\lambda d)^{\theta+(N-2)/2}}\biggr)\lambda\|v\|,\end{aligned} \tag{A.11}$$

where $\theta > 0$ is a positive constant.

PROOF. In fact, arguing as Rey [19] (see (3.20)–(3.22) there), we have

$$\begin{aligned}&\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-1-\tau} v \\ &= \int_\Omega Q(y)|U_{x,\lambda}|^{2^*-1-\tau} v + \int_\Omega Q(y)(|PU_{x,\lambda}|^{2^*-1-\tau} - |U_{x,\lambda}|^{2^*-1-\tau})v \\ &= \int_\Omega Q(y)|U_{x,\lambda}|^{2^*-1-\tau} v + O\biggl(\frac{\|v\|}{(\lambda d)^{\theta+(N-2)/2}}\biggr) \\ &= Q(x)\int_\Omega |U_{x,\lambda}|^{2^*-1-\tau} v \\ &\quad + \int_\Omega (Q(y)-Q(x))|U_{x,\lambda}|^{2^*-1-\tau} v + O\biggl(\frac{1}{(\lambda d)^{\theta+(N-2)/2}}\biggr)\|v\| \\ &= Q(x)\int_\Omega |U_{x,\lambda}|^{2^*-1-\tau} v + O\biggl(\sum_{j=1}^{k} \frac{|D^{(j)} Q(x)|}{\lambda^j} + \frac{1}{(\lambda d)^{\theta+(N-2)/2}}\biggr)\|v\|.\end{aligned} \tag{A.12}$$

But

$$\text{(A.13)}\quad Q(x)\int_\Omega |U_{x,\lambda}|^{2^*-1-\tau}v = \lambda^{-(N-2)/2\tau}\int_{\mathbb{R}^N} U_{0,1}^{2^*-1-\tau}v\bigg(\frac{y}{\lambda}+x\bigg)\,dy = \lambda^{-(N-2)/2\tau}\int_{\mathbb{R}^N}[U_{0,1}^{2^*-1-\tau}-U_{0,1}^{2^*-1}]v\bigg(\frac{y}{\lambda}+x\bigg)\,dy = O(\tau)\|v\|.$$

Combining (A.12) and (A.13), we get (A.9). Since $|\partial PU_{x,\lambda}/\partial\lambda| \le C\lambda^{-1}U_{x,\lambda}$ and $|\partial PU_{x,\lambda}/\partial x_j| \le C\lambda U_{x,\lambda}$, we can prove (A.10) and (A.11) in a similar way.□

Lemma A.4. *There is a $\sigma > 0$, such that*

$$\int_\Omega PU_{x,\lambda}v = O\bigg(\frac{1}{\lambda^{1+\sigma}}\bigg)\|v\|,\qquad \int_\Omega \frac{\partial PU_{x,\lambda}}{\partial\lambda}v = O\bigg(\frac{1}{\lambda^{2+\sigma}}\bigg)\|v\|.$$

Proof. For the proof of Lemma A.4 we refer to the paper [19, (3.19), p. 18]. □

Appendix B

In this section we assume that $x \in \Omega$ satisfies $d = d(x,\partial\Omega) \ge d_\circ > 0$, and $v = v(x,\lambda) \in E_{x,\lambda}$ satisfies

$$\text{(B.1)}\qquad \|v\| = O\bigg(\tau + \frac{|DQ(x)|}{\lambda} + \lambda^{-1-\sigma}\bigg),$$

where $\sigma > 0$ is a small constant.

Lemma B.1. *Suppose that $N \ge 4$ and if $N = 4$, then $\varepsilon = 0$. We have*

$$\text{(B.2)}\quad \bigg\langle I'_\tau(\alpha PU_{x,\lambda}+v), \frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle = \frac{N-2}{22^*}Q(x)A\frac{\tau}{\lambda^{1+(N-2)\tau/2}} + \frac{\Delta Q(x)}{2^*N\lambda^{3+(N-2)\tau/2}} - \frac{(N-2)H(x,x)}{2\lambda^{N-1+(N-2)\tau/2}}\int_{\mathbb{R}^N} U_{0,1}^{2^*-1} + \frac{2\varepsilon}{\lambda^3}\int_{\mathbb{R}^N} U_{0,1}^2 + o\bigg(\frac{1}{\lambda^3}\bigg),$$

where $\lambda^3 o(1/\lambda^3) \to 0$ as $\tau \to 0$, $\alpha \to 1$.

Proof. We have

$$\text{(B.3)}\quad \bigg\langle I'_\tau(\alpha PU_{x,\lambda}+v), \frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle = \bigg\langle \alpha PU_{x,\lambda}, \frac{\partial PU_{x,\lambda}}{\partial\lambda}\bigg\rangle - \varepsilon\int_\Omega(\alpha PU_{x,\lambda}+v)\frac{\partial PU_{x,\lambda}}{\partial\lambda} - \int_\Omega Q(y)|\alpha PU_{x,\lambda}+v|^{2^*-1}\frac{\partial PU_{x,\lambda}}{\partial\lambda}.$$

By (B.5) in [19], we have

$$\left\langle PU_{x,\lambda}, \frac{\partial PU_{x,\lambda}}{\partial\lambda}\right\rangle = \frac{(N-2)BH(x,x)}{2\lambda^{N-1}} + O\left(\frac{1}{\lambda^N}\right), \tag{B.4}$$

where $B = \int_{\mathbb{R}^N} U_{0,1}^{2^*-1}$.

Similarly to (B.16) in [19], using Lemma A.4, we have

$$\begin{aligned}\int_\Omega (PU_{x,\lambda} + v)\frac{\partial PU_{x,\lambda}}{\partial\lambda} &= -2B_1\lambda^{-3} + O\left(\frac{\|v\|}{\lambda^{2+\sigma}}\right)\\ &= -2B_1\lambda^{-3} + O\left(\frac{\tau}{\lambda^{2+\sigma}} + \frac{1}{\lambda^{3+\sigma}}\right),\end{aligned} \tag{B.5}$$

where $B_1 = \int_{\mathbb{R}^N} U_{0,1}^2$. On the other hand it follows from Lemma A.3 that

$$\begin{aligned}&\int_\Omega Q(y)|\alpha PU_{x,\lambda} + v|^{2^*-1-\tau}\frac{\partial PU_{x,\lambda}}{\partial\lambda}\\ &\quad= \int_\Omega Q(y)|\alpha PU_{x,\lambda}|^{2^*-1-\tau}\frac{\partial PU_{x,\lambda}}{\partial\lambda}\\ &\qquad+ (2^*-1-\tau)\int_\Omega Q(y)|\alpha PU_{x,\lambda}|^{2^*-2-\tau}\frac{\partial PU_{x,\lambda}}{\partial\lambda}v + O\left(\frac{\|v\|^2}{\lambda}\right)\\ &\quad= \int_\Omega Q(y)|\alpha PU_{x,\lambda}|^{2^*-1-\tau}\frac{\partial PU_{x,\lambda}}{\partial\lambda}\\ &\qquad+ O\left(\frac{\tau^2}{\lambda} + \frac{|DQ(x)|^2}{\lambda^3} + \frac{1}{\lambda^{3+\sigma}}\right).\end{aligned} \tag{B.6}$$

Also, by Proposition 1 in [19], we have

$$\begin{aligned}&\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-1-\tau}\frac{\partial PU_{x,\lambda}}{\partial\lambda}\\ &\quad= \int_\Omega Q(y)|U_{x,\lambda}|^{2^*-1-\tau}\frac{\partial PU_{x,\lambda}}{\partial\lambda}\\ &\qquad- (2^*-1-\tau)\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-2-\tau}\varphi_{x,\lambda}\frac{\partial PU_{x,\lambda}}{\partial\lambda}\\ &\qquad+ O\left(\frac{\|\varphi_{x,\lambda}\|_\infty^2}{\lambda}\int_\Omega U_{x,\lambda}^{2^*-2-\tau}\right)\\ &\quad= \int_\Omega Q(y)|U_{x,\lambda}|^{2^*-1-\tau}\frac{\partial PU_{x,\lambda}}{\partial\lambda}\\ &\qquad- (2^*-1-\tau)\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-2-\tau}\varphi_{x,\lambda}\frac{\partial PU_{x,\lambda}}{\partial\lambda} + O\left(\frac{1}{\lambda^{3+\sigma}}\right).\end{aligned} \tag{B.7}$$

Besides,

$$\begin{aligned}&\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-1-\tau}\frac{\partial PU_{x,\lambda}}{\partial\lambda}\\ &\quad= \int_\Omega Q(y)|U_{x,\lambda}|^{2^*-1-\tau}\frac{\partial U_{x,\lambda}}{\partial\lambda} - \int_\Omega Q(y)|U_{x,\lambda}|^{2^*-1-\tau}\frac{\partial\varphi_{x,\lambda}}{\partial\lambda} + O\left(\frac{1}{\lambda^N}\right).\end{aligned} \tag{B.8}$$

In view of the symmetry of $U_{x,\lambda}$ and $\partial U_{x,\lambda}/\partial\lambda$, we have

$$
\begin{aligned}
\text{(B.9)}\quad &\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-1-\tau}\frac{\partial U_{x,\lambda}}{\partial\lambda}\\
&= Q(x)\int_\Omega |U_{x,\lambda}|^{2^*-1-\tau}\frac{\partial U_{x,\lambda}}{\partial\lambda} + \int_\Omega \langle DQ(x), y-x\rangle |U_{x,\lambda}|^{2^*-1-\tau}\frac{\partial U_{x,\lambda}}{\partial\lambda}\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N}\langle D^2Q(x)(y-x), y-x\rangle |U_{x,\lambda}|^{2^*-1-\tau}\frac{\partial U_{x,\lambda}}{\partial\lambda} + O\biggl(\frac{1}{\lambda^4}\biggr)\\
&= \frac{Q(x)}{2^*-\tau}\frac{\partial}{\partial\lambda}\int_{\mathbb{R}^N}|U_{x,\lambda}|^{2^*-\tau}\\
&\quad + \frac{\Delta Q(x)}{2N}\frac{1}{2^*-\tau}\frac{\partial}{\partial\lambda}\int_{\mathbb{R}^N}|y|^2|U_{x,\lambda}|^{2^*-\tau} + O\biggl(\frac{1}{\lambda^4}\biggr)\\
&= -\frac{N-2}{2(2^*-\tau)}Q(x)\frac{\tau}{\lambda^{1+(N-2)\tau/2}}\int_{\mathbb{R}^N}U^{2^*-\tau}\\
&\quad - \biggl(1+\frac{N-2}{4}\tau\biggr)\frac{\Delta Q(x)}{(2^*-\tau)N\lambda^{3+(N-2)\tau/2}}\int_{\mathbb{R}^N}|y|^2U^{2^*-\tau} + O\biggl(\frac{1}{\lambda^4}\biggr)\\
&= -\frac{N-2}{22^*}Q(x)A\frac{\tau}{\lambda^{1+(N-2)\tau/2}} - \frac{\Delta Q(x)}{2^*N\lambda^{3+(N-2)\tau/2}}\int_{\mathbb{R}^N}|y|^2U^{2^*}\\
&\quad + O\biggl(\frac{\tau^2}{\lambda}+\frac{\tau}{\lambda^3}+\frac{1}{\lambda^4}\biggr).
\end{aligned}
$$

Following the proof of (B.13) in [19], it is easy to show that

$$
\text{(B.10)}\qquad \int_\Omega Q(y)|U_{x,\lambda}|^{2^*-1-\tau}\frac{\partial\varphi_{x,\lambda}}{\partial\lambda} = -\frac{N-2}{2}\frac{BQ(x)H(x,x)}{\lambda^{N-1+(N-2)\tau/2}} + O\biggl(\frac{1}{\lambda^N}\biggr).
$$

Substituting (B.9) and (B.10) into (B.8) we obtain

$$
\begin{aligned}
\text{(B.11)}\quad &\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-1-\tau}\frac{\partial PU_{x,\lambda}}{\partial\lambda}\\
&\qquad = -\frac{N-2}{22^*}Q(x)A\frac{\tau}{\lambda^{1+(N-2)\tau/2}} - \frac{\Delta Q(x)}{2^*N\lambda^{3+(N-2)\tau/2}}\int_{\mathbb{R}^N}|y|^2U^{2^*}\\
&\qquad\quad + \frac{(N-2)BH(x,x)}{2\lambda^{N-1+(N-2)\tau/2}} + O\biggl(\frac{\tau^2}{\lambda}+\frac{\tau}{\lambda^3}+\frac{1}{\lambda^4}\biggr).
\end{aligned}
$$

By Proposition 1 in [19] we have

$$
\begin{aligned}
\text{(B.12)}\qquad &(2^*-1-\tau)\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-2-\tau}\varphi_{x,\lambda}\frac{\partial PU_{x,\lambda}}{\partial\lambda}\\
&\quad = (2^*-1-\tau)\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-2-\tau}\varphi_{x,\lambda}\frac{\partial U_{x,\lambda}}{\partial\lambda}\\
&\qquad - (2^*-1-\tau)\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-2-\tau}\varphi_{x,\lambda}\frac{\partial\varphi_{x,\lambda}}{\partial\lambda}\\
&\quad = (2^*-1-\tau)\frac{1}{\lambda^{(N-2)/2}}\int_\Omega Q(y)|U_{x,\lambda}|^{2^*-2-\tau}H(y,x)\frac{\partial U_{x,\lambda}}{\partial\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&+ O\left(\frac{1}{\lambda^{N-1+2\theta}}\right) \\
&= \frac{Q(x)H(x,x)}{\lambda^{(N-2)/2}} \frac{\partial \lambda^{-(N-2)(1+\tau)/2}}{\partial \lambda} \int_{\mathbb{R}^N} U^{2^*-1-\tau} + O\left(\frac{1}{\lambda^{3+\sigma}}\right) \\
&= -\frac{N-2}{2} \frac{Q(x)H(x,x)}{\lambda^{N-1+(N-2)\tau/2}} B + O\left(\frac{1}{\lambda^{3+\sigma}}\right).
\end{aligned}
$$

Combining (B.7), (B.11) and (B.12) we obtain

$$
\begin{aligned}
\text{(B.13)} \quad &\int_\Omega Q(y)|PU_{x,\lambda}|^{2^*-1-\tau} \frac{\partial PU_{x,\lambda}}{\partial \lambda} \\
&= -\frac{N-2}{22^*} Q(x) A \frac{\tau}{\lambda^{1+(N-2)\tau/2}} - \frac{\Delta Q(x)}{2^* N \lambda^{3+(N-2)\tau/2}} \int_{\mathbb{R}^N} |y|^2 U^{2^*} \\
&\quad + \frac{(N-2)BH(x,x)}{\lambda^{N-1+(N-2)\tau/2}} + O(\lambda^{-(3+\sigma)}).
\end{aligned}
$$

Inserting (B.13) into (B.6), we get

$$
\begin{aligned}
\text{(B.14)} \quad &\int_\Omega Q(y)|PU_{x,\lambda} + v_\varepsilon|^{2^*-1-\tau} \frac{\partial PU_{x,\lambda}}{\partial \lambda} \\
&= -\frac{N-2}{22^*} Q(x) A \frac{\tau}{\lambda^{1+(N-2)\tau/2}} - \frac{\Delta Q(x)}{2^* N \lambda^{3}} \int_{\mathbb{R}^N} |y|^2 U^{2^*} \\
&\quad + \frac{(N-2)BH(x,x)}{\lambda^{N-1+(N-2)\tau/2}} + O\left(\frac{\tau^2}{\lambda} + \frac{\tau}{\lambda^3} + \frac{1}{\lambda^{3+\sigma}}\right).
\end{aligned}
$$

Then Lemma B.1 follows from (B.3)–(B.5) and (B.14). □

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