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STABLE MAPS OF GENUS ZERO TO FLAG SPACES

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Dedicated to Professor Jürgen Moser

0. Introduction

Topological quantum field theory led recently to a spectacular progress in numerical algebraic geometry. It was shown that generating functions of certain charactertistic numbers of modular spaces of stable algebraic curves with labelled points satisfy remarkable differential equations of KP-type (E. Witten, M. Kontsevich). In a later series of developments, this was generalized, partly conjecturally, to the spaces of maps of curves into algebraic varieties leading to the Mirror Conjecture and the construction of quantum cohomology.

The key technical notion in the context of algebraic geometry is that of a *stable map* introduced by M. Kontsevich (cf. [8] and [1]) following the earlier work by M. Gromov in symplectic geometry. It provides a natural compactification of spaces of maps, in the same way as stable curves compactify moduli spaces.

We will be working over a ground field. Let W be an algebraic variety.

Definition 0.1. A stable map (to W) is a structure $(C; x_1, \dots, x_n; f)$ consisting of the following data.

(a) $(C; x_1, \dots, x_n)$ is a connected complete reduced curve with $n \geq 0$ labelled pairwise distinct non–singular points x_i and at most ordinary double singular points.

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(b) $f : C \to W$ is a morphism having no non–trivial first order infinitesimal automorphisms identical on W and x_i's (stability). This means that every irreducible component of C of genus zero (respectively, 1) has at least three (respectively, one) special points (inverse images of singular and labelled points) on its normalization.

A family of stable maps parametrized by a noetherian scheme S is a structure consisting of a flat proper morphism $\pi : \mathcal{C} \to S$, n sections $x_i : S \to \mathcal{C}$, and a morphism $f : \mathcal{C} \to W$ whose restriction to each geometric fiber of π is a stable map in the sense of the previous Definition.

Families of stable maps form an algebraic stack in the sense of [2].

For fixed $n \geq 0$, $g \geq 0$ and an algebraic homology class of dimension two β, we denote by $\overline{\mathcal{M}}_{g,n}(W,\beta)$ the substack of maps for which g is the arithmetic genus of C and $\beta = f_*([C])$. For the proof of the following theorem see [8], [1] and [3].

THEOREM 0.2.

(a) *If W is projective, then $\overline{\mathcal{M}}_{g,n}(W,\beta)$ is a proper separated algebraic stack of finite type.*

(b) *If we assume that W is convex in the following sense: $H^1(C, f^*(\mathcal{T}_W)) = 0$ for any stable map $f : C \to W$ of genus zero, then the stacks $\overline{\mathcal{M}}_{0,n}(W,\beta)$ are smooth.*

We denote by $\mathcal{M}_{0,n}(W,\beta)$ the big cell of this stack over which C is smooth. The complement of this cell is a divisor with normal crossings.

In addition, the spaces of geometric points of these stacks are represented by algebraic schemes, crude moduli spaces of stable maps, $\overline{M}_{g,n}(W,\beta)$ and $M_{g,n}(W,\beta)$ respectively. If

$$(C; x_1, \ldots, x_n; f)$$

is a stable map, we denote by $[(C; x_1, \ldots, x_n; f)]$, or simply $[f]$, the corresponding point. Two maps $[f^{(i)}]$, $i = 1, 2$, define the same point if and only if there is an isomorphism $g : C^{(1)} \to C^{(2)}$ such that $g(x_i^{(1)}) = x_i^{(2)}$ for all i and $g \circ f^{(2)} = f^{(1)}$.

This paper is a sequel to [9].

We will consider mostly the case $g = 0$, $W = G/P$ (generalized flag spaces). Although flag spaces are convex, the respective spaces of stable maps are not smooth but only orbifolds in general. For a recent account of these spaces see [10]. Our goal is to calculate their virtual Poincaré polynomials, or rather an appropriate generating function for these polynomials. This calculation generalizes the one made for $\overline{M}_{0,n}$ in [9] and [4]. Remarkably, up to a change of variables, this function satisfies the same universal differential equation as that in [9], the dependence on W being reflected only in the initial condition which involves the Eisenstein series of $W = G/P$ (see 2.3 below).

1. Stratification of the space of stable maps

1.1. Virtual Euler–Poincaré maps. Let $\mathcal{R}$ be a commutative associative $\mathbf{Q}$-algebra, Var the category of algebraic varieties, not necessarily complete, smooth, or irreducible, over a fixed ground field.

We will call a map $Ob\,Var \to \mathcal{R}: \ V \mapsto [V]$ a *virtual Euler–Poincaré map* if the following conditions are satisfied:

(i) $[V]$ depends only on the isomorphism class of V.
(ii) (Additivity) If $V = \coprod_i V_i$ is a finite disjoint union of locally closed subsets ("strata"), then $[V] = \sum_i [V_i]$.
(iii) (Multiplicativity) $[\prod_j V_j] = \prod_j [V_j]$.

From (ii) and (iii) it follows that if $V \to B$ is a Zariski locally trivial fiber space with fiber F, then $[V] = [B][F]$.

If we work over $\mathbf{C}$, and $\mathcal{R} = \mathbf{Q}[q]$ (respectively, $\mathcal{R} = \mathbf{Q}[r,s]$) the main examples are virtual Poincaré (respectively, Hodge) polynomials:

$$P_V(q) := \sum_{i,j} (-1)^{i+j} \dim(\mathrm{gr}_W^j H_c^i(V)) q^j, \tag{1.1}$$

$$H_V(r,s) := \sum_{i,j,k} (-1)^{i+j+k} h^{j,k}(H_c^i(V)) r^j s^k, \tag{1.2}$$

and also the virtual Euler characteristics $\chi(V) = P_V(-1)$. These properties were for the first time systematically used by Danilov and Khovanskiĭ in the toric geometry.

Recently C. Soulé and H. Gillet [5] established that the map $V \mapsto$ *class of the motive $h^*(V)$ of V in the K_0-ring of Grothendieck's motives* extends from projective smooth varieties to Var and becomes an Euler–Poincaré map. For a somewhat different construction see [6].

On the subcategory generated by the (dual) Tate motive one can identify the Gillet–Soulé map with virtual Poincaré map putting $h^*(\mathbf{P}^1) = q^2 + 1$, and calculate the latter via point count over $\mathbf{F}_{q^2}$. Most of our calculations are in fact restricted to this situation.

We will have also to localize $\mathcal{R}$, most notably by inverting $[\mathrm{PGL}(2)] = q^2(q^4-1)$. Since the Euler characteristic of this manifold vanishes, we have to apply a limiting procedure $q^2 \to 1$ producing logarithms and infinite ramification in our formulas for generating functions of χ, as already happened in [9].

1.2. Poincaré polynomials of $\underline{\mathrm{Map}}_\beta(\mathbf{P}^1, W)$. If W is a flag manifold, the scheme of maps of $\mathbf{P}^1$ to W landing in β is a smooth (generally non-complete) manifold. Its Gillet–Soulé motive lies in the Tate's subring, and the generating function

$$E(W,z) := \sum_{\beta\in B} [\underline{\mathrm{Map}}_\beta(\mathbf{P}^1, W)] z^\beta \tag{1.3}$$

is rational in z. Here $B \subset N$ means the effective subsemigroup of the lattice N of algebraic homology classes of curves, and z^β are formal monomials, elements of the dual lattice M written multiplicatively.

In fact, in the $\mathbf{F}_{q^2}$-avatar, (1.3) essentially coincides with an appropriate Eisenstein series discussed e.g. in [7]. If we introduce a complex vector variable $s \in M_{\mathbf{C}}$ and replace z^β by $q^{-2(\beta.s)}$, the poles of Eisenstein series will lie on the complexified walls of the ample cone $B^t_{\mathbf{R}}$ in N shifted by $-K_W$, and reflections with respect to the walls will generate the functional equations.

EXAMPLE 1.2.1. We have

$$E(\mathbf{P}^n, z) = \frac{1-q^{2n+2}}{1-q^2} \cdot \frac{1-q^2 z}{1-q^{2n+2} z}. \tag{1.4}$$

In fact, we denote by N_d the number of non–zero $(n+1)$-tuples

$$(f_0(t_0,t_1), \dots, f_n(t_0,t_1))$$

of coprime forms of degree d over $\mathbf{F}_{q^2}$ divided by $|\mathbf{F}^*_{q^2}| = q^2-1$. Since the number of all such $(n+1)$-tuples is $q^{2(n+1)(d+1)}-1$ and the map $((f_0,\dots,f_n),g) \mapsto (f_0 g, \dots, f_n g)$ with fixed $g \neq 0$ and degrees of f_i, g has all fibers of cardinality q^2-1, we have

$$\sum_{k=0}^{d} N_{d-k}(q^{2(k+1)}-1) = q^{2(k+1)(d+1)} - 1,$$

from which (1.4) easily follows.

Identifying $M = \mathrm{Pic}\,\mathbf{P}^n$ with $\mathbf{Z}$ via $\mathcal{O}(1) \mapsto 1$ and putting $z = q^{-2s}$ one sees that the real pole of (1.4) is $s = n+1 = -K_{\mathbf{P}^n}$.

1.3. Big cells of stable map spaces. We will say that a stable map $(C; x_1, \dots, x_n; f)$ to W belongs to the big cell $M_{0,k}(W,\beta)$ if $C \cong \mathbf{P}^1$, and $f_*([C]) = \beta$. We will explain how these big cells are related to $\underline{\mathrm{Map}}_\beta(\mathbf{P}^1, W)$.

(a) $\beta = 0$. In this case, $k \geq 3$, and the map

$$(\mathbf{P}^1; x_1, \dots, x_k; f) \mapsto (f(x_1), [(\mathbf{P}^1; x_1, \dots, x_k)])$$

induces the identification

$$M_{0,k}(W,0) \cong W \times M_{0,k}. \tag{1.5}$$

In particular, when W is a point, we get simply $M_{0,k}$.

(b) $\beta \neq 0$. In this case, a choice of three points x_1, x_2, x_3 in $\mathbf{P}^1$ defines an isomorphism

$$M_{0,3}(W,\beta) \cong \underline{\mathrm{Map}}_\beta(\mathbf{P}^1, W). \tag{1.6}$$

Let $G^{(i)} \subset \mathrm{PGL}(2)$ be the subgroup fixing $x_1, \dots, x_i$, $i = 0, \dots, 3$. Then $G^{(i)}$ freely acts upon $M_{0,3}(W,\beta)$, and we have

$$M_{0,i}(W,\beta) \cong \underline{\mathrm{Map}}_\beta(\mathbf{P}^1, W)/G^{(i)}, \quad i \leq 3. \tag{1.7}$$

Finally, for $k \geq 4$ the forgetful map

$$M_{0,k}(W,\beta) \to M_{0,3}(W,\beta)$$

identifies $M_{0,k}(W,\beta)$ with a locally trivial fibration over $\underline{\mathrm{Map}}_\beta(\mathbf{P}^1, W)$ with fiber

$$M_{0,k} = \left((\mathbf{P}^1)^k \setminus \bigcap_{i<j} \Delta_{ij} \right) \Big/ \mathrm{PGL}(2).$$

To summarize, we have the following formula for the virtual Euler–Poincaré class of $M_{0,k}(W,\beta)$ valid for $\beta = 0$ as well:

PROPOSITION 1.3.1. *We have*

$$[M_{0,k}(W,\beta)] = [\underline{\mathrm{Map}}_\beta(\mathbf{P}^1, W)] \binom{[\mathbf{P}^1]}{k} k! \frac{1}{[\mathrm{PGL}(2)]}. \tag{1.8}$$

In fact,

$$[M_{0,k}] = \binom{[\mathbf{P}^1] - 3}{k-3} (k-3)!, \qquad [\mathrm{PGL}(2)] = \binom{[\mathbf{P}^1]}{3} 3!.$$

We will now construct strata of the space of stable maps numbered by marked trees. Their virtual Euler–Poincaré classes will be expressed via products of those of big cells.

1.4. Trees. In this paper, a *tree* τ is a finite connected simply connected CW-complex of dimension 1 or 0 (one-vertex tree). A *flag* of a tree is a pair consisting of a vertex and an adjoining edge. The *valency* $|v|$ of a vertex v is the number of flags containing v. The sets of vertices (respectively, edges, flags) of τ are denoted V_τ (respectively, E_τ, F_τ).

For a set S, $\mathcal{P}(S)$ denotes the set of its subsets. As above, B is the semigroup of classes of effective algebraic curves in a flag manifold W.

DEFINITION 1.4.1. A (k, W)-marking of a tree τ is a map

$$\mu : \ V_\tau \to B \times \mathcal{P}(1, \dots, k) : \ v \mapsto (\beta_v, S_v),$$

satisfying the following conditions:

(a) The family $\{S_v \mid v \in V_\tau, S_v \neq \emptyset\}$ forms a partition of $\{1, \ldots, k\}$ into pairwise disjoint subsets. (For $k = 0$, we interpret $\mathcal{P}$ as one-element set, and this condition as empty).

(b) If $\beta_v = 0$, then $|v| + |S_v| \geq 3$.

There is an obvious notion of isomorphism of marked trees.

1.4.1. Type of a stable map. Let $(C; x_1, \ldots, x_k; f)$ be a stable map of genus 0 to W. Its *combinatorial type* is, by definition, the isomorphism class of the dual tree of C with the marking depending on f. We can describe it as follows. First of all, we put

$$V_\tau = \{\text{irreducible components } C_v \text{ } of \text{ } C\},$$
$$E_\tau = \{\text{intersection points of irreducible components}\}.$$

Thus a flag of τ is a pair (*intersection point of two components, one of these components*). Furthermore,

$$\beta_v := f_*([C_v]), \quad S_v = \{i \mid x_i \in C_v\}.$$

Obviously, $\beta_v = 0$ means that $f(C_v)$ is a point so that condition 1.4.1(b) means stability.

1.5. Maps of fixed type. We consider a combinatorial type (τ, μ). For any vertex $v \in V_\tau$, we construct the stack $M_{0,S_v}(W, \beta_v)$ parametrizing stable maps of $\mathbf{P}^1$ with points labelled by S_v, of class β_v. For any $t \in S_v$, there exists a canonical evaluation map $ev_t : M_{0,S_v}(W, \beta_v) \to W$, $[f] \mapsto f(x_t)$ where x_t is the point marked by t. We put

$$M_{0,k}(W, \beta, \tau, \mu) = \prod_W M_{0,S_v}(W, \beta_v),$$

where $\prod_W$ means the partial fibered product over W which leaves in the total product only those families of stable maps $(f_v \mid v \in V_\tau)$ for which $f_v(s) = f_w(t)$ whenever s, t are halves of the same edge of τ.

THEOREM 1.5.1. *Let W be a generalized flag variety. We put*

$$N(W, \beta) := \frac{[M_{0,3}(W, \beta)]}{[W][PGL(2)]} = \frac{[\underline{\mathrm{Map}}_\beta(\mathbf{P}^1, W)]}{[W][PGL(2)]}. \tag{1.9}$$

Then

$$[M_{0,k}(W, \beta, \tau, \mu)] = [W] \prod_{v \in V_\tau} \varepsilon(\beta_v, |v| + k_v) N(W, \beta_v) \binom{[\mathbf{P}^1]}{|v| + k_v} (|v| + k_v)! \tag{1.10}$$

where $\mu: \ v \mapsto (\beta_v, S_v)$, $k_v = |S_v|$, $\beta = \sum \beta_v$, *and* $\varepsilon(\beta, n) = 0$ *for* $\beta = 0$, $n = 0, 1, 2$, *and* 1 *otherwise.*

Sketch of the proof. For $\beta = 0$, we have a projection $M_{0,k}(W, 0, \tau, \mu) \to W$, $(C; x_1, \dots, x_k; f) \mapsto f(C)$. Its fiber is the stratum of the moduli space of stable curves with labelled points of the given combinatorial type (τ, μ); we notice that μ now is just a map $v \mapsto S_v$. This fiber is isomorphic to $\prod_{v \in V_\tau} M_{0,|v|}$. The fibration is locally trivial over $[W]$. Hence

$$[M_{0,k}(W, 0, \tau, \mu)] = [W] \prod_{v \in V_\tau} \binom{[\mathbf{P}^1] - 3}{|v + k_v| - 3} (|v + k_v| - 3)!.$$

One easily sees that this coincides with (1.10).

For $\beta \neq 0, \tau$ one-vertex that is, $C = \mathbf{P}^1$, (1.10) follows from (1.8).

Finally, in the remaining cases we represent a stable map $f : C \to W$ as a vector of stable maps of irreducible components $f_v : C_v \to W$. If we fix one component, the rest will be constrained by incidence conditions. This accounts for the necessity to divide by $[W]$ in each v-factor; we incorporated this division in Definition (1.9) of $N(W, \beta)$. The exterior factor $[W]$ in (1.10) appears because one component can be moved freely. $\square$

2. Generating function of the space of stable maps

2.1. The problem. In this section, we set to calculate the following series in $\mathcal{R}[[t, z]]$ (we remind that monomials in z belong to a semigroup ring):

$$\Phi_W(t, z) := \sum_{\tau/(\mathrm{iso})} \frac{1}{|\mathrm{Aut}\,\tau|} \sum_{\substack{V_\tau \to B \times N \\ v \mapsto (\beta_v, k_v)}} [W] \times \prod_{v \in V_\tau} \frac{t^{k_v}}{k_v!} z^{\beta_v} \varepsilon(\beta_v, |v| + k_v) N(W, \beta_v) \binom{[\mathbf{P}^1]}{|v| + k_v} (|v| + k_v)!. \tag{2.1}$$

We retain the notation from Section 1; in particular, ε is introduced in order to exclude inadmissible markings.

To motivate this definition, we consider the following more natural generating function:

$$\widetilde{\Phi}_W(t, z) := \sum_{\beta, k} [\overline{M}_{0,k}(W, \beta)] \frac{t^k}{k!} z^\beta. \tag{2.2}$$

We assume in addition that $[\overline{M}_{0,k}(W, \beta)]$ in (2.2) means an "orbifold" virtual class of $[\overline{M}_{0,k}(W, \beta)]$ which is defined on the category of small algebraic stacks and, besides the usual additivity and multiplicativity properties postulated in 1.1, satisfies the following condition: *if* V *is smooth,* G *a finite group freely acting upon* V, *then* $[V/G] = [V]/|G|$.

In this case (2.2) coincides with (2.1). To see this, one has to consider the stratified covering of $\overline{M}_{0,k}(W,\beta)$ by $M_{0,k}(W,\beta,\tau,\mu)$ where (τ,μ) runs over all admissible marked trees with $\sum_v \beta_v = \beta$, $\sum_v k_v = k$, and $M_{0,k}(W,\beta.\tau,\mu)$ are moduli spaces of rigidified stable maps. The claim follows from the fact that Aut τ acts freely on $\coprod_\mu M_{0,k}(W,\beta,\tau,\mu)$.

THEOREM 2.2. *We denote by $\varphi^0 = \varphi_W(t,z)$ the unique root in $\mathcal{R}[[t,z]]$ of the following equation:*

$$\frac{[\mathbf{P}^1]}{[W]}E(W,z)(1+t+\varphi^0)^{[\mathbf{P}^1]-1} = \varphi^0\frac{[\mathbf{P}^1]-1}{[\mathbf{P}^1]-2}+\frac{t}{[\mathbf{P}^1]-2}+\frac{1}{([\mathbf{P}^1]-1)([\mathbf{P}^1]-2)}, \tag{2.3}$$

satisfying the condition

$$\varphi^0 \equiv \sum_{\beta\in\mathrm{ind}} N(W,\beta)z^\beta \mathrm{mod}\ (z,t)^2,$$

where the summation is taken over indecomposable elements of B. Then we have

$$\Phi_W(t,z) = -\frac{[\mathbf{P}^1]-1}{2[\mathbf{P}^1]}(\varphi^0)^2+\frac{\varphi^0}{[\mathbf{P}^1]}-\frac{t^2}{2[\mathbf{P}^1]}, \tag{2.4}$$

$$\frac{\partial\Phi_W}{\partial t}(t,z) = \varphi^0. \tag{2.5}$$

PROOF. We first rearrange the inner sum and product in (2.1):

$$\Phi_W(t,z) = \sum_{\tau/(\mathrm{iso})}\frac{[W]}{|\mathrm{Aut}\ \tau|}\prod_{v\in V_\tau}\sum_{\beta,k\geq 0}\frac{t^k}{k!}z^\beta \times \varepsilon(\beta,|v|+k)N(W,\beta)\binom{[\mathbf{P}^1]}{|v|+k}(|v|+k)!. \tag{2.6}$$

Furthermore, for a fixed $v\in V_\tau$,

$$\begin{aligned}\sum_{\beta,k}\frac{t^k}{k!}z^\beta\varepsilon(\beta,|v|+k)N(W,\beta)\binom{[\mathbf{P}^1]}{|v|+k}(|v|+k)! \\ &= \Big(\sum_\beta N(W,\beta)z^\beta\Big)\Big(\sum_{k\geq 0}\frac{t^k}{k!}\binom{[\mathbf{P}^1]}{|v|+k}(|v|+k)!\Big)-\varepsilon_{|v|} \\ &= \frac{E(W,z)}{[\mathrm{PGL}(2)][W]}\sum_{k\geq 0}\frac{t^k}{k!}\binom{[\mathbf{P}^1]}{|v|+k}(|v|+k)!-\varepsilon_{|v|},\end{aligned} \tag{2.7}$$

where $\varepsilon_n = 0$ for $n\geq 3$, and

$$\varepsilon_n = \sum_{k=0}^{2-n}N(W,0)\frac{t^k}{k!}\binom{[\mathbf{P}^1]}{n+k}(n+k)! \quad \text{for } n\leq 2. \tag{2.8}$$

To calculate the resulting sum over marked trees we can now apply the formalism of [8], [9]. We introduce one more formal variable φ and consider the formal potential

$$S(\varphi) := -\frac{\varphi^2}{2} + \sum_{n=0}^{\infty} \frac{C_n}{n!}\varphi^n, \tag{2.9}$$

$$C_n := \frac{E(W,z)}{[\mathrm{PGL}(2)][W]} \sum_{k=0}^{\infty} \frac{t^k}{k!}(n+k)!\binom{[\mathbf{P}^1]}{n+k} - \varepsilon_n. \tag{2.10}$$

We have

$$\begin{aligned} S(\varphi) = &- \frac{\varphi^2}{2} + \frac{E(W,z)}{[\mathrm{PGL}(2)][W]} \sum_{n,k=0}^{\infty} \varphi^n t^k \binom{n+k}{k}\binom{[\mathbf{P}^1]}{n+k} - \sum_{n=0}^{2} \frac{\varepsilon_n}{n!}\varphi^n \\ &= \frac{E(W,z)}{[\mathrm{PGL}(2)][W]}(1+t+\varphi)^{[\mathbf{P}^1]} - \varphi^2 \frac{[\mathbf{P}^1]-1}{2([\mathbf{P}^1]-2)} \\ &\quad - \varphi\left(\frac{1}{([\mathbf{P}^1]-1)([\mathbf{P}^1]-2)} + \frac{t}{[\mathbf{P}^1]-2}\right) \\ &\quad - \left(\frac{1}{[\mathbf{P}^1]([\mathbf{P}^1]-1)([\mathbf{P}^1]-2)} + \frac{t}{([\mathbf{P}^1]-1)([\mathbf{P}^1]-2)} \right. \\ &\quad \left. + \frac{t^2}{2([\mathbf{P}^1]-2)}\right). \end{aligned} \tag{2.11}$$

According to a general formula of perturbation theory (cf. [8], [9]) we have

$$\Phi_W(t,z) = S^{\mathrm{crit}} := S(\varphi^0),$$

where φ^0 is an appropriate formal solution of $\frac{\partial}{\partial\varphi}S(\varphi) = 0$ which we will now identify. Differentiating (2.11) in φ we obtain:

$$\frac{E(W,z)}{[W]}[\mathbf{P}^1](1+t+\varphi^0)^{[\mathbf{P}^1]-1} = \varphi^0\frac{[\mathbf{P}^1]-1}{[\mathbf{P}^1]-2} + \frac{t}{[\mathbf{P}^1]-2} + \frac{1}{([\mathbf{P}^1]-1)([\mathbf{P}^1]-2)},$$

which is (2.3). Rewriting this equation as

$$\varphi^0 = C_1 + C_2\varphi^0 + C_3\frac{(\varphi^0)^2}{2} + \dots,$$

and taking into account that C_1 modulo $(t,z)^2$ starts with terms linear in z we see that there exists a unique solution $\varphi^0 \in \mathcal{R}[[t,z]]$ such that $\varphi^0 \equiv C_1 \bmod (t,z)^2$. In view of (2.10), this coincides with the congruence in the statement of the Theorem.

It remains to calculate $S(\varphi^0)$. Multiplying (2.3) by $(1+t+\varphi^0)/[\mathbf{P}^1]$ and simplifying, we get (2.4).

Finally, derivating (2.3) and (2.4) in t, we can obtain (2.5): we reproduce this calculation below in simplified notation. □

2.3. Comments and supplements. We will now consider the case $\mathcal{R} = \mathbf{Q}[q]$, $[V] = P_V(q)$ (the virtual Poincaré polynomial of V). Equations (2.3) and (2.4) take the form

$$\frac{q^2+1}{P_W(q)}E(W,z)(1+t+\varphi^0)^{q^2} = \varphi^0\frac{q^2}{q^2-1} + \frac{t}{q^2-1} + \frac{1}{q^2(q^2-1)}, \tag{2.12}$$

$$\Phi_W(t,z) = -\frac{q^2}{2(q^2+1)}(\varphi^0)^2 + \frac{1}{q^2+1}\varphi^0 - \frac{1}{2(q^2+1)}t^2. \tag{2.13}$$

(a) *A differential equation for* φ^0. Derivating (2.12) in t and multiplying the result by $(1+t+\varphi^0)/q^2$ we get the differential equation

$$(1-q^2\varphi^0)\varphi^0_t = (q^2+1)\varphi^0 + t. \tag{2.14}$$

Up to a simple variable change $\varphi^0 = \psi - t$, this is the same equation as (0.7) (and (0.15)) in [9]:

$$(1+q^2-q^2\psi)\psi_t = 1+\psi. \tag{2.15}$$

Its universality is remarkable: in (2.14) there is no dependence on W and z (below it will be encoded in the choice of constant in a general solution of (2.14)), and in [9] it emerged also in a calculation of the generating function for arbitrary configuration spaces $X[n]$.

(b) *Euler characteristics:* $q^2 = 1$. A formal substitution $q^2 = 1$ into (2.11) is impossible, however a limiting procedure gives:

$$(1+t+\varphi^0)\log\,(1+t+\varphi^0) = 2\varphi^0 + t, \tag{2.16}$$

and $\Phi_W(t,z)$ becomes

$$\Phi_W(t,z) = -\frac{1}{4}(\varphi^0)^2 + \frac{1}{2}\varphi^0 - \frac{1}{4}t^2. \tag{2.17}$$

(c) *General solution of* (2.14). We can apply to (2.14) Proposition 1.5.1 in [9]. We put

$$x = t + \frac{q^2+1}{q^2}, \quad y = q^2\varphi^0 - 1, \quad w = \frac{y}{x}.$$

The general solution of (2.14) in implicit form is given by

$$Cx = (w+1)^{1/(q^2-1)}(w+q^2)^{q^2/(1-q^2)}. \tag{2.18}$$

This makes evident the ramification structure of φ for $q^2 \neq 1$. Constant C in our case is a function of z which can be calculated by considering (2.18) at $t = 0$ that is, $x = (q^2+1)/q^2$. Function $w(z) = (q^4\varphi^0(0,z) - q^2)/(q^2+1)$ is found from (2.12) at $t = 0$.

(d) As in [9], the derivative of Φ_W in t is simpler than Φ_W itself. Indeed, from (2.13) and (2.14) we have:

$$\begin{aligned}\frac{\partial \Phi_W}{\partial t} &= -\frac{q^2}{q^2+1}\varphi^0\varphi^0_t + \frac{1}{q^2+1}\varphi^0_t - \frac{t}{q^2+1} \\ &= -\frac{1}{q^2+1}\big(\varphi^0_t - (q^2+1)\varphi^0 - t\big) + \frac{1}{q^2+1}\varphi^0_t - \frac{t}{q^2+1} = \varphi^0.\end{aligned}$$

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