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ON THE PROBLEM OF REALIZATION OF A GIVEN GAUSSIAN CURVATURE FUNCTION

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Dedicated to Jürgen Moser

1. Introduction

The Gaussian curvature of a smooth surface embedded into the Euclidean space is a smooth function on the surface. We investigate below the local realization problem: given a germ of a smooth function of two variables, find a surface whose Gaussian curvature is the given function.

It is well known that any function germ $g(x,y)$ is realizable as the Gaussian curvature of the surface $z = f(x,y)$ if the curvature value at the central point is not vanishing. It is also known that it is realizable (in the same sense), if the curvature is vanishing at the central point, but its differential does not vanish. In this case, the parabolic curve is smooth.

In the case of a singular parabolic curve, the problem is more difficult. We shall see that any parabolic curve singularity occurs for a suitable surface.

Theorem 1. *For any smooth function of two variables vanishing at its critical point of finite multiplicity, there exists a smooth surface in the Euclidean* 3*-space, whose Gaussian curvature coincides with the given function at a neighbourhood of the given point* (*provided that the surface is identified with the plane by a suitable local diffeomorphism, depending on the function*).

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I recall that the *multiplicity* μ of a critical point 0 of function f is the dimension of the quotient vector-space of the space of formal power series in (x, y) modulo the ideal generated by the partial derivatives of the function:

$$\mu = \dim_{\mathbb{R}} \mathbb{R}[[x, y]]/\{a(x, y)f_x + b(x, y)f_y\}.$$

Finite multiplicity critical points of holomorphic functions are just complex isolated critical points. The codimension of the set of the functions having a critical point of infinite multiplicity is infinite. Thus typical finite-dimensional families of functions contain no function having a critical point of infinite multiplicity.

Corollary 1. *Any singularity of finite multiplity μ of a plane curve is realizable as a singularity of the parabolic curve on a surface in the Euclidean 3-space (up to a diffeomorphism).*

Remark 1. The Theorem and the Corollary hold for analytical functions and diffeomorphisms as well as for the holomorphic ones or for the infinitely smooth ones and even for the finitely smooth ones (the number of the derivatives in the last case grows with μ).

Remark 2. I have no counterexamples to the statement of Theorem 1 for infinite multiplicity germs.

In the formal series setting, one can prove more.

Theorem 2. *For any formal power series $g(x, y) = ax+by+\ldots$ there exists a formal power series $f(x, y)$ such that the formal power series of the Gaussian curvature of the formal surface $z = f(x, y)$ coincides with g.*

In Theorem 2, the identification of the surface with the plane (x, y) is fixed by the choice of the coordinates (x, y, z) in the 3-space.

Corollary 2. *Any formal singularity of a plane curve $g(x, y) = 0$ is realizable as the singularity of the parabolic curve of some formal surface $z = f(x, y)$.*

Remark 3. I have no counterexamples (even for $\mu = \infty$) in the smooth case (neither for Theorem 2 nor for its Corollary).

Theorem 1 follows from Theorem 2, as it is explained below (in Section 3). The same arguments (in a simpler form) prove the following results.

Theorem 3. *For any function $g(x, y)$ vanishing at a critical point of finite multiplicity, there exists a function $f(x, y)$, whose Hessian*

$$h = h[f] = f_{xx}f_{yy} - (f_{xy})^2,$$

coindices with g at a neighbourhood of the point up to a suitable smooth change of variables $g(x, y) = h(X(x, y), Y(x, y))$.

REMARK 4. Corollary 1 follows already from Theorem 3 (which is a simplified version of Theorem 1) since the Hessian vanishes exactly along the parabolic curve.

THEOREM 4. *For any formal series* $g(x, y) = ax + by + \dots$, *there exists a formal series* f *whose Hessian is* $g = h[f]$.

REMARK 5. Corollary 2 follows already from Theorem 4 (which is a simplified version of Theorem 2).

REMARK 6. The proofs of these results provide in fact more: one might replace in Theorems 3 and 4 the Hessian $h = h[f]$ by any function (series) of the form, say,

$$g[f] = hM(x, y, f_x, f_y, f, h), \quad M(0) \neq 0,$$

provided that $f(0) = f_x(0) = f_y(0) = 0$ (there exist many other possibilities).

This Remark implies Theorems 1 and 2, since the Gaussian curvature admits such a representation (in this case $M = M(f_x, f_y)$, see Section 4 below).

REMARK 7. Theorems 1–4 have higher dimensional versions (describing the Hessians of the functions of n variables and the Jacobians of the Gaussian mappings from the hypersurfaces in $\mathbb{R}^{n+1}$ to the n-sphere).

REMARK 8. Whether the statements of Theorems 3 and 4 hold for infinite mutiplicity critical points and for smooth functions f and g respectively is not known to me (see the Remarks 1 and 2 above).

The extension of the results of this paper to the (formally more general) case of the Jacobians of the Lagrangian mappings in symplectic geometry is automatic (since Lagrangian germs have gradient representatives).

2. Proof of Theorem 4

We shall look for the series

$$(*) \qquad f = f_2 + f_3 + \dots \quad (f_i = \text{terms of degree } i),$$

starting from $f_2 = y^2/2$. This choice of f_2 is the crucial point of the present paper: the remaining part of the proof consists of standard Cauchy–Kowalewsky type calculations.

The terms of the Hessian $h[f] = h_1 + h_2 + \dots$ are given by the obvious formulae

$$h_1 = f_{3,xx}, \quad h_2 = f_{4,xx} + (f_{3,xx} f_{3,yy} - f_{3,xy}^2),$$

of the form

$$(**) \qquad h_k = f_{k+2,xx} + P_k([f_m]), 3 \leq m < k + 2),$$

where p_k is a (quadratic) polynomial with respect to the second derivatives $(f_{m,xx},\ f_{m,xy},\ f_{m,yy})$ of homogeneous functions f_m.

The system of equations for the functions f_i that we get is triangular:

$$h_1[f_3] = g_1, \quad h_2[f_3, f_4] = g_2, \dots, \quad h_k[f_3, \dots, f_{k+2}] = g_k, \dots .$$

It remains to solve the equation

$$(***) \qquad f_{k+2,xx} = g_k - P_k([f_3], \dots, [f_{k+1}]),$$

(whose right hand side is known) with respect to $f_{k+2} = \sum a_{p,q} x^p y^q$, $p+q = k+2$. Denoting $g_k - P_k = \sum b_{p,q} x^p y^q$, $p+q = k$, we solve the equation $(***)$, taking

$$a_{p,q} = b_{p-2,q}/p(p-1) \quad \text{if } p \geq 2,$$

$a_{0,q} = a_{1,q} = 0$. Theorem 4 is proved. □

REMARK 9. The extension of Theorem 4, mentioned in Remark 6, is proved by the same reasoning. Indeed, the multiplication of the Hessian by the multiplier $M = c + M_1 + M_2 + \dots$, $c \neq 0$, preserves the triangular character of the system $h[f]M[f, h] = g$ for f.

To study the n-dimensional hypersurfaces and functions of n variables (Remark 7) one should start from $f_2 = \pm y_1^2/2 \pm \dots \pm y_{n-1}^2/2$.

3. Proofs of Theorems 3 and 1

We deduce Theorem 3 from Theorem 4.

Let function g have at the origin the critical value 0 of finite multiplicity μ. For any N the proof of Theorem 4 (in paragraph 2 above) provides a polynomial f such that $h[f] = g + r$, where the remainder r has at the origin a zero of order at least N.

If $N \geq \mu + 2$, function $g + r$ is reducible to g by a smooth (analytical, holomorphic…) change of variables (see e.g., [1]). This proves Theorem 3.

Theorem 1 follows similarly from Theorem 2 and from the finite $(\mu + 2)$ determinacy property of the finite multiplicity critical point (used in the proof of Theorem 3 above).

4. Proof of Theorem 2

We start from an obvious remark.

LEMMA. *The Gaussian curvature g of the surface, defined in some Cartesian orthonormal coordinates (x, y, z) in the Euclidean space as the graph of a function $z = f(x, y)$, is related to the Hessian h of the function f by the formula*

$$g = h[f]/E^4[f], \quad \textit{where } E^2 = 1 + (f_x)^2 + (f_y)^2.$$

Proof. Function E is the lenght of the normal vector $n = (-f_x, -f_y, 1)$ of the surface. The area element dS of the surface in inclined to the horizontal direction at the same angle as the angle of deviation of the normal from the vertical direction. Hence $dS = E dx \wedge dy$.

The Gaussian mapping G from the surface to the sphere is the product of the gradient mapping F (sending (x, y) to $(u = -f_x,\ v = -f_y)$ from the surface to the horizontal plane $\{u, v, 1\}$, (tangent to the unite sphere at $(0, 0, 1)$) followed by the central projection P from the horizontal plane to the sphere: $G = P \circ F$.

The area element of the sphere, $d\omega$, induces on the horizontal plane the form $P^* d\omega = E^{-3} du \wedge dv$ (the factor E^{-2} is the contribution of the fact that the point of the plane is E-times farther from the origin than the point of the sphere, and an additional factor E^{-1} is provided by the inclination of the horizontal plane to the radial direction).

By definition of the Gaussian curvature we get

$$g = \frac{G^* d\omega}{dS} = \frac{J^*(E^{-3}\, du \wedge dv)}{E\, dx \wedge dy} = E^{-4} h,$$

since the Hessian of f is the Jacobian of its gradient mapping F: $F^*\, du \wedge dv = h[f]\, dx \wedge dy$. The Lemma is thus proved. □

Proof of Theorem 2. To prove Theorem 2, we look for the series (∗) as in Section 2. We get

$$E^{-4}[f] = 1 + e_2[f_2] + e_3([f_2], [f_3]) + \dots,$$

e_k being the homogeneous part of degree k of the power series E^{-4} in (x, y).

The form e_k is a weighted homogeneous polynomial in the first derivatives of functions $f_2, \dots, f_k$ (higher order terms, starting from f_{k+1}, contribute noting to e_k):

$$e_2[f_2] = 2y^2, \quad e_3([f_2], [f_3]) = -4y f_{3,y}, \dots, \quad e_k = e_k([f_2], \dots, [f_k]).$$

It follows that the system

$$h[f] E^{-4}[f] = g,$$

of the equations for the unknown f_k (which we should solve to prove Theorem 2, conformally to the Lemma) is triangular.

Indeed, equating the homogeneous terms in the product

$$(h_1 + h_2 + \dots)(1 + e_2 + \dots) = g_1 + g_2 + \dots,$$

we get for the unknowns f_k the sequence of equations

$$h_1 = g_1, \quad h_2 = g_2, \quad h_3 + h_1 e_2 = g_3, \dots, \quad h_k + Q_k([h_{<k}], [e_{<k}]) = g_k.$$

Substituing the expression $(**)$ from the proof of Theorem 4 to h_k, we get for the unknown functions f_k the system of equations of the form

$$f_{k+2,xx} = -R_k([f_{<k+2}]) + g_k \quad (k \geq 1),$$

(with known R_k and g_k). We solve these equations in the same way as equations $(***)$ were solved in paragraph 2, finding successively $f_3, f_4, \dots$. Theorem 2 is thus proved. □

As it was explained in paragraph 3, Theorem 1 follows.

5. Parabolic curves at a flattening point

We call *the flattening points* of the surface $z = f(x, y)$ the points where the second differential of f vanishes. In suitable (cartesian orthonormal) coordinates the Taylor series of f at a flattening point takes the form

$$f = f_3 + f_4 + \dots \quad (f_0 = f_1 = f_2 = 0).$$

A typical example is the generalized monkey saddle $D_4^{\pm}$: $f = 3x^2y \pm y^3$. The flattening points are critical points of the Gaussian curvature, where the critical value vanishes.

Theorem 5. *The second differential of the Gaussian curature at a flattening point cannot be a nonzero nonnegative quadratic form (in particular, the flattening point cannot be a nondegenerate minimum point of the Gaussian curvature), while all the other quadratic forms are realizable as the second differentials of the Gaussian curvature at the flattening points.*

Proof. Let $f_3 = ax^3 + 3bx^2y + 3cxy^2 + dy^3$. The Hessian of this function is

$$h_2 = 36(Ax^2 + Bxy + Cy^2),$$

where $A = ac - b^2$, $B = ad - bc$, $C = bd - c^2$.

We choose the (Cartesian orthonormal) coordinates reducing the quadratic form h_2 to the normal form ($B = 0$). In these coordinates $ac = A + b^2$, $bd = C + c^2$, $ad = bc$. Hence we have

$$(A + b^2)(C + c^2) = b^2c^2, \quad AC + Ac^2 + Cb^2 = 0.$$

If the form $Ax^2 + Cy^2$ is positive definite ($A > 0$, $C > 0$), the last equality is impossible. The positive semidefinite case ($A > 0$, $C = 0$) is impossible too, since then $c = 0$ and hence $A + b^2 = 0$.

If one of the coefficients, say A, is negative, the required realization is provided by

$$b = \pm\sqrt{-A}, \quad a = c = 0, \quad d = C/b.$$

Theorem 5 is thus proved. □

REMARK 10. Our formulae imply that the degenerate forms h_2 are realized only by degenerate forms $f_3 = ax^3 + 3bx^2y$, and that the zero-form h_2 is realized only by the bidegenerate forms $f_3 = ax^3$.

THEOREM 6. *The parabolic curve singularity at a flattening point of a surface cannot be diffeomorphic to the E_6 singularity (to the singularity of the curve $x^3 + y^4$ at the origin).*

PROOF. The equation of the parabolic curve of the surface $z = f(x, y)$ is $h[f](x, y) = 0$. Choosing a coordinate system for which $f = f_3 + f_4 + \dots$, we should have $h_2 = 0$ to realize the E_6 singularity of h. From Remark 10 above, we get (in suitable coordinates) $f_3 = ax^3$. We can normalize a to be $1/6$ ($a \neq 0$, since for E_6 $h_3 \sim x^3 \neq 0$).

Now compute h_3. Since we should get the E_6 singularity for $h[f]$, it has the form $h_3 = (px + qy)^3$. But for $f_2 = 0$, $f_3 = x^3/6$ we get $h_3 = xf_{4,yy}$. Hence $q = 0$, $h_3 = Px^3$, $P \neq 0$. Now we get $f_{4,yy} = Px^2$, $f_4 = x^2e_2(x, y)$,

$$h_4 = xf_{5,yy} + f_{4,xx}f_{4,yy} - (f_{4,xy})^2.$$

Since f_4 is divisible by x^2, $f_{4,yy}$ and $f_{4,xy}$ are divisible by x, and thus h_4 is divisible by x.

Hence the sum of the terms of degrees 3 and 4 of the Taylor series of $h[f]$ is equal to

$$H(x, y) = h_3 + h_4 = Px^3 + xR_3(x, y).$$

We shall prove now that such a function cannot have an E_6 singularity (whose 4th degree Taylor polynomial should be equal $X^3 + Y^4$ for some local coordinates (X, Y), defined by $(x = \alpha X + \beta Y + \dots, y = \gamma X + \delta Y = \dots)$).

We denote the expression of H in new coordinates by

$$K(X, Y) = H(x(X, Y), y(X, Y)).$$

Equating the terms of degree 3 to X^3, we get

$$P(\alpha X + \beta y)^3 = X^3.$$

Thus $\beta = 0$, $x = \alpha X +$ (higher order terms). It follows that all the terms of degree 4 in the series K are divisible by X. Indeed, the terms of degree 4, coming from Px^3, have the form

$$T_4 = (3Px^2(x - \alpha X))_4 = 3P(x^2)_2(x - \alpha X)_2.$$

The quadratic part of the expansion of x^2 in the power series with argument X, Y is equal to α^2X^2 since $\beta = 0$. Hence the polynomial T_4 is divisible by X^2.

The fourth degree terms coming from xR_3 do contain X for the same reason ($x = \alpha X +$ higher order terms). Hence the 4th degree Taylor polynomial of K is divisible by X. Thus it cannot be equal to $X^3 + Y^4$. Theorem 6 is proved. □

REMARK 11. I do not know which curve singularities are realizable as the parabolic curve singularity at a flattening point. This question (together with similar questions on the singularities of the Hessian of the Gaussian curvature and of the Jacobian of a Lagrangian mapping) might be interesting also for higher flattenings.

References

[1] V. I. ARNOLD, S.M. GUSEIN-ZADE AND A.N. VARCHENKO, *Singularities of Differentiable Maps*, vol. 1, Nauka, Moscow, 1982; English translation, Birkhäuser, 1985.

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