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ON THE SOLVABILITY OF A TWO POINT BOUNDARY VALUE PROBLEM AT RESONANCE II

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1. Introduction

Let $k \geq 1$ be a fixed integer. We consider the boundary value problem

$$u'' + k^2 u + g(x,u) = h(x) \quad \text{in } (0,\pi), \quad u(0) = u(\pi) = 0, \tag{1_k}$$

where $g : (0,\pi)\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, $g(x,u)$ is measurable in $x \in (0,\pi)$ for each $u \in \mathbb{R}$ and continuous in $u \in \mathbb{R}$ for a.e. $x \in (0,\pi)$, $h \in L^1(0,\pi)$ is given. We assume throughout this paper that

(H1) For each $r > 0$, there exists $a_r \in L^1(0,\pi)$ such that

$$|g(x,u)| \leq a_r(x) \quad \text{for a.e. } x \in (0,\pi) \text{ and } |u| \leq r.$$

(H2) There exists $\Gamma \in L^1(0,\pi)$ such that

$$\|\Gamma\|_{L^1} \leq 2k \tag{2}$$

and

$$\limsup_{|u|\to\infty} |g(x,u)/u| \leq \Gamma(x) \tag{3}$$

uniformly for a.e. $x \in (0,\pi)$.

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The solvability of the problem (1_k) has been studied for Γ in $L^\infty(0,\pi)$ (see [2], [7] and the references therein). Existence theorems for a solution to (1_k) when $k=1$ under a growth condition in terms of the L^1 bound of Γ are proved in [6]. In this paper we continue the study of [6] by treating the problem for the general case $k \geq 2$ and improving the L^1 bound for Γ when $k=1$. Our main result is Theorem 1 in Section 3, which is obtained under a Landesman–Lazer condition (see (7) below) originated in [8]. In Section 4 we improve the solvability conditions when $k=1$ by assuming $\|\Gamma\|_{L^1} \leq 4$. The proof of Theorem 1, which becomes more involved when $k \geq 2$, is based on some inequalities of the Lyapunov type obtained in [5] and the Leray–Schauder's fixed point theorem formulated by Granas as a nonlinear alternative in [2].

As in [6] we shall make use of the real Banach spaces $L^p(0,\pi)$, $C[0,\pi]$ and $C^1[0,\pi]$, and the Sobolev spaces $H_0^1(0,\pi)$ and $W^{2,1}(0,\pi)$, with the norms distinguished by appropriate subscripts. We recall the compact imbedding of $H_0^1(0,\pi)$ into $C[0,\pi]$. By a solution of (1_k), we mean a function $u \in H_0^1(0,\pi)$ solving the differential equation in (1_k) in the sense of distribution. It follows from the standard regularity arguments that $u \in W^{2,1}(0,\pi)$ and satisfies the differential equation in (1_k) a.e. on $(0,\pi)$.

2. Preliminaries

In this section we give some auxiliary results which provide important steps in the proofs below. We first state in the following Lemma 1 two inequalities of the Lyapunov type which extend [6, Lemma 1]. We refer to [5] for their proofs.

LEMMA 1. *Let $p \in L^1(0,\pi)$ such that either $p \geq 0$ or $p \leq 0$ a.e. on $(0,\pi)$, and let the problem*

$$\begin{aligned} &v'' + k^2 v + p(x)v = 0 \quad \text{in } (0,\pi), \\ &v(0) = v(\pi) = 0, \end{aligned}$$

have a nontrivial solution $v \in W^{2,1}(0,\pi)$.

(a) *If $\|p\|_{L^1} \leq 2k$, then $p = 0$ a.e. on $(0,\pi)$, so that $v = \alpha \sin kx$ for some $\alpha \in \mathbb{R} \setminus \{0\}$.*
(b) *If $k=1$ and $\|p\|_{L^1} \leq 4$, then v has no zero in $(0,\pi)$.*

Before giving the next lemma, we introduce the following notation. For $v \in W^{2,1}(0,\pi) \cap H_0^1(0,\pi)$, we expand v into the sine series

$$v = \sum_{n=1}^{\infty} b_n \sin nx$$

and denote

$$v = v^- + v^0 + v^+, \quad v^\perp = v - v^0,$$

where $v^-, v^0, v^+ \in W^{2,1}(0,\pi) \cap H_0^1(0,\pi)$ are defined by $v^- = 0$ if $k = 1$ and

$$\begin{aligned} v^- &= \sum_{n=1}^{k-1} b_n \sin nx \quad \text{if } k \geq 2, \\ v^0 &= b_k \sin kx. \end{aligned} \tag{4}$$

Lemma 2. *Let $h \in L^1(0,\pi)$. Then there exist ε, $\delta > 0$ such that*

$$\delta \|(v^\perp)'\|_{L^2}^2 \leq \|v'\|_{L^2} \|h\|_{L^1}$$

for any $v \in W^{2,1}(0,\pi) \cap H_0^1(0,\pi)$ such that

$$\begin{aligned} &v'' + k^2 v + p(x) v = h(x) \quad \text{in } (0,\pi), \\ &v(0) = v(\pi) = 0, \end{aligned}$$

where $p \in L^1(0,\pi)$ is a function satisfying $\|p\|_{L^1} < \varepsilon$ and either $p \geq 0$ or $p \leq 0$ a.e. on $(0,\pi)$.

Proof. We follow an idea first introduced in [9] and developed, among others, in [2] and [7]. By the pairwise orthogonality of v^-, v_0 and v^+ in $H_0^1(0,\pi)$, it is easy to verify that

$$\begin{aligned} \int_0^\pi h(x)(v^- + v^0 - v^+)\, dx &= \int_0^\pi [k^2 (v^-)^2 - {(v^-)'}^2]\, dx \\ &+ \int_0^\pi p(x)(v^- + v^0)^2\, dx + \int_0^\pi [{(u^+)'}^2 - k^2 (u^+)^2 - p(x)(u^+)^2]\, dx. \end{aligned} \tag{5}$$

The second integral on the right-hand side of (5) is nonnegative. Moreover, using the sine series of the functions given in (4), we see that there exist δ_1, $\delta_2 > 0$ which depend only on k such that

$$\int_0^\pi [k^2 (v^-)^2 - {(v^-)'}^2]\, dx \geq \delta_1 \|(v^-)'\|_{L^2}^2,$$

and

$$\int_0^\pi [{(u^+)'}^2 - k^2 (u^+)^2]\, dx \geq \delta_2 \|(v^+)'\|_{L^2}^2.$$

Since

$$\left| \int_0^\pi p(x)(v^+)^2\, dx \right| \leq \|p\|_{L^1} \|v^+\|_C^2 \leq \pi \|p\|_{L^1} \|(v^+)'\|_{L^2}^2,$$

and

$$\|v^- + v^0 - v^+\|_C \leq \sqrt{\pi} \|v'\|_{L^2},$$

the result follows.

3. Solvability theorems for $k \geq 2$

We assume throughout this section that $k \geq 2$. Our main result is the following

Theorem 1. *Let* $g : (0,\pi) \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function satisfying the conditions* (H1), (H2). *If there exist* $r > 0$ *and* $a, b \in L^1(0,\pi)$ *such that*

$$\begin{aligned} g(x,u) &\geq b(x) \quad \text{if } u \geq r, \\ g(x,u) &\leq a(x) \quad \text{if } u \leq -r, \end{aligned} \tag{6}$$

for a.e. $x \in (0,\pi)$, *then the problem* (1_k) *is solvable for any* $h \in L^1(0,\pi)$ *satisfying*

$$\int_0^\pi h(x)v(x)\,dx < \int_{v>0} g_+(x)v(x)\,dx + \int_{v<0} g_-(x)v(x)\,dx, \tag{7}$$

for $v(x) = \pm \sin kx$, *where* $g_+(x) = \liminf_{n\to\infty} g(x,u)$, $g_-(x) = \limsup_{n\to-\infty} g(x,u)$ *for* $x \in (0,\pi)$.

Proof. The proof follows the scheme introduced in [9] and [10], and widely used since. Let $0 < \gamma \leq 1$ be fixed. We consider the boundary value problems

$$\begin{aligned} &u'' + k^2 u + (1-t)\gamma u + tg(x,u) = t\,h(x) \quad \text{in } (0,\pi), \\ &u(0) = u(\pi) = 0, \end{aligned} \tag{8_k}$$

for $0 < t \leq 1$, which becomes the original problem (1_k) when $t = 1$.

We suppose for the moment that there exists $R > 0$ such that $\|u\|_C < R$ for all possible solutions u to the problem (8_k) for some $0 < t < 1$ and use this to finish proving the theorem. For any $h \in L^1(0,\pi)$, the linear problem

$$w'' + k^2 w + \gamma w = h(x), \quad w(0) = w(\pi) = 0,$$

has a unique solution $w \in W^{1,2}(0,\pi) \cap H_0^1(0,\pi)$, because by the choice of γ the corresponding homogeneous problem has only the trivial solution. We define $F : L^1(0,\pi) \to C[0,\pi]$, $Fh = w$, which is a compact linear operator by the compact imbedding of $H_0^1(0,\pi)$ into $C[0,\pi]$. We define $G : C[0,\pi] \to L^1(0,\pi)$ by

$$(Gu)(x) = h(x) + \gamma u - g(x,u(x)),$$

which by (H1) is continuous and maps bounded sets into bounded sets. Let $T = F \circ G : C[0,\pi] \to C[0,\pi]$. Then T is a compact map and the problem (8_k) is equivalent to the operator equation

$$u = tTu,$$

for $0 < t \leq 1$ which by assumption has no solution on the boundary of the ball $B_R(0) = \{u \in C[0,\pi] : \|u\|_C \leq R\}$ for $0 < t < 1$. It follows from the nonlinear alternative of Granas (see [3, Chapter 2, Theorem 5.1] that the operator equation $u = Tu$, or equivalently the original problem (1_k) has a solution in $B_R(0)$.

It remains to show that solutions to (8_k) for $0 < t < 1$ have an *a priori* bound in $C[0,\pi]$. To this end, we first choose $r > 0$ such that

$$g(x,u)/u \le \Gamma(x) + 1,$$

as well as the two inequalities in (6) hold for $|u| \ge r$. Let $\theta : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $0 \le \theta \le 1$ on $\mathbb{R}$ and $\theta(u) = 0$ for $|u| \le r$, $\theta(u) = 1$ for $|u| \ge 2r$. We define

$$g_1(x,u) = \begin{cases} \min\{g(x,u) + |b(x)|, (\Gamma(x)+1)u\}\theta(u) & \text{if } u \ge 0, \\ \max\{g(x,u) - |a(x)|, (\Gamma(x)+1)u\}\theta(u) & \text{if } u \le 0, \end{cases}$$

and $g_2(x,u) = g(x,u) - g_1(x,u)$. Then $g_1, g_2 : (0,\pi) \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions. Moreover, g_1 satisfies (H1) and

$$0 \le g_1(x,u)/u \le \Gamma(x) + 1, \tag{9}$$

for a.e. $x \in (0,\pi)$ and $u \in \mathbb{R}$, where we define $g_1(x,u)/u = 0$ if $u = 0$; g_2 is dominated by a function in $L^1(0,\pi)$, that is, there exits $c \in L^1(0,\pi)$ such that $|g(x,u)| \le c(x)$ for a.e. $x \in (0,\pi)$ and $u \in \mathbb{R}$. Thus we also have

$$\limsup_{|u|\to\infty} g_1(x,u)/u = \limsup_{|u|\to\infty} g(x,u)/u \le \Gamma(x), \tag{10}$$

for a.e. $x \in (0,\pi)$ and $u \in \mathbb{R}$.

Now we argue by contradiction and suppose that there exists a sequence $\{u_n\}$ in $W^{2,1}(0,\pi) \cap H_0^1(0,\pi)$ and a corresponding sequence $\{t_n\}$ in $(0,1)$ such that u_n is a solution to (8_k) when $t = t_n$, and $\|u_n\|_C \ge n$ for $n \ge 1$. Let $v_n = u_n/\|u_n\|_C$. Then $\|v_n\|_C = 1$ and

$$\begin{aligned} &v_n'' + k^2 v_n + p_n(x) v_n = h_n(x) \quad \text{in } (0,\pi), \\ &v_n(0) = v_n(\pi) = 0, \end{aligned} \tag{11}$$

where

$$\begin{aligned} p_n(x) &= (1-t_n)\gamma + t_n g_1(x,u_n(x))/u_n(x), \\ h_n(x) &= t_n[h(x) - g_2(x,u_n(x))]/\|u_n\|_C. \end{aligned} \tag{12}$$

Clearly $\lim_{n\to\infty} h_n = 0$ in $L^1(0,\pi)$. By (9) we have

$$0 \le p_n(x) \le \Gamma(x) + 1, \tag{13}$$

for a.e. $x \in (0,\pi)$ and $n \ge 1$. It follows from the Dunford–Pettis theorem that the sequence $\{p_n\}$ has a subsequence which converges weakly to a function p in $L^1(0,\pi)$. Moreover, by the Mazur theorem $0 \le p(x) \le \Gamma(x)+1$ for a.e. $x \in (0,\pi)$. From (11) we see that v_n'' is dominated by a function in $L^1(0,\pi)$ independent of n. Since each v_n' vanishes somewhere in $(0,\pi)$, the sequence $\{v_n'\}$ is equicontinuous and uniformly bounded on $[0,\pi]$. Hence the sequence $\{v_n\}$ is also equicontinuous and uniformly bounded on $[0,\pi]$. It follows from the Arzela–Ascoli theorem that $\{v_n\}$ has a subsequence which converges in $C^1[0,\pi]$. We assume without any loss of generality that $\{p_n\}$ converges weakly to p in $L^1(0,\pi)$, $t_n \to t_0$ and there exists $v \in C^1[0,\pi]$ such that $\{v_n\}$ converges to v in $C^1[0,\pi]$ and so does also in $H_0^1(0,\pi)$. Letting $n\to\infty$ in (11), we have

$$v'' + k^2 v + p(x)v = 0, \quad v(0) = v(\pi) = 0, \tag{14_k}$$

where by (10), (13) and the Lebesgue theorem

$$\|p\|_{L^1} = \lim_{n\to\infty} \|p_n\|_{L^1} \le 2k. \tag{15}$$

Since v is a nontrivial solution to (14_k), it follows from Lemma 1 (a) that $p=0$ a.e. on $(0,\pi)$, so that either $v=\sin kx$ or $v=-\sin kx$. Consequently $t_0=1$.

We consider the sequence $\{v_n^0\}$ as defined in (4) which is contained in the one-dimensional vector subspace generated by $\sin kx$. Obviously $\{v_n^0\}$ also converges to v in $C^1[0,\pi]$. Taking the inner product in $L^2(0,\pi)$ of (8_k) when $u=u_n$ and $t=t_n$ with v_n^0, we have

$$\begin{aligned} t_n\left[\int_0^\pi h(x)v_n^0\,dx - \int_0^\pi g(x,u_n)v_n^0\,dx\right] &= (1-t_n)\gamma\int_0^\pi u_n v_n^0\,dx \\ &= (1-t_n)\gamma\|u_n\|_C\int_0^\pi (v_n^0)^2\,dx \ge 0, \end{aligned}$$

and so

$$\begin{aligned} \int_0^\pi h(x)v_n^0\,dx &\ge \int_0^\pi g(x,u_n)v_n^0\,dx \\ &= \int_{v>0} g(x,u_n)v_n^0\,dx + \int_{v<0} g(x,u_n)v_n^0\,dx. \end{aligned} \tag{16}$$

Using the inequality

$$|w(x)/\sin x| \le (\pi/2)\|w'\|_C \quad \text{for } x\in[0,\pi], \tag{17}$$

valid for all $w\in C^1[0,\pi]$ with $w(0)=w(\pi)=0$, we see that for $x\in(0,\pi)$, if $v(x)>0$, then $v_n(x)>0$ for n large enough, so that $u_n(x)\to\infty$; if $v(x)<0$, then $v_n(x)<0$ for n large enough, so that $u_n(x)\to-\infty$. We suppose for the moment that there exists a function $f\in L^1(0,\pi)$ such that for n large enough,

$$g(x,u_n)v_n^0 \ge f(x) \quad \text{for a.e. } x\in(0,\pi). \tag{18}$$

By taking the limits inferior on both sides of (16) and applying the Fatou Lemma, we would have

$$\int_0^\pi h(x)v\,dx \geq \liminf_{n\to\infty}\int_{v>0} g(x,u_n)v_n^0\,dx + \liminf_{n\to\infty}\int_{v<0} g(x,u_n)v_n^0\,dx$$
$$\geq \int_{v>0} g_+(x)v\,dx + \int_{v<0} g_-(x)v\,dx,$$

which contradicts the Landesman–Lazer condition (7).

It suffices to prove (18). Since $\lim_{n\to\infty}\|p_n\|_{L^1} = \|p\|_{L^1} = 0$, by Lemma 2 there exists $\delta > 0$ such that for n large enough

$$\delta\|(v_n^\perp)'\|_{L^1}^2 \leq \|v_n'\|_{L^2}\|h_n\|_{L^1}.$$

As noted before, $\{v_n'\}$ is uniformly bounded on $[0,\pi]$ and so it follows from the definition of h_n that there exists $\beta > 0$ such that

$$\|u_n\|_C|v_n^\perp(x)|^2 \leq 2\beta,$$

and so

$$u_n(x)v_n^0(x) \geq -\|u_n\|_C|v_n(x) - v_n^0(x)|^2/2 = -\|u_n\|_C|v_n^\perp(x)|^2/2 \geq -\beta$$

for $x \in [0,\pi]$. Hence

$$g(x,u_n(x))v_n^0(x) = (g_1(x,u_n(x))/u_n(x))u_n(x)v_n^0(x) + g_2(x,u_n(x))v_n^0(x)$$
$$\geq -\beta(\Gamma(x)+1) - c(x),$$

for a.e. $x \in (0,\pi)$. Thus it suffices to choose $f(x) = -\beta(\Gamma(x)+1) - c(x)$. This completes the proof of the theorem.

We see in the proof of Theorem 1 above that only the parts of Lemmas 1 and 2 in which $p \in L^1(0,\pi)$ satisfies $p \geq 0$ a.e. on $(0,\pi)$ are used. By applying the other parts of Lemmas 1 and 2 we obtain similarly the following

Theorem 2. *Let $g : (0,\pi)\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function satisfying the conditions* (H1), (H2). *If there exist $r > 0$ and $a, b \in L^1(0,\pi)$ such that*

$$g(x,u) \leq b(x) \quad \text{if } u \geq r,$$
$$g(x,u) \geq a(x) \quad \text{if } u \leq -r,$$

for a.e. $x \in (0,\pi)$, then the problem (1_k) is solvable for any $h \in L^1(0,\pi)$ satisfying

$$\int_0^\pi h(x)v(x)\,dx > \int_{v>0} g_+(x)v(x)\,dx + \int_{v<0} g_-(x)v(x)\,dx,‘$$

for $v(x) = \pm\sin kx$, where $g_+(x) = \limsup_{n\to\infty} g(x,u)$, $g_-(x) = \liminf_{n\to-\infty} g(x,u)$ for $x \in (0,\pi)$.

Clearly the Landesman–Lazer conditions are essential for Theorems 1 and 2 to hold. It would be interesting to obtain solvability conditions for (1_k) if the equality holds in place of one of the inequalities. We refer to [4] for a solvability result without assuming a Landesman–Lazer condition when g is bounded.

4. Solvability conditions for $k = 1$

When $k = 1$ the solvability conditions for the problem (1_k) obtained in Section 3 can be significantly improved. The following Theorem 3, which is obtained under assumptions with or without a Landesman–Lazer condition, extends the main results of [6].

Theorem 3. *Let $g : (0,\pi) \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the conditions* (H1), (H2) *except* (2) *which is replaced by*

$$\|\Gamma\|_{L^1} \le 4. \tag{19}$$

(a) *If there exist $r > 0$ and a, $b \in L^1(0,\pi)$ such that* (6) *holds for a.e. $x \in (0,\pi)$, then the problem (1_1) is solvable for any $h \in L^1(0,\pi)$ satisfying*

$$\int_0^\pi g_-(x)\sin x\,dx < \int_0^\pi h(x)\sin x\,dx < \int_0^\pi g_+(x)\sin x\,dx, \tag{20}$$

where g_+ and g_- are defined as in Theorem 1*;*

(b) *If*

$$g(x,u)u \ge 0 \quad \text{for } u \in \mathbb{R}, \tag{21}$$

then the problem (1_1) is solvable for any $h \in L^1(0,\pi)$ satisfying $\int_0^\pi h(x) \sin x\,dx = 0$.

Proof. The existence of a solution to (1_1) is proved by the theorem of Granas as in the proof of Theorem 1. It requires an *a priori* bound for the solutions of (8_1) for $0 < t < 1$. This is obtained in exactly the same way as in the proof of Theorem 1 up to the step where we have a nontrivial solution $v \in W^{2,1}(0,\pi) \cap H_0^1(0,\pi)$ to the problem (14_1). It follows from the weaker assumption (19) that $\|p\|_{L^1} \le 4$ instead of (15). By Lemma 1(b) we know that v has no zero in $(0,\pi)$. Moreover, $0 < t_0 \le 1$ since (14_1) with $p = \gamma$ a.e. on $(0,\pi)$ cannot have a nontrivial solution.

We assume that $v > 0$ on $(0,\pi)$; the case in which $v < 0$ on $(0,\pi)$ can be treated similarly. Using the inequality (17), we obtain that $u_n > 0$ on $(0,\pi)$ for n large enough, so that $u_n(x) \to \infty$ for $x \in (0,\pi)$. In the following we consider only n large enough. Taking the inner product in $L^2(0,\pi)$ of (8_1) when $u = u_n$ and $t = t_n$ with $\sin x$, we have

$$t_n\left[\int_0^\pi h(x)\sin x\,dx - \int_0^\pi g(x,u_n(x))\sin x\,dx\right] = (1-t_n)\gamma\int_0^\pi u_n(x)\sin x\,dx > 0. \tag{22}$$

If the assumption in (a) holds, then it follows from (H1) and the first inequality in (6) that $g(x,u_n(x))$ is bounded from below by a function in $L^1(0,\pi)$ independent of n. By (22) and the Fatou lemma we would have

$$\int_0^\pi h(x)\sin x\,dx \geq \int_0^\pi g_+(x)\sin x\,dx,$$

which contradicts the second inequality in (20). If the assumption in (b) holds, then by (22) again we would have

$$\int_0^\pi g(x,u_n(x))\sin x\,dx < 0,$$

which contradicts (21). This completes the proof of the theorem.

It is clear from the proof of Theorem 3 that we can obtain new solvability conditions for (1_1) by making combinations out of conditions (20) and (21). For example, under the general hypothesis of Theorem 3 we have that

(c) If there exist $r > 0$, $b \in L^1(0,\pi)$ such that the first inequality in (6) holds for a.e. $x \in (0,\pi)$ and

$$g(x,u) \leq 0 \quad \text{for } u \leq 0,$$

then the problem (1_1) is solvable for any $h \in L^1(0,\pi)$ satisfying

$$0 \leq \int_0^\pi h(x)\sin x\,dx < \int_0^\pi g_+(x)\sin x\,dx,$$

where g_+ is defined as in Theorem 1.

We refer to [1] for a result similar to Theorem 3(b) under slightly restricted conditions.

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