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ONE-POINT SINGULAR SOLUTIONS TO THE NAVIER–STOKES EQUATIONS

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Dedicated to Olga Ladyzhenskaya

1. Introduction

Stationary or self similar solutions with suitable homogeneity often play a crucial role in the regularity theory of nonlinear problems, which are physically or geometrically interesting. This has been manifested in the regularity theory of harmonic maps and minimal surfaces. The local partial regularity theorem in [CKN] implies that there are no self-similar solutions with small local energy (also see [TX] for generalizations). Making use of some arguments in [NRS], Tsai has ruled out the existence of any self-similar solutions with a finite local energy. Yet it is unclear whether or not solutions of the incompressible Navier–Stokes equation in three space dimensions would develop singularities in finite time. Therefore, it may be still interesting to construct special solutions of the 3-dimensional Navier–Stokes equation.

In this note, we construct a one-parameter family of explicit smooth solutions of the 3-dimensional impressible Navier–Stokes equation on $\mathbb{R}^3 \setminus p$, where p is any given point. These solutions are axisymmetric, homogeneous of degree -1. They are steady solutions to the Navier–Stokes equations and also solve the self-similar

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form of the Navier–Stokes equations. Such solutions are unique in the class of axisymmetric flows. They should provide ansatz at infinity for possible singular solutions of the Navier–Stokes equation or exterior problems for a stationary 3-dimensional Navier–Stokes equation. Their construction will be given in the next section. We refer the readers to [CP], [GK] for some related works. It should be noted that for special parameters, our explicit solutions become the known solutions for a jet regarded as emerging from a point source ([LL]). However, our approach seems more general as it also yields uniqueness of the solutions and can be applied even to ideal fluids.

In the third section, we will show that 3-dimension steady incompressible Euler equations do not admit solutions of this type. Instead, a class of more singular homogeneous of degree -1 axisymmetirc solutions will be presented for the invisible systems.

2. Explicit solutions for viscous flows

In this section, we will derive explicit formulas for a one-parameter family of singular solutions of the 3-dimensional Navier–Stokes equations, which is steady, axisymmetric, homogeneous of degree -1, and regular everywhere except at a given point (such a solutions will be called a *one-point singular solution*. Furthermore, we will prove that our solution formulas yield all possible one-point singular solutions for viscous axisymmetric flows. More precisely, we will show the following theorem.

Theorem 1. *All the one-point singular solutions to the* 3*-dimensional incompressible Navier–Stokes equations, which are singular at* (x_1^0, x_2^0, x_3^0) *and symmetric about* x_1*-axis, are given by the following explicit formula:*

$$\vec{u}(\vec{x}) = 2\bigg(\frac{cr^2 - r(x_1 - x_1^0) + c(x_1 - x_1^0)^2}{r(cr - (x_1 - x_1^0))^2}, \frac{(x_2 - x_2^0)(c(x_1 - x_1^0) - r)}{r(cr - (x_1 - x_1^0))^2}, \frac{(x_3 - x_3^0)(c(x_1 - x_1^0) - r)}{r(cr - (x_1 - x_1^0))^2}\bigg), \tag{2.1}$$

$$p(\vec{x}) = \frac{4(c(x_1 - x_1^0) - r)}{r(cr - (x_1 - x_1^0))^2}, \tag{2.2}$$

where $r = \sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + (x_3 - x_3^0)^2}$*, and* c *is an arbitrary constant such that*

$$|c| > 1. \tag{2.3}$$

Proof. Due to the translation invariance of the Navier–Stokes equations, we may assume that the singular point is the origin, i.e. $\vec{x}^0 = (x_1^0, x_2^0, x_3^0) = (0, 0, 0)$. We will also use the notation

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2} = |\vec{x}|, \quad s = \frac{x_1}{r}. \tag{2.4}$$

Then our main task is to look for solutions to the $3D$ Navier–Stokes equations of the form.

$$\vec{u}(\vec{x}) \equiv (u_1(\vec{x}), u_2(\vec{x}), u_3(\vec{x})) = \frac{1}{r}\left(f(s), \frac{x_2}{r}g(s) + \frac{x_3}{r}k(s), \frac{x_3}{r}g(s) - \frac{x_2}{r}k(s)\right), \tag{2.5}$$

$$p(\vec{x}) = \frac{1}{r^2}h(s), \tag{2.6}$$

where $f(s)$, $g(s)$, $k(s)$ and $h(s)$ are to be determined C^2-smooth and bounded functions on $-1 \le s \le 1$, so that the ansatz (2.5)–(2.6) solves the Navier–Stokes everywhere except at $r = 0$. This leads to a system of second order ordinary differential equations for $(f(s), g(s), k(s), h(s))$ as follows.

Direct calculations show that

$$\nabla u_1 = \Big(\frac{1}{r^2}(-sf(s) + (1-s^2)f'(s)), - \frac{x_2}{r^3}(f(s) + sf'(s)), -\frac{x_3}{r^3}(f(s) + sf'(s))\Big), \tag{2.7}$$

$$\vec{u}\cdot\nabla u_1 = \frac{1}{r^3}(-sf^2(s) + (1-s^2)f(s)f'(s) - (1-s^2)gf - s(1-s^2)gf'), \tag{2.8}$$

$$\Delta u_1 = \frac{1}{r^3}[(1-s^2)f']', \tag{2.9}$$

$$\nabla p = \frac{1}{r^3}\left([(1-s^2)h']', -\frac{x_2}{r}(2h + sh'), -\frac{x_3}{r}(2h+sh')\right), \tag{2.10}$$

where $f' = \frac{d}{ds}(f(s))$, etc. Then the first component of the momentum conservation is equivalent to

$$t((1-s^2)f')' - ((1-s^2)h)' + sf^2 - (1-s^2)ff' + (1-s^2)gf + s(1-s^2)gf' = 0. \tag{2.11}$$

Next, one calculates that

$$\vec{u}\cdot\nabla u_2 = \frac{x_2}{r_4}[-2sgf + (1-s^2)g'f + (2s^2-1)g^2 - s(1-s^2)gg' - k^2] + \frac{x_3}{r^4}[-2skf + (1-s^2)k'f + 2s^2kg - s(1-s^2)gk'], \tag{2.12}$$

$$\Delta u_2 = \frac{x_2}{r^4}[(1-s^2)g]'' + \frac{x_3}{r^4}[(1-s^2)k]''. \tag{2.13}$$

It follows from (2.10), (2.12) and (2.13) that the second component of the momentum conservation in the Navier–Stokes equations is equivalent to the following system:

$$((1-s^2)g)'' + 2sfg - (1-s^2)g'f - (2s-1)g^2 + s(1-s^2)gg' + 2h + sh' + k^2 = 0, \tag{2.14}$$

$$((1-s^2)k)'' + 2skf - (1-s^2)k'f - 2s^2kg + s(1-s^2)gk' = 0. \tag{2.15}$$

By symmetry, it should be clear that system (2.14)–(2.15) is also equivalent to the third component of the momentum conservation in the Navier–Stokes equations. Finally, one checks easily that the continuity equation div $\vec{u} = 0$ becomes

$$(1-s^2)f' - sf - s(1-s^2)g' + 2s^2 g = 0. \tag{2.16}$$

Thus, to prove Theorem 1, one needs to study all regular ($C^2([-1,1])$) solutions $(f,g,h,k)(s)$ to the system of ordinary differential equations, (2.11) and (2.14)–(2.16). To this end, we will simplify the systems (2.11) and (2.14)–(2.16) into a simple integrable system by the following steps.

Step 1. Multiplying (2.14) by s and substracting the resulting equations from (2.11), one obtains after some manipulations that

$$(h - sf - (1-s^2)g)' = -sk^2 + g(f - sg), \tag{2.17}$$

where equations (2.11) have been used. We set

$$h = H + sF + G, \quad f = F + sG, \quad g = G, \quad k = K. \tag{2.18}$$

Then (2.17), (2.16), and (2.14)–(2.15) are transformed respectively into

$$H' = -sK^2 + GF, \tag{2.19}$$

$$(1-s^2)F' = sF - G, \tag{2.20}$$

$$\begin{aligned}((1-s^2)G)'' + 2s(FG) - (1-s^2)G'F + G^2 + 2G + sG' \\ + 2sF + s(sF)' + 2H + sH' + K^2 = 0,\end{aligned} \tag{2.21}$$

$$((1-s^2)K)'' + 2s(KF) + (s^2-1)K'F = 0. \tag{2.22}$$

Using equations (2.19)–(2.20) repeatedly, one can reduce (2.21) and (2.22) to

$$((1-s^2)(G - H - sF))'' + (s(s^2-1)K^2)' + (s^2+1)K^2 = 0, \tag{2.23}$$

$$((1-s^2)K)'' + ((s^2-1)KF)' + (sF - G)K = 0, \tag{2.24}$$

respectively. We now proceed to solve the systems (2.19)–(2.20) and (2.23)–(2.24).

Step 2. We claim that if $(f,g,h,k)(s)$ is a regular solution in $C^2([-1,1])$ to (2.11) and (2.14)–(2.16), then

$$K(s) \equiv 0 \quad \forall s \in [-1,1]. \tag{2.25}$$

Indeed, if $(f,g,h,k) \in C^2([-1,1])$ solves (2.11) and (2.14)–(2.16), then (2.19)–(2.20) and (2.23)–(2.25) are solved by $(F,G,H,K) \in C^2([-1,1])$. In particular, (2.24) together with (2.20), shows that

$$((1-s^2)K)'' - (((1-s^2)K)F)' + (1-s^2)KF' = 0. \tag{2.26}$$

We set $\mu(s) = (1-s^2)K(s)$. Then $\mu(s)$ solves

$$\mu''(s) - \mu'(s)F(s) = 0, \quad \mu(-1) = \mu(1) = 0. \tag{2.27}$$

However, the problem (2.27) admits only a trivial smooth solution. Thus, $\mu(s) \equiv 0$. Hence $K(s) \equiv 0$. Consequently it remains to solve the following system:

$$H' = GF, \tag{2.28}$$

$$(1-s^2)F' = sF - G, \tag{2.29}$$

$$((1-s^2)(G-H-sF))'' = 0. \tag{2.30}$$

Step 3. We now integrate system (2.28)–(2.30) explicitly. First, multiplying (2.29) by F and using (2.28), one derives that

$$\Big(H(s) + \frac{1}{2}(1-s^2)F^2(s)\Big)' = 0.$$

It follows that there exists an integration constant C_1 such that

$$H(s) + \frac{1}{2}(1-s^2)F^2(s) = C_1. \tag{2.31}$$

Next, equation (2.30) implies that there exist constants C_2 and C_3 such that

$$(1-s^2)(G-H-sF) = C_2 + C_3 s, \quad s \in [-1,1]. \tag{2.32}$$

As a consequence of (2.32) and the fact that $(G,H,F) \in C^2([-1,1])$, one has $C_2 = C_3 = 0$ so that

$$G(s) = H(s) + sF(s), \quad \forall s \in [-1,1]. \tag{2.33}$$

Finally, it follows from (2.29), (2.33) and (2.31) that

$$(1-s^2)F'(s) = \frac{1}{2}(1-s^2)F^2(s) - C_1, \quad s \in [-1,1]. \tag{2.34}$$

Since $F \in C^2([-1,1])$, so (2.14) shows $C_1 = 0$, and consequently,

$$H(s) + \frac{1}{2}(1-s^2)F^2(s) \equiv 0, \quad \forall s \in [-1,1], \tag{2.35}$$

and

$$F'(s) = \frac{1}{2}F^2(s), \quad s \in [-1,1]. \tag{2.36}$$

Equation (2.36) can be integrated to obtain

$$F(s) = \frac{2}{C-s}, \quad s \in [-1,1], \tag{2.37}$$

with C an arbitrary constant. Combining (2.37) with (2.35) yields that

$$H(s) = \frac{2(s^2-1)}{(C-s)^2}, \quad s \in [-1,1]. \tag{2.38}$$

Together with (2.33), this yields

$$G(s) = \frac{2(Cs-1)}{(C-s)^2}, \quad s \in [-1,1]. \tag{2.39}$$

Returning to the original variables (f,g,h,k) (2.18), we arrive at

$$f(s) = \frac{2}{(C-s)^2}(C-2s+Cs^2), \tag{2.40}$$

$$g(s) = \frac{2(Cs-1)}{(C-s)^2}, \tag{2.41}$$

$$h(s) = \frac{4(Cs-1)}{(C-s)^2}, \tag{2.42}$$

$$k(s) \equiv 0, \tag{2.43}$$

and the solution will be regular if and only if

$$|C| > 1. \tag{2.44}$$

Now the formulas (2.1)–(2.3) follow directly from (2.5)–(2.6), and (2.40)–(2.44).

The proof of Theorem 1 is completed. □

We conclude this section by pointing out that our previous analysis also gives all the point-singular self-similar solutions to the 3-dimensional Navier–Stokes equations, which are symmetric and homogeneous of degree -1. Indeed, self-similar solutions of the Navier–Stokes equation are solutions of the form (see [Le], [NRS], [Ts])

$$\begin{aligned}(\vec{U}(\vec{x}',t), P(\vec{x}',t)) \\ = \bigg(\frac{1}{\sqrt{2a(T-t)}}\vec{u}\bigg(\frac{\vec{x}'-\vec{x}'_0}{\sqrt{2a(T-t)}}\bigg), \frac{1}{2a(T-t)}p\bigg(\frac{\vec{x}'-\vec{x}'_0}{\sqrt{2a(T-t)}}\bigg)\bigg),\end{aligned} \tag{2.45}$$

where $T \in \mathbb{R}^1$, $\vec{x}'_0 \in \mathbb{R}^3$ being a fixed point, $a > 0$ (< 0) if $t < T$ $(t > T)$, and $\vec{u} = (u_1,u_2,u_3)$ and p are defined in $\mathbb{R}^3$. We set

$$\vec{x} = \frac{\vec{x}'-\vec{x}'_0}{\sqrt{2a(T-t)}} \tag{2.46}$$

to be the self-similar variable. Then the governing equations for $(\vec{u},p)(\vec{x})$ are

$$a(\vec{u}+(\vec{x}\cdot\nabla)\vec{u}) + (\vec{u}\cdot\nabla)\vec{u} + \nabla p = \Delta\vec{u} \quad \text{in } \mathbb{R}^3, \tag{2.47}$$

$$\operatorname{div}\vec{u} = 0. \tag{2.48}$$

If we require that the self-similar flow is homogeneous of degree -1 and axisymmetric (with x_1-axis as symmetry axis) with a point singularity at the origin, then $(\vec{u},p)(\vec{x})$ are given by formulas (2.5)–(2.6) with (f,g,h,k) given in (2.40)–(2.43), as can be easily checked. Thus we have shown

Theorem 2. *Any self-similar solution to the Navier–Stokes equations, which is symmetric about x_1'-axis and homogeneous of degree -1 with singular point at $\vec{x}_0' = (x_{10}', x_{20}', x_{30}')$, is given by the following formula*

$$\vec{u}(\vec{x}', t) = \left(\frac{cr^2 - r(x_1' - x_{10}') + c(x_1' - x_{10}')^2}{(cr - (x_1' - x_{10}'))^2 r}, \frac{(x_1' - x_{20}')(c(x_1' - x_{10}') - r)}{(cr - (x_1' - x_{10}'))^2 r}, \frac{(x_3' - x_{30}')(c(x_1' - x_{10}') - r)}{(cr - (x_1' - x_{10}'))^2 r} \right),$$

$$p(\vec{x}', t) = \frac{4(c(x_1' - x_{10}') - r)}{r(cr - (x_1' - x_{10}'))^2},$$

where $r = |\vec{x}' - \vec{x}_0'| = \sqrt{(x_1' - x_{10}')^2 + (x_2' - x_{20}')^2 + (x_3' - x_{30}')^2}$, *and c is any constant such that $|c| > 1$.*

3. Singular solutions for inviscid flows

We now study steady axisymmetric, homogeneous of degree -1, solutions to the 3-dimensional incompressible Euler equations

$$\partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \nabla p = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}^n, \tag{3.1}$$

$$\operatorname{div} \vec{u} = 0. \tag{3.2}$$

We will show that in contrast to the viscous flows, the inviscid Euler equations do not possess nontrivial steady axisymmetric, homogeneous of degree -1, solutions which are regular everywhere except at a single point. Furthermore, it is shown that for the inviscid Euler equations, (3.1)–(3.2), all nontrivial steady axisymmetric solutions, which are homogeneous of degree -1, are singular everywhere along the axis of symmetry, and explicit formulas for such singular solutions are obtained. More precisely, we have the following theorem.

Theorem 3. *Let's onsider, steady, homogeneous of degree -1, and axisymmetric solutions to the 3-dimensional incompressible Euler equations* (3.1)–(3.2). *Then,*

1. *there exists no such solution which is regular (C^2) everywhere except at a point, unless it is a trivial solution;*
2. *all such solutions are singular along the axis of symmetry, and are given by the following explicit formulas*

$$\vec{u}(\vec{x}) = \frac{1}{r}\left(F(s) - sG(s), \frac{x_2}{r}G(s) + \frac{x_3}{r}\frac{C_1}{1-s^2}, \frac{x_3}{r}G(s) - \frac{x_2}{r}\frac{C_1}{1-s^2} \right), \tag{3.3}$$

$$p(\vec{x}) = \frac{1}{r^2}\frac{C_3 + C_2 s}{1-s^2}, \tag{3.4}$$

where $s = x_1/r$, $r = \sqrt{x_1^2 + x_3^2 + x_3^2}$,

$$(3.5)\qquad \begin{aligned} F(s) &= \pm\sqrt{C_4(rs^2) - (C_1^2 + 2C_3 + 2C_2 s)}/(1-s^2), \\ G(s) &= -sF(s) - ((1-s^2)F(s))', \end{aligned}$$

where C_1, C_2, C_3 *and* C_4 *are arbitrary constants such that*

$$(3.6)\qquad (C_4 - C_1^2 - 2C_3) - 2C_2 s - C_4 s^2 \geq 0, \quad s \in (-1,1],$$

here we assume that the axis of symmetry is x_1*-axis;*

3. *all such solutions, which are integrable near the axis of symmetry, are given by*

$$(3.7)\qquad \vec{u}(\vec{x}) = (C_5/\sqrt{x_2^2 + x_3^2}, 0, 0), \quad \vec{x} \in \mathbb{R}^3,$$

$$(3.8)\qquad p(\vec{x}) = 0.$$

Proof. As in Section 2, all the steady solutions to the Euler equations (3.2)–(3.2), which are symmetric about x_1-axis and homogeneous of degree -1, can be written in the form (2.5)–(2.6) with f, g, k, h to be determined. We substitute the expressions (2.5)–(2.6) into the equations (3.1)–(3.2) to obtain, in the same way as for (2.11), (2.14)–(2.16), that

$$(3.9)\qquad (1-s^2)h)' - (1-s^2)ff' - s(1-s^2)gf' - (1-s^2)fg - sf^2 = 0,$$

$$(3.10)\qquad (1-s^2)fg' - s(1-s^2)gg' - 2sfg + (2s^2-1)g^2 - 2h - sh' - k^2 = 0,$$

$$(3.11)\qquad (1-s^2)f' - s(1-s^2)g' - sf + 2s^2 g = 0,$$

$$(3.12)\qquad -2sfk + (1-s^2)fk' + 2s^2 kg - s(1-s^2)gk' = 0.$$

To solve system (3.9)–(3.12) explicitly, we set

$$(3.13)\qquad F(s) = f(s) - sg(s), \quad G(s) = g(s), \quad H(s) = h(s), \quad K(s) = k(s).$$

We can then transform system (3.9)–(3.12) into the following equivalent system

$$(3.14)\qquad H' = -sK^2 + GF,$$

$$(3.15)\qquad (1-s^2)F' = sF - G,$$

$$(3.15)\qquad (1-s^2)(FG)' = 4sFG + 2H + (1-s^2)K^2,$$

$$(3.16)\qquad ((1-s^2)K)'F = 0,$$

see the derivation of (2.19)–(2.22). System (3.14)–(3.17) can be solved explicitly by the following steps.

Step 1. It holds that

$$(3.18)\qquad K(s) = \frac{C_1}{1-s^2} \quad \forall s \in (-1,1),$$

for an arbitrary constant C_1. This follows from the claim that

$$((1-s^2)K(s))' = 0 \quad \forall s \in (-1,1). \tag{3.19}$$

(3.19) can be verified as follows. If (3.19) is false, then there exists a subinterval $[a,b] \subset (-1,1)$ such that

$$((1-s^2)K(s))' \neq 0 \quad \forall s \in [a,b]. \tag{3.20}$$

This, together with (3.17), shows that $F(s) \equiv 0$ for all $s \in [a,b]$. Hence, $G(s) \equiv 0$ on $[a,b]$ due to (3.15), and so, equations (3.14) and (3.16) become

$$H'(s) = -sK^2(s), \quad s \in [a,b], \tag{3.21}$$

$$2H(s) + (1-s^2)K^2(s) = 0. \tag{3.22}$$

It follows from (3.21) and (3.22) that

$$((1-s^2)H(s))' = 0 \quad \forall s \in [a,b].$$

Hence $(1-s^2)H(s) = C_0$ on $[a,b]$, for some constant C_0. This and (3.21) yield that

$$(1-s^2)K(s) = \sqrt{-2C_0} \quad \text{on } [a,b],$$

which implies that

$$((1-s^2)K(s))' = 0 \quad \text{on } [a,b].$$

This contradicts (3.20). Hence (3.19) holds, so does (3.18).

Step 2. There exist two arbitrary constants C_2 and C_3 such that

$$H(s) = \frac{C_3 + C_2 s}{(1-s^2)}, \quad s \in (-1,1). \tag{3.23}$$

To see this, we rewrite equation (3.16), by using equation (3.14), as

$$(1-s^2)H''(s) - 4sH'(s) - 2H(s) = 2s(s^2-1)K(s)K'(s) + 4s^2K^2(s) \tag{3.24}$$

on $(-1,1)$. Due to the special form of $K(s)$ in (3.18), one checks easily that

$$2s(s^2-1)K(s)K'(s) + 4s^2K^2(s) \equiv 0 \quad \text{on } (-1,1).$$

Hence (3.24) becomes

$$((1-s^2)H(s))'' \equiv 0 \quad \text{on } (-1,1),$$

which shows (3.23) immediately.

Step 3. We are now in the position to derive the explicit expressions for $F(s)$ and $G(s)$. Indeed, multiplying (3.15) by $F(s)$ and using (3.14), one can derive

$$\left(\frac{1}{2}(1-s^2)F^2(s) + H(s)\right)' = sK^2(s) \quad \text{on } (-1.1). \tag{3.25}$$

This, together with (3.18) and (3.23), shows that

$$(1-s^2)^2F^2(s) = C_4(1-s^2) - (C_1^2 + 2C_3 + 2C_2 s)$$

on $(-1,1)$, for an arbitrary constant C_4. Hence, as long as

$$C_4(1-s^2) - (C_1^2 + 2C_3 + 2C_2 s) \geq 0 \quad \text{on } [-1,1], \tag{3.26}$$

we obtain the formula for $F(s)$ as

$$F(s) = \pm \frac{\sqrt{C_4(1-s^2) - (C_1^2 + 2C_3 + 2C_2 s)}}{1-s^2}, \quad s \in (-1,1). \tag{3.27}$$

With $F(s)$ as determined, we can solve for $G(s)$ from (3.15) to get

$$G(s) = sF + (s^2-1)F' = ((s^2-1)F(s))' - sF(s) \quad \text{on } (-1,1). \tag{3.28}$$

Using (3.27), one gets

$$\begin{aligned} G(s) = {} & \pm \frac{s}{(s^2-1)}\sqrt{C_4(1-s^2) - (C_1^2 + 2C_3 + 2Cs)} \\ & \pm \frac{C_4 s - C_2}{\sqrt{C_4(1-s^2) - (C_1^2 + 2C_3 + 2C_2 s)}}, \end{aligned} \tag{3.29}$$

for $s \in (-1,1)$.

We can now conclude Theorem 3 by reading off from the formulas (3.26), (3.28), (3.18), (3.19), and their derivations. The proof is completed. □

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