Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 11, 1998, 103–113

ON THE THREE CRITICAL POINTS THEOREM

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1. Introduction

Let φ be a C^1 real function defined on $\mathbb{R}^m$. We assume that φ is coercive (i.e. $\varphi(x) \to \infty$ as $||x|| \to \infty$). It is well-known that under these assumptions φ reaches a minimum at some point x_0. Let now x_1 be a critical point of φ which is not a global minimum. M. A. Krasnosel'skiĭ [10] made the following observations: if x_1 is a nondegenerate singular point of the vector field $\nabla\varphi$ (i.e. the topological index $\operatorname{ind}(\nabla\varphi(x_1), 0)$ is different from zero), then φ admits a third critical point. In the sequel this statement became known as the "Three Critical Points Theorem" (TCPT).

The above result of Krasnosel'skiĭ was extended to the context of Banach spaces (see [1], [4], [8], [17]). Another generalization was obtained by Chang [5], [6] using the methods of Morse theory (the condition $\operatorname{ind}(\nabla\varphi(x_1), 0) \neq 0$ is replaced by the weaker assumption of nontriviality of Morse critical groups at x_1). Also, Brezis and Nirenberg [3] gave a very useful variant of TCPT for applications using the principle of local linking (see also [12]). In this paper we shall give a proof of TCPT based on a "strong" deformation lemma (see Lemma 2.1 below) thus avoiding standard minimax techniques. In contrast to the previous work in this field, we prove in fact the Lusternik–Schnirel'man type

1991 *Mathematics Subject Classification.* 58E05, 45G10, 47H30.

Key words and phrases. Critical point theory, three critical point theorem, essential ctitical value, Hammerstein integral equation.

Partially supported by the International Soros Science Education Program and the Belorussian Fond of Fundamental Investigations.

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alternative, that is, for a functional φ there exist either three distinct critical values or the set of minimum points is "plentiful".

As an application of our result we prove a theorem on the existence of two nontrivial solutions for a Hammerstein integral equation.

2. Definitions, preliminary results, and statement of the main theorem

Let $\mathbb{X}$ be a Banach space. In what follows we shall assume that φ is a C^1-functional which is bounded from below, satisfying the Palais–Smale condition ((PS)-condition): *any sequence* (x_n) *such that* $|\varphi(x_n)| < c$ *and* $\|\nabla\varphi(x_n)\| \to 0$ *as* $n \to \infty$ *has a convergent subsequence.* It is well known that, under these assumptions, φ is coercive and has a minimum on $\mathbb{X}$ (see e.g. [3]).

Let us denote by

$$m = \inf_{\mathbb{X}} \varphi$$

the minimum of φ over $\mathbb{X}$ and by

$$M = \{x \in \mathbb{X} \mid \varphi(x) = m\}$$

the set of minimum points of φ. Moreover, by

$$\varphi^c = \{x \in \mathbb{X} \mid \varphi(x) \le c\}$$

we denote the Lebesgue set of the functional φ for the value $c \in \mathbb{R} \cup \{\infty\}$, where it is assumed that $\varphi^\infty = \mathbb{X}$.

Let $A \subseteq B \subseteq \mathbb{X}$. A continuous map $h : [0,1] \times A \to B$ such that $h_0(x) = x$ for all $x \in A$ is said to be a *deformation* of A in B. The set A is *contractible* in B if there exists a deformation h_t of A in B such that $h_1(A) = \{p\}$, where p is a point in B. In the case $A = B$ we say that B is *contractible in itself* (*to a point*). The set A is a *deformation retract* of B if there is a deformation h_t of the set B in itself such that $h_1(B) \subseteq A$ and $h_1(x) = x$ for all $x \in A$. The set A is called a *strong deformation retract* of B, if A is a deformation retract and, moreover, $h_t(x) = x$ for all $x \in A$ and $t \in [0,1]$. It is well known that, if A is a (strong) deformation retract of B, then A and B have the same homotopy type. The converse does not hold in general (see, however, Lemma 2.2 below).

For C^1-functionals satisfying the (PS)-condition the following deformation lemma is well-known (for a proof see e.g. [6]).

Lemma 2.1. *If we assume that the interval* $[a,b] \subseteq \mathbb{R} \cup \{\infty\}$ *does not contain critical values of* φ, *then* φ^a *is a strong deformation retract of* φ^b.

We now indicate some elementary consequences of Lemma 2.1.

Proposition 2.1. *Let* $[a,b] \subseteq \mathbb{R} \cup \{\infty\}$ *and assume that the Lebesgue set* φ^a *is not a strong deformation retract of* φ^b. *Then* φ *has a critical value* $c \in [a,b]$.

Proposition 2.2. *Let $a \in \mathbb{R}$ be such that φ^a is not contractible in itself to a point. Then φ has a critical value $c \geq a$.*

In the sequel it will be useful for us to distinguish a particular class of critical values, obtained by Proposition 2.1 and 2.2.

Definition 2.1. A value $c \in \mathbb{R}$ of the functional φ is called an *essential critical value* if there exist arbitrarily small numbers $\varepsilon > 0$ such that the Lebesgue set $\varphi^{c-\varepsilon}$ is not a strong deformation retract of the Lebesgue set $\varphi^{c+\varepsilon}$.

Remark 2.1. From Lemma 2.1 it follows that an essential critical value of a C^1-functional φ satisfing the (PS)-condition is indeed a critical value of φ. The notion of essential critical values is in fact topological and can therefore be extended to continuous (not necessarily C^1) functionals. A related notion for continuous functionals in metric spaces was introduced and studied in [7].

A version of TCPT can be obtained as follows.

Theorem 2.1 (Three Critical Values Theorem, TCVT). *If φ has an essential critical value $c > m$, then either φ admits at least three distinct critical values, or the set of minimum points M is not contractible in itself. In particular, φ has at least three critical points.*

In order to prove Theorem 2.1 we need some more information about the structure of the Lebesgue sets of the functional φ. We recall that a metric space C is an *absolute neighbourhood retract* (ANR) if for any closed subset $A \subseteq B$ of a metric space B we have that any continuous map $f : A \to C$ has a continuous extension over some neighbourhood $\mathcal{U}_A$ of A in B. Any Banach manifold (with boundary) and, in particular, any Lebesgue set corresponding to a regular value of a C^1-functional φ is an ANR (see e.g. [14]).

Very important for us is the fact that, for ANR's, the notions of deformation retract and strong deformation retract coincide. In fact, the following stronger assertion holds (see [16]).

Lemma 2.2. *Let B be an* ANR*, $A \subseteq B$ a closed subset of B such that A is also an* ANR*. Then A is a strong deformation retract of B if and only if the inclusion $i : A \hookrightarrow B$ is a homotopy equivalence.*

With the aid of Lemma 2.2 one gets some useful information about the homotopy type of the Lebesgue sets of the functional φ in a neighbourhood of an essential critical value.

Lemma 2.3. *Let $c \in \mathbb{R}$ be an isolated and essential critical value of φ. Then there exist arbitrarily small numbers $\varepsilon > 0$ such that at least one of the two Lebesgue sets $\varphi^{c-\varepsilon}$ and $\varphi^{c+\varepsilon}$ is not contractible in itself.*

Proof. We fix $\varepsilon > 0$ in such a way that the interval $[c - \varepsilon, c + \varepsilon]$ does not contain critical values of φ different from c, and the Lebesgue set $\varphi^{c-\varepsilon}$ is not a strong deformation retract of the Lebesgue set $\varphi^{c+\varepsilon}$. Since $c + \varepsilon$ and $c - \varepsilon$ are regular values of φ, the Lebesgue sets $\varphi^{c-\varepsilon}$ and $\varphi^{c+\varepsilon}$ are closed ANR's and, by Lemma 2.2, we have that the inclusion $i : \varphi^{c-\varepsilon} \hookrightarrow \varphi^{c+\varepsilon}$ is not a homotopy equivalence.

We assume now that both $\varphi^{c-\varepsilon}$ and $\varphi^{c+\varepsilon}$ are contractible in itself. In this case any continuous map $f : \varphi^{c-\varepsilon} \to \varphi^{c+\varepsilon}$, and, in particular, the inclusion map is a homotopy equivalence [16], a contradiction. □

We recall now a classical result from Lusternik–Schnirel'man Theory (see e.g. [15]). In what follows $\mathrm{Cat}_B(A)$ stands for the category of a set $A \subseteq B$ in B.

Lemma 2.4. *Let B be an* ANR *and $A \subseteq B$. Then there exist a neighbourhood $\mathcal{U}_A$ of A in B such that* $\mathrm{Cat}_B(\mathcal{U}_A) = \mathrm{Cat}_B(A)$.

In order to prove the next lemma we use a particular case of Lemma 2.4. Namely let us assume, under the assumptions of Lemma 2.4, that set A is contractible in itself. Then $\mathrm{Cat}_B(A) = 1$ and, since subsets of B of category one are contractible in B, there exists a neighbourhood $\mathcal{U}_A$ of A which is contractible in B.

Lemma 2.5. *Let m be an isolated critical value of φ and let M be contractible in itself. Then there exist arbitrary small numbers $\varepsilon > 0$ such that the Lebesgue set $\varphi^{m+\varepsilon}$ is contractible in itself.*

Proof. We fix $\varepsilon > 0$ in such a way that the interval $[m, m + \varepsilon]$ does not contain critical values of φ different from m. This implies that $\varphi^{m+\varepsilon}$ is an ANR, and, since $M \subseteq \varphi^{m+\varepsilon}$ is contractible in itself, by Lemma 2.4 there exists a neighbourhood $\mathcal{U}_M$ of M in $\varphi^{m+\varepsilon}$ which is contractible in $\varphi^{m+\varepsilon}$.

Let us show now that there exists a $\delta \in (0, \varepsilon)$ such that

$$M \subseteq \varphi^{m+\delta} \subseteq \mathcal{U}_M \subseteq \varphi^{m+\varepsilon}.$$

In fact, if this is not the case, there exists a minimizing sequence $(x_n) \subseteq \varphi^{m+\delta} \setminus \mathcal{U}_M$. Now, since φ is a C^1-functional satisfying the (PS)-condition it follows that (x_n) has a subsequence converging to some $x_0 \in M$ (see e.g. [3]). Obviously, $x_0 \notin \mathcal{U}_M$. This contradicts the fact that $M \subseteq \mathcal{U}_M$.

Finally, by Lemma 2.1, $\varphi^{m+\delta} \subseteq \mathcal{U}_M$ is a strong deformation retract of $\varphi^{m+\varepsilon}$ and $\mathcal{U}_M$ is contractible in $\varphi^{m+\varepsilon}$; from this it follows immediately that $\varphi^{m+\varepsilon}$ is contractible in itself. □

3. Proof of the main result (TCVT)

We start by assuming that m and c are isolated critical values of φ, for otherwise the theorem is proved. Since $c > m$ is an essential critical value of φ, by Lemma 2.3 there exist arbitrarily small numbers $\varepsilon > 0$ such that at least one of the two Lebesgue sets $\varphi^{c+\varepsilon}$ and $\varphi^{c-\varepsilon}$ is not contractible in itself.

Let us assume first, that $\varphi^{c+\varepsilon}$ is not contractible in itself. From Proposition 2.2 it follows that φ has a third critical value $c_1 > c$.

Now we assume that $\varphi^{c-\varepsilon}$ is not contractible in itself and that the set of minimum point M is contractible in itself. Then, by Lemma 2.5, there exist sufficiently small numbers $\eta > 0$ such that the Lebesgue set $\varphi^{m+\eta}$ is contractible in itself. On the other hand, $\varphi^{m+\eta}$ cannot be a strong deformation retract of the noncontractible set $\varphi^{c-\varepsilon}$. Thus, by Proposition 2.1, functional φ has a third critical value $c_1 \in (m, c)$.

The TCPT as stated is perhaps too general and not very handy to be used in the study of nonlinear problems. It is clear, however, that any statement implying the existence of an essential critical value $c > m$ of a functional φ will give a corresponding variant of TCPT. In order to illustrate this point let us give the following two results.

Proposition 3.1. *Let x_0 be a strict local minimum of the functional φ. Then $c = \varphi(x_0)$ is an essential critical value of φ.*

Proof. Let B_ϱ (S_ϱ) be the ball (sphere) in the space $\mathbb{X}$ with the center at x_0 and radius $\varrho > 0$. Since φ is a C^1-functional satisfying the (PS)-condition and x_0 is a strict local minimum of φ, it follows (see e.g. [3]) that for each sufficiently small $\varrho > 0$

$$\inf_{S_\varrho} \varphi(x) = c_\varrho > c, \quad \inf_{B_\varrho} \varphi(x) = c.$$

Obviously, c_ϱ tends to c as ϱ tends to zero. We fix $\varepsilon = \varepsilon_\varrho \in (0, c_\varrho - c)$. Then the component of the Lebesgue set $\varphi^{c+\varepsilon}$ which contains the point x_0 does not meet the Lebesgue set $\varphi^{c-\varepsilon}$. Hence, $\varphi^{c-\varepsilon}$ is not a strong deformation retract of $\varphi^{c+\varepsilon}$. Since ε can be chosen arbitrarily small, this means that $c = \varphi(x_0)$ is an essential critical value of φ. □

Proposition 3.2. *Let $\mathbb{X} = \mathbb{V}^- \dot{+} \mathbb{V}^+$, where $0 < \dim \mathbb{V}^- < \infty$ and assume that there exist an arbitrarily small $\varrho > 0$ such that*

$$\inf_{||x||=\varrho} \varphi(x) > 0 \quad (x \in \mathbb{V}^+), \tag{1}$$

$$\sup_{||x||=\varrho} \varphi(x) < 0 \quad (x \in \mathbb{V}^-). \tag{2}$$

Then $c = 0$ is an essential critical value of φ.

Proof. First, we note that $\varphi(0) = 0$ and, moreover, 0 is a critical point of φ under conditions (1) and (2).

Let B_ϱ^- (S_ϱ^-) be the ball (sphere) in the subspace $\mathbb{V}^-$ with zero center and radius $\varrho > 0$, and B_ϱ^+ (S_ϱ^+) be the ball (sphere) in the subspace $\mathbb{V}^+$ with zero center and radius $\varrho > 0$. We set

$$c_\varrho^- = \sup_{S_\varrho^-} \varphi(x), \qquad c_\varrho^+ = \inf_{S_\varrho^+} \varphi(x).$$

Obviously, c_ϱ^- and c_ϱ^+ tend to zero as ϱ tends to zero. We fix $\varrho > 0$ and $\varepsilon = \varepsilon_\varrho \in (0, \min\{c_\varrho^-, c_\varrho^+\})$ such that (1) and (2) hold. Let $h : [0,1] \times \varphi^\varepsilon \to \varphi^\varepsilon$ be a deformation for which

$$h_t(x) = x \quad (x \in \varphi^{-\varepsilon},\ t \in [0,1]).$$

Consider the continuous mapping $\eta : [0,1] \times B_\varrho^- \to \mathbb{R} \times \mathbb{V}^-$ defined by means of the formula

$$\eta(t,x) = (||Qh_t(x)||, Ph_t(x)),$$

where P and Q are orthoprojectors on the subspaces $\mathbb{V}^-$ and $\mathbb{V}^+$, correspondingly. (This construction is borrowed from the preprint version of [13]). Then $\eta(0,x) = (0,x)$ for $x \in B_\varrho^-$, and $\eta(t,x) = (0,x)$ for $x \in S_\varrho^-$ and $t \in [0,1]$. Moreover, by virtue of (1) and the choice of ε, the relation $(\varrho, 0) \notin \operatorname{Im} \eta$ is true.

Now we show that there exist $\sigma \in [0, \varrho]$ and $y \in B_\varrho^-$ such that $\eta(1,y) = (\sigma, 0)$. In fact, the homotopy $H : [0,1] \times ([-1,1] \times B_\varrho^-) \to \mathbb{R} \times \mathbb{V}^-$ defined by

$$H_t(s,x) = \eta(t,x) + (s,0),$$

satisfies the conditions

$$\begin{aligned} &H_0(s,x) = (s,x) \quad ((s,x) \in [-1,1] \times B_\varrho^-), \\ &(0,0) \notin H_t(\partial([-1,1] \times B_\varrho^-)) \quad (t \in [0,1]). \end{aligned}$$

Thus, $\deg(H_1(\,\cdot\,,\,\cdot\,), [-1,1] \times B_\varrho^-, (0,0)) = 1$, and therefore $\eta(1,y) = (\sigma, 0)$ for some suitable $y \in B_\varrho^-$. But in this case $h_1(y) \notin \varphi^{-\varepsilon}$ and this means that $\varphi^{-\varepsilon}$ is not a strong deformation retract of φ^ε and, since $\varepsilon = \varepsilon_\varrho$ can be chosen arbitrarily small, $c = 0$ is an essential critical value of φ. □

Remark 3.1. Under the hypotheses of Proposition 3.2, for the point zero, considered as a critical point of the vector field $\nabla\varphi$, we may have $\operatorname{ind}(\nabla\varphi, 0) = 0$. Related examples can be given already in $\mathbb{R}^3$.

Remark 3.2. More general principles on the existence of essential critical values can be obtained with the aid of different kinds of linking conditions (see e.g. [3], [12], [13]).

4. Nontrivial solutions of Hammerstein equations

We apply the results above to the problem of the existence of nontrivial solutions to a Hammerstein nonlinear integral equation

$$x(t) = \int_\Omega k(t,s)\, f(s, x(s))\, ds, \tag{3}$$

where $\Omega \subseteq \mathbb{R}^m$ is a bounded domain, $k(t,s) : \Omega \times \Omega \to \mathbb{R}$ is a measurable, symmetric kernel and the function $f(s,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory conditions.

Equation (3) can be rewritten in the operator form $x = KFx$, where F is the nonlinear superposition operator generated by the Caratheodory function $f(s,u)$, i.e.

$$Fx(s) = f(s, x(s)),$$

and K is the linear integral operator

$$Kx(t) = \int_\Omega k(t,s)x(s)\,ds,$$

generated by the kernel $k(t,s)$.

In what follows we assume that K is a selfadjoint positive definite and compact operator acting in L_2. In particular, the spectrum $\sigma(K)$ of K consists of a countable set of positive characteristic eigenvalues $\lambda_1 > \lambda_2 > \dots$ of finite multiplicity, and zero is the only point of accumulation.

Let $\langle\cdot\,,\,\cdot\rangle$ and $||\cdot|| = \langle\cdot\,,\,\cdot\rangle^{1/2}$ stand for the scalar product and norm in L_2, respectively, and let $\varphi : L_2 \to \mathbb{R}$ be the Golomb functional generated by K and F, i.e.,

$$\varphi(h) = \frac{1}{2}\langle h, h\rangle - \int_\Omega \phi(s, Hh(s))\,ds,$$

where $H = K^{1/2}$ is the square root of K and

$$\phi(s,u) = \int_0^u f(s,v)\,dv,$$

is the potential generated by $f(s,u)$.

Let us assume that for some $p \in [2,\infty)$ and $p' = p/(p-1)$, the operator K acts and is compact as an operator from $L_{p'}$ into L_p. Furthermore, we suppose that operator F acts from L_p into $L_{p'}$; the latter means that function $f(s,u)$ satisfies an inequality

$$|f(s,u)| \le m(s) + b|u|^{p-1},$$

with $m(s) \in L_p$ and $b < \infty$.

Under these assumptions (see [9], [18]), the Golomb functional φ is C^1-smooth in L_2, and each critical point h^* of φ defines a solution $x^* = Hh^*$ to the Hammerstein equation (3).

In the case when the nonlinearity $\phi(s,u)$ satisfies the following one-sided growth conditions

$$\phi(s,u) \le n(s) + \frac{1}{2}au^2,$$

with $n(s) \in L_1$ and $a < ||K||^{-1}$, the Golomb functional is lower bouded and satisfies the (PS)-condition. Therefore, φ attains its minimum in L_2 so that equation (3) has at least one solution in L_p (see e.g. [9], [18]).

Now assume that function $f(s,u)$ may be represented in a form

$$f(s,u) = a(s)u + w(s,u), \tag{4}$$

where $a(s) \in L_{p/(p-2)}$ and $w(s,u)$ satisfies the following special condition: for each $\varepsilon > 0$ there exist $a_\varepsilon(s) \in L_{p/(p-2)}$ and b_ε such that $||a_\varepsilon||_{L_{p/(p-2)}} < \varepsilon$ and

$$|w(s,u)| \le a_\varepsilon(s)|u| + b_\varepsilon |u|^{p-1},$$

the latter guarantees that the Golomb functional φ is twice differentiable at zero and

$$\nabla^2\varphi(0) = I - \widetilde{H}AH, \tag{5}$$

where $\widetilde{H}$ is the natural extension of H on $L_{p'}$ and

$$Ax(s) = a(s)x(s).$$

It should be pointed out that the spectrum $\sigma(\widetilde{H}AH)$ of the operator $\widetilde{H}AH$ in L_2 coincides with the spectrum $\sigma(KA)$ of the operator KA in L_p.

In the case when the spectrum of the operator KA does not belong to the interval $(-\infty, 1]$ zero is a critical point of the Golomb functional, but it is not a minimum point. Hence, equation (3) has at least one nontrivial solution (see [19]).

Under some additional assumptions on the nonlinearity the TCVT gives the existence of at least two nontrivial solutions of the Hammerstein equation (3).

We set

$$\omega(s,u) = \int_0^u w(s,t)\,dt. \tag{6}$$

Theorem 4.1. *Let us assume that $\sigma(KA) \cap [1,\infty) \ne \emptyset$, and that one of the following three conditions holds:*

(a) $1 \notin \sigma(KA)$.

(b) $1 \in \sigma(KA)$ *and the condition*

$$\omega(s,u) \ge a|u|^p \quad (|u| \le u(s)), \qquad -\omega(s,u) \le L|u|^p \quad (|u| > u(s)), \tag{7}$$

holds for some $a > 0$, $L < \infty$, and some measurable positive function $u(s)$.

(c) $1 \in \sigma(KA)$, *eigenfunctions of KA corresponding to eigenvalues from $[1,\infty)$ belong to $L_{\widetilde{p}}$, and the condition*

$$\omega(s,u) \ge a|u|^{\widetilde{p}} \quad (|u| \le u(s)), \qquad -\omega(s,u) \le L|u|^{\widetilde{p}} \quad (|u| > u(s)), \tag{8}$$

holds for some $a > 0$, $p < \widetilde{p} < \infty$, and some measurable positive function $u(s)$.

Then the Hammerstein equation (3) *has at least two nontrivial solutions which belong to* L_p.

Proof. We will show that $c = 0$ is an essential critical value of the Golomb functional φ in the Hilbert space $\mathbb{X} = L_2$ and that $\inf_{\mathbb{X}} \varphi < 0$.

As a matter of fact, we will actually show that the conditions of Proposition 3.2 are fulfilled. Indeed, let us rewrite φ in the form

$$\varphi(h) = \frac{1}{2}\,\langle (I - \widetilde{H}AH)h, h\rangle \;-\; \psi(h),$$

where

$$\psi(h) = \int_\Omega \omega(s, Hh(s))\,ds.$$

We notice that $\psi(h) = o(||h||^2)$ as $||h|| \to 0$ and the Hessian of φ at zero is given by $\nabla^2\varphi(0) = I - \widetilde{H}AH$. Under the assumptions of Proposition 3.2 we have $\mathbb{X} = \mathbb{V}^- \oplus \mathbb{V}^+$, where $\mathbb{V}^+$ is the positive invariant subspace of $I - \widetilde{H}AH$ and $\mathbb{V}^-$ is its orthogonal complement. With this choice of $\mathbb{V}^+$, φ satisfies condition (1) of Proposition 3.2. It only remains to show that also condition (2) of Proposition 3.2 is satisfied as well. Also $0 < \dim \mathbb{V}^- < \infty$ holds.

First, we assume that (a) holds. Then (2) is automatically satisfied since $1 \notin \sigma(KA)$ implies that operator $I - \widetilde{H}AH$ is invertible and therefore $\mathbb{V}^-$ is the negative invariant subspace of $I - \widetilde{H}AH$.

Now we assume that (b) holds. In this case the Hessian $\nabla^2\varphi(0) = I - \widetilde{H}AH$ has a nontrivial kernel and $\mathbb{V}^-$ is the direct sum of the invariant negative subspace and the kernel itself. It is not hard to see that condition (2) holds if we show, under assumption (7), the existence of $\varrho > 0$ such that $\psi(h) > 0$ for any $0 < ||h|| \le \varrho$, $h \in \mathbb{V}^-$.

By virtue of (7)

$$\begin{aligned}
\int_\Omega \omega(s, Hh(s))\,ds &= \int_{\Omega\setminus D(h)} \omega(s, Hh(s))\,ds + \int_{D(h)} \omega(s, Hh(s))\,ds \\
&\ge a \int_{\Omega\setminus D(h)} |Hh(s)|^p\,ds - L \int_{D(h)} |Hh(s)|^p\,ds \\
&= a \int_\Omega |Hh(s)|^p\,ds - (a+L) \int_{D(h)} |Hh(s)|^p\,ds \\
&= a||Hh(s)|L_p||^p - (a+L) \int_{D(h)} |Hh(s)|^p\,ds,
\end{aligned}$$

where

$$D(h) = \{s \in \Omega : \, |Hh(s)| > u(s)\}.$$

Taking into account that there exists a positive number ν such that

$$||Hh(s)|L_p|| \ge \nu||h|| \quad (h \in \mathbb{V}^-),$$

and, furthermore, there exists a function $e(s) \in L_p$ such that

$$|Hh(s)| \leq ||h|| e(s) \quad (h \in \mathbb{V}^-),$$

we obtain

$$\psi(h) = \int_\Omega \omega(s, Hh(s))\, ds \geq \bigg(a\nu^p - (a+L) \int_{T(||h||)} e(s)^p\, ds\bigg) ||h||^p, \tag{9}$$

where

$$T(r) = \{s \in \Omega : re(s) > u(s)\}.$$

It is evident that $\operatorname{mes} T(r) \to 0$ as $r \to 0$. Thus, this relation and the absolute continuity property of the Lebesgue integral imply that

$$\lim_{r \to 0} \int_{T(r)} e(s)^p\, ds = 0.$$

Hence, there exists a $\varrho > 0$ such that the inequality (9) implies the inequality $\psi(h) > 0$ ($||h|| \leq \varrho$) and, therefore,

$$\varphi(h) = \frac{1}{2} \langle (I - \widetilde{H}AH)h, h \rangle \; - \; \psi(h) < 0 \quad (||h|| \leq \varrho, \; h \in \mathbb{V}^-).$$

In other words, condition (2) of Proposition 3.2 is satisfied.

The case when condition (c) holds is proved in a similar way; it is sufficient to remark that H in this case acts from L_2 into $L_{\widetilde{p}}$. □

Finally, let us remark that Theorem 4.1 can be generalized to Hammerstein equations with kernels which have a finite number of negative eigenvalues and also to systems of Hammerstein equations on ideal spaces of vector-functions (see e.g. [2]).

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Manuscript received February 10, 1997

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