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MULTIPLICITY OF BIFURCATION POINTS FOR VARIATIONAL INEQUALITIES VIA CONLEY INDEX

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1. Introduction

The present work deals with nonlinear eigenvalue problems of the following type:

$$
(1.1)\qquad \begin{cases} a(u,v-u)+\langle P'(u),v-u\rangle \geq \lambda b(u,v-u) \quad \forall v\in\mathbb{K},\\ (u,\lambda)\in\mathbb{K}\times\mathbb{R}, \end{cases}
$$

where a,b are bilinear symmetric, $P'(0)=P''(0)=0$, and $\mathbb{K}$ is a closed convex set containing 0. In particular, we search for the λ's such that $(0,\lambda)$ accumulates solutions (u_n,λ_n) with $u_n\neq 0$. It is known that such λ's are eigenvalues of the 0-asymptotic problem, namely there exists $u\neq 0$ such that (u,λ) solves

$$
(1.2)\qquad \begin{cases} a(u,v-u)\geq \lambda b(u,v-u) \quad \forall v\in\mathbb{K}_0,\\ (u,\lambda)\in\mathbb{K}_0\times\mathbb{R}, \end{cases}
$$

where $\mathbb{K}_0=\overline{\bigcup_{t>0} t\mathbb{K}}$ is a closed convex cone. The typical problem one has to face is twofold: (1) find eigenvalues (which is nontrivial, unless $\mathbb{K}_0$ is a linear space): (2) ensure that some eigenvalues are bifurcation points (which is not always true, as counterexamples show). Much work has been done in this context; see [11], [15]–[18], [21]–[30], [32]–[34] and the references therein for a more complete picture of the situation.

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In this paper some conditions are found which provide the existence of two eigenvalues at which bifurcation occurs; these eigenvalues may coincide, giving rise in this case to two "bifurcation branches". Such conditions consist in a double "linking behaviour", in two consecutive dimensions: for the precise statement see Theorem 4.7. To prove this result we use a nonsmooth variational framework which has been very fruitful in treating problems of this kind (see [4], [7], [8], [11]–[13], [19], [20]). For finding eigenvalues we look for lower critical points of a suitable (natural) nonsmooth function; the problem of bifurcation is reduced to proving some stability under perturbations of such points. In this particular case we use the Conley index for the evolution flows associated with the function, regarding the critical points as invariant sets (the rest points) in the flow. So if we find some region with nontrivial index (due to the variational nature of the flow) such a region must contain a critical point; furthermore, bifurcation is related to the continuation property of the Conley index, in a continuous family of flows. For this latter aspect we have to check that the family of flows we are involved in actually satisfy the conditions for having the continuation property: this is done, for a general class of nonsmooth functions, in Section 1.

In Section 5 we show an application of the previous theorems to a problem of eigenvalues for a semilinear elliptic variational inequality.

Finally, we wish to point out that, due to the fact that we cannot distinguish the two critical points by the value of the function (that is, by the eigenvalue) the use of a tool like the Conley index (or something like the Morse index, in a nonsmooth setting) seems to be necessary.

2. Conley index for flows associated with $\mathcal{C}(p,q)$-functions

Throughout this section X will denote an open subset of a Hilbert space L with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ and norm $\|\cdot\|$, P will be a metric space (of parameters). We use the notation $B_L(u,R)$ and $B_P(\varrho,\delta)$ to denote the open balls in L and P respectively. Moreover, we consider a family $(f_\varrho)_{\varrho\in P}$ of lower semicontinuous functionals, $f_\varrho : X \to \mathbb{R}\cup\{\infty\}$ for all ϱ in P. We denote by $\mathcal{D}(f_\varrho)$ the domain $\{u \mid f_\varrho(u) < \infty\}$ and by $\partial^- f_\varrho(u)$ the (Fréchet) subdifferential of f_ϱ at a point u such that $u \in \mathcal{D}(f_\varrho)$ (see e.g. [13]).

DEFINITION 2.1. We say that $(f_\varrho)_\varrho$ are *equi*-$\mathcal{C}(p,q)$ if there exist two continuous functions $p,q : \mathcal{D} \to \mathbb{R}$, where $\mathcal{D} = \{(\varrho,u) \mid \varrho\in P,\ u\in\mathcal{D}(f_\varrho)\}$, such that

$$f_\varrho(v) \geq f_\varrho(u) + \langle\alpha, v-u\rangle - (p(\varrho,u)\|\alpha\| + q(\varrho,u))\|v-u\|^2 \qquad \forall\varrho\in P,\ \forall u,v\in\mathcal{D}(f_\varrho),\ \forall\alpha\in\partial^- f_\varrho(u). \tag{2.1}$$

Definition 2.2. We say that $(f_\varrho)_{\varrho\in P}$ is Γ-*continuous* if for all sequences (ϱ_n) converging in P to a point ϱ we have

$$f_\varrho = \Gamma^-(L)\text{-}\lim_{n\to\infty} f_{\varrho_n} \tag{2.2}$$

(for the notion of Γ-convergence, or variational convergence, of a sequence of functions we refer the reader to [1], [9], [10]).

Definition 2.3. We say that f_ϱ are *equi-locally-coercive* if for every sequence (ϱ_n) converging in P to a point ϱ and for every sequence (u_n) in X such that u_n and $f_{\varrho_n}(u_n)$ are bounded, there exists a subsequence (u_{n_k}) which converges to a point u in X.

The following two lemmas can be proved, with easy adaptations, as the corresponding results in Theorem 4.9 of [13] (see also [27]).

Lemma 2.4. *Assume $(f_\varrho)_\varrho$ to be equi-$\mathcal{C}(p,q)$ and Γ-continuous. Let $\varrho_n \to \varrho$ in P, $u_n \to u$ in X, $\alpha_n \overset{L}{\rightharpoonup} \alpha$ (weakly in L), $f_{\varrho_n}(u_n)$ be bounded above and $\alpha_n \in \partial^- f_{\varrho_n}(u_n)$ for all n. Then*

$$f_{\varrho_n}(u_n) \to f_\varrho(u), \quad \alpha \in \partial^- f_\varrho(u).$$

Lemma 2.5. *Assume $(f_\varrho)_\varrho$ to be equi-$\mathcal{C}(p,q)$, Γ-continuous and equi-locally-coercive. Then for every (ϱ,u) in $\mathcal{D}$ there exist $T(\varrho,u)>0$ and a unique $\mathcal{U}_{\varrho,u} : [0,T(\varrho,u)[\, \to X$ absolutely continuous such that*

$$\begin{cases} \mathcal{U}_{\varrho,u}(0) = u, \\ t \mapsto f_\varrho(\mathcal{U}_{\varrho,u}(t)) \text{ is nonincreasing in } [0,T(0,\varrho)[, \\ -\mathcal{U}_{\varrho,u}(t) \in \partial^- f_\varrho(\mathcal{U}_{\varrho,u}(t)) \text{ for almost all } t \in [0,T(0,\varrho)[. \end{cases}$$

Furthermore, if $\varrho_n \to \varrho$ in P, $u_n \to u$ in X, $f_{\varrho_n}(u_n)$ is bounded and $t_n \to t$ in $\mathbb{R}$ with $t < T(\varrho,u)$, then eventually $t_n < T(\varrho_n,u_n)$ and $\mathcal{U}_{\varrho_n,u_n}(t_n) \to \mathcal{U}_{\varrho,u}(t)$ in X. If in addition $f_{\varrho_n}(u_n) \to f_\varrho(u)$, then also $f_{\varrho_n}(\mathcal{U}_{\varrho_n,u_n}(t_n)) \to f_\varrho(\mathcal{U}_{\varrho,u}(t))$.

Remark 2.6. Let c be a real number and set, for ϱ in P

$$\begin{aligned} &X_\varrho^{(c)} = \{u \in X \mid f_\varrho(u) \le c\}, \\ &\omega_\varrho(u) = \sup\{T>0 \mid \mathcal{U}_{\varrho,u} \text{ exists on } [0,T[\}, \\ &\Phi_\varrho(u,t) = \mathcal{U}_{\varrho,u}(t). \end{aligned} \tag{2.3}$$

Then $(X_\varrho^{(c)}, \omega_\varrho, \Phi_\varrho)$ is a continuous family of local unilateral flows, as defined in [26], [27]. Moreover (due to the equi-local-coerciveness), the compactness assumption (N.2) of [27] is fulfilled for every bounded set N.

Now we wish to check the validity of assumption (C) of [27], in some sense a continuity property of the "moving" domains $X_\varrho^{(c)}$. To find some general condition for having property (C) we start with a definition (cf. [5]).

Definition 2.7. Let V be a subset of L. We say that V is *p-convex* if there exists a a continuous function $p : V \to \mathbb{R}$ such that

$$\langle \nu, v-u\rangle \le p(u)\|\nu\|\cdot\|v-u\|^2 \quad \forall u,v\in V,\ \forall \nu \in N_u(V),$$

where $N_u(V)$ is the normal cone to V at u, defined as $N_u(V) = \partial^- I_V(u)$ and I_V is the indicator function of V which is zero on V and ∞ outside V.

If $(V_\varrho)_{\varrho\in P}$ is a family of subsets of L we say that $(V_\varrho)_\varrho$ are *equi-p-convex* if there exists $p : \mathcal{V} \to \mathbb{R}$ continuous, where $\mathcal{V} = \{(\varrho, v) \mid \varrho \in P,\ v \in V_\varrho\}$, such that

$$\langle \nu, v-u\rangle \le p(\varrho,u)\|\nu\|\cdot\|v-u\|^2 \quad \forall \varrho\in P,\ \forall u,v\in V_\varrho,\ \forall \nu \in N_u(V_\varrho). \tag{2.4}$$

Lemma 2.8. *Let $(V_\varrho)_\varrho$ be equi-p-convex and assume that*

$$\begin{gathered}\text{for every } (\varrho_n)_n \text{ converging to } \varrho \text{ in } P \text{ and every bounded set } B,\\ B\cap V_{\varrho_n} \to B\cap V_\varrho \quad \text{in the Hausdorff metric.}\end{gathered} \tag{2.5}$$

Then

(a) *the function $d : P\times L \to \mathbb{R}$ defined by $d(\varrho,u) = \operatorname{dist}(u, V_\varrho)$ is continuous; furthermore, if $\varrho_n \to \varrho$, then $d(\,\cdot\,,\varrho_n) \to d(\,\cdot\,,\varrho)$ uniformly on every bounded set;*

(b) *given ϱ_0 in P and a compact subset S of V_{ϱ_0}, there exist $\delta, R > 0$ such that for all $\varrho \in B_P(\varrho_0,\delta)$ and $u \in B_L(S,R)$ there exists a unique $\pi_\varrho(u)$ in V_ϱ such that*

$$\|\pi_\varrho(u) - u\| \le \|v-u\| \quad \forall v \in V_\varrho$$

(the projection of minimal distance on V_ϱ). Furthermore, δ, R can be chosen in such a way that

$$\|\pi_\varrho(u') - \pi_\varrho(u'')\| \le 2\|u'-u''\|, \quad \|\pi_\varrho(u) - u\| \le 2R$$

for $\varrho \in B_P(\varrho_o,\delta)$ and $u,u',u'' \in B_L(S,R)$. In particular, $(\varrho,u) \mapsto \pi_\varrho(u)$ is continuous.

Proof. To prove (a) let ϱ',ϱ'' be in P, u',u'' in L, B a bounded set in L and denote by h the Hausdorff distance:

$$h(A,B) = \max(h^*(A,B), h^*(B,A)), \quad h^*(A,B) = \max\{\operatorname{dist}_L(x,B) \mid x\in A\}.$$

For all $v' \in B\cap V_{\varrho'}$ and $v'' \in B\cap V_{\varrho''}$ we have

$$\begin{aligned}\operatorname{dist}_L(u',v') &\le \operatorname{dist}_L(u',u'') + \operatorname{dist}_L(u'',v'') + \operatorname{dist}_L(v'',v')\\ &\le \operatorname{dist}_L(u',u'') + \operatorname{dist}_L(u'',v'') + h(B\cap V_{\varrho'}, B\cap V_{\varrho''}),\end{aligned}$$

hence, taking inf's,

$$\operatorname{dist}_L(u', B \cap V_{\varrho'}) - \operatorname{dist}_L(u'', B \cap V_{\varrho''}) \le \operatorname{dist}_L(u', u'') + h(B \cap V_{\varrho'}, B \cap V_{\varrho''}),$$

which implies the conclusion, since we can exchange (u', ϱ') with (u'', ϱ'') and since $\operatorname{dist}_L(u', V_{\varrho'}) = \operatorname{dist}_L(u', B \cap V_{\varrho'})$ (and the same for u''), if B is a sufficiently large ball.

We prove (b). Since S is a compact set, we can find $R', \delta' > 0$ and $\overline{p} \ge 0$ such that for all (ϱ, u) in $B_P(\varrho_0, \delta') \times B_L(S, R')$ with $u \in V_\varrho$, we have $p(\sigma, u) \le \overline{p}$ so that

$$(2.6) \quad \langle \nu, v - u \rangle_L \le \overline{p} \|v - u\|_L^2 \|\nu\| \qquad \forall \varrho \in B_P(\varrho_0, \delta'),\ \forall u, v \in V_\varrho \cap B_L(S, R'),\ \forall \nu \in N_u(V_\varrho).$$

We can suppose $R' < \overline{p}/4$ and also $B_L(S, R') \subset X$, since X is open. Now take $R'' = R'/4$. We claim that there exists a $\delta'' \le \delta'$ such that for all ϱ in $B_P(\varrho_0, \delta'')$ and u in $B_L(S, \delta'')$,

$$\operatorname{dist}_L(u, V_\varrho) < \operatorname{dist}_L(u, X \setminus B_L(S, \delta')).$$

To see this just observe that, for u in $B_L(S, \delta'')$,

$$\operatorname{dist}_L(u, X \setminus B_L(S, \delta')) \ge 3R'/4,$$

and applying (a) one has $\operatorname{dist}_L(S, X_\varrho) \to 0$ as $\varrho \to \varrho_0$. So we can choose δ'' in such a way that $\operatorname{dist}_L(S, V_\varrho) \le R'/4$ for all ϱ in $B_L(S, \delta'')$, and the assertion is proved. Finally, set $R = R''/4$; arguing as before we can find a $\delta > 0$ such that for all ϱ in $B_P(\varrho_o, \delta)$ and u in $B_L(S, R)$,

$$\operatorname{dist}_L(u, V_\sigma) < \operatorname{dist}_L(u, X \setminus B_L(S, R'')).$$

Now, applying Proposition 2.6 of [5], we see that, for all ϱ in $B_P(\varrho_0, \delta)$ and u in $B_P(S, \delta'')$, the projection $\pi_\varrho(u)$ of minimal distance on V_ϱ exists. Since, by construction,

$$\operatorname{dist}_L(u, V_\varrho) < \operatorname{dist}_L(u, X \setminus B_L(S, R')) < 2R'' < R' < \overline{p}/2,$$

it follows that $\pi_\sigma(u) \in B_L(S, R')$ and

$$\|\pi_\varrho(v) - \pi_\varrho(u)\| \le 2\|v - u\|$$

for all ϱ in $B_P(\varrho_0, \delta)$ and v, u in $B_L(S, R'')$ (see Proposition 2.9 of [5]). For the same reasons, if $u \in B_L(S, R)$, then $\pi_\varrho(u) \in B_L(S, R'')$, so also $\pi_\varrho(u)$ can be projected on $X_{\varrho'}$ for ϱ' in $B_P(\varrho_0, \delta)$.

Now we prove that the map $(\varrho, u) \mapsto \pi_\varrho(u)$ is continuous in $B_P(\varrho_0, \delta) \times B_L(S, R)$. For this let $(\varrho_n, u_n) \to (\varrho, u)$; we have

$$
\begin{aligned}
&\langle u_n - \pi_{\varrho_n}(u_n), \pi_\varrho(u) - \pi_{\varrho_n}(u_n)\rangle_L \\
&\quad = \langle u_n - \pi_{\varrho_n}(u_n), \pi_\varrho(u) - \pi_{\varrho_n}(\pi_\varrho(u))\rangle_L \\
&\qquad + \langle u_n - \pi_{\varrho_n}(u_n), \pi_{\varrho_n}(\pi_\varrho(u)) - \pi_{\varrho_n}(u_n)\rangle_L \\
&\quad \le \|u_n - \pi_{\varrho_n}(u_n)\|_L \|\pi_\varrho(u) - \pi_{\varrho_n}(\pi_\varrho(u))\|_L \\
&\qquad + \overline{p}\|u_n - \pi_{\varrho_n}(u_n)\|_L \|\pi_{\varrho_n}(\pi_\varrho(u)) - \pi_{\varrho_n}(u_n)\|_L^2 \\
&\quad \le 2R'' h(V_\varrho \cap B_L(S,R'), V_{\varrho_n} \cap B_L(S,R')) \\
&\qquad + 4R''\overline{p}(\|\pi_{\varrho_n}(\pi_\varrho(u)) - \pi_\varrho(u)\|_L^2 + \|\pi_\varrho(u) - \pi_{\varrho_n}(u_n)\|_L^2) \\
&\quad \le 2R'' h(V_\varrho \cap B_L(S,R'), V_{\varrho_n} \cap B_L(S,R')) \\
&\qquad + 4R''\overline{p}(h(X_\varrho \cap B_L(S,R'), X_{\varrho_n} \cap B_L(S,R')))^2 + \|\pi_\varrho(u) - \pi_{\varrho_n}(u_n)\|_L^2/4 \\
&\quad = o(n) + \|\pi_\varrho(u) - \pi_{\varrho_n}(u_n)\|^2/4
\end{aligned}
$$

where $o(n) \to 0$ as $n \to \infty$; we have used the fact that $u_n - \pi_{\varrho_n}(u_n) \in N_{\pi_{\varrho_n}(u_n)}(V_{\varrho_n})$ and (2.6). In the same way one proves that

$$\langle u - \pi_\varrho(u), \pi_{\varrho_n}(u_n) - \pi_\varrho(u)\rangle_L \le o'(n) + \|\pi_\varrho(u) - \pi_{\varrho_n}(u_n)\|_L^2/4$$

with $o'(n) \to 0$, which added to the first yields

$$
\begin{aligned}
\|\pi_\varrho(u) - \pi_{\varrho_n}(u_n)\|_L^2 &\le \langle u - u_n, \pi_\varrho(u) - \pi_{\varrho_n}(u_n)\rangle_L \\
&\qquad + \|\pi_\varrho(u) - \pi_{\varrho_n}(u_n)\|_L^2/2 + o''(n) \\
&\le \|u - u_n\|_L^2 + 3\|\pi_\varrho(u) - \pi_{\varrho_n}(u_n)\|_L^2/4 + o''(n)
\end{aligned}
$$

with $o''(n) \to 0$, which implies $\|\pi_\varrho(u) - \pi_{\varrho_n}(u_n)\|_L \to 0$. The rest of the assertion is straightforward.

Proposition 2.9. *If $(V_\varrho)_\varrho$ are equi-p-convex and satisfy the assumption* (2.5) (*local Hausdorff continuity*)*, then given $\varrho_0 \in P$ and a compact subset S of V_{ϱ_0}, there exist $\delta, R > 0$ and a continuous function $\Psi : B_P(\varrho_0,\delta) \times B_L(S,R) \times [0,1] \to X$ with the properties:*

(a) *$\Psi(\varrho,u,0) = u$ for all $\varrho \in B_P(\varrho_0,\delta)$ and $u \in B_L(S,R)$,*
(b) *$\Psi(\varrho,\Psi(\varrho,u,t),s) = \Psi(\varrho,u,t+s)$ for all $\varrho \in B_P(\varrho_0,\delta)$, $u \in B_L(S,R)$ and $t,s \in [0,1]$ such that $\Psi(u,t) \in B_L(S,R)$ and $t+s \in [0,1]$,*
(c) *$\Psi(\varrho,u,t) = u$ for all $\varrho \in B_P(\varrho_0,\delta)$, $u \in V_\varrho$ and $t \in [0,1]$,*
(d) *$\Psi(\varrho,u,1) \in V_\varrho$ for all $\varrho \in B_P(\varrho_0,\delta)$ and $u \in B_L(S,R)$.*

Proof. Let R,δ,π be as in Lemma 2.8; it suffices to define

$$\Psi(\varrho,u,t) = u + (2Rt \wedge \|\pi_\varrho(u) - u\|)\frac{\pi_\varrho(u) - u}{\|\pi_\varrho(u) - u\|}$$

and check the claimed properties.

PROPOSITION 2.10. *Let $(f_\varrho)_\varrho$ be equi-$\mathcal{C}(p,q)$, equi-locally-coercive and Γ-continuous. Moreover, let $\varrho \in P$ and $c \in \mathbb{R}$, and assume that*

$$\forall u \in \mathcal{D}(f_{\varrho_0}) \text{ with } f_{\varrho_0}(u) = c, \quad 0 \notin \partial^- f_{\varrho_0}(u)$$

(c is not a critical value for f_{ϱ_0}). Then:

(a) *for all $u_0 \in X$, $R > 0$ there exist $\varepsilon, \delta, \sigma > 0$ such that*

$$\forall \varrho \in B_P(\varrho_0, \delta), \forall u \in B_L(u_0, R) \text{ with } f_\varrho(u) \in [c-\varepsilon, c+\varepsilon], \forall \alpha \in \partial^- f_\varrho(u),$$
$$\|\alpha\|_L \geq \sigma;$$

(b) *if $\varrho_n \to \varrho_0$ then*

$$I_{X^{(c)}_{\varrho_0}} = \Gamma^-(X)\text{-}\lim_{n\to\infty} I_{X^{(c)}_{\varrho_n}};$$

this implies that (2.5) holds for $(X^{(c)}_\varrho)_\varrho$, since $(I_{X^{(c)}_\varrho})_\varrho$ is equi-locally-coercive;

(c) *for all $u_0 \in X$ and $R > 0$ there exists $\delta > 0$ such that the family of sets $(X^{(c)}_\varrho)_{\varrho \in B_P(\varrho_0,\delta)}$ are equi-p-convex in $B_L(u_0, R) \cap W$.*

PROOF. (a) By contradiction assume that the conclusion is false. Then we could find $u_0 \in X$, $R > 0$, a sequence $(\varrho_n)_n$ converging to ϱ_0, a sequence $(u_n)_n$ in $X \cap B_L(u_0, R)$ and a sequence $(\alpha_n)_n$ in L such that

$$f_{\varrho_n}(u_n) \to c, \quad \alpha_n \to 0.$$

By the coerciveness assumption we can suppose that $u_n \to u$ in W. Using Lemma 2.4 we get $f_{\varrho_0}(u) = c$ and $0 \in \partial^- f_{\varrho_0}(u)$, which contradicts the assumption.

(b) We have to prove two facts.

• If $u_n \to u$ in X then $\liminf_{n\to\infty} I_{X^{(c)}_{\varrho_n}} \geq I_{X^{(c)}_{\varrho_0}}$. But if the previous liminf is ∞, then the assertion is trivial; otherwise we have eventually $f_{\varrho_n}(u_n) \leq c$ and therefore $f_{\varrho_0}(u) \leq \liminf_{n\to\infty} f_{\varrho_n}(u_n) \leq c$, using the Γ-continuity of $(f_\varrho)_\varrho$.

• Given u in X we have to find a sequence $(u_n)_n$ converging to u with $I_{X^{(c)}_{\varrho_n}}(u_n) \to I^{(c)}_{\varrho_0}(u)$ (namely with $f_{\varrho_n}(u_n) \leq c$ for all n). Using the properties of $(f_\varrho)_\varrho$, we find $(u'_n)_n$ converging to u with $f_{\varrho_n}(u'_n) \to f_{\varrho_0}(u) = c$. Now let $\varepsilon, \delta, \sigma > 0$ be as in part (a), relative to $u_0 = u$ and $R = 1$ (for instance); we can suppose that $\varrho_n \in B_P(\varrho_0, \delta)$, $u'_n \in B_L(u, 1)$ and $f_{\varrho_n}(u'_n) \in [c-\varepsilon, c+\varepsilon]$ for all n. Denote by $\mathcal{U}_n$ the curve $\mathcal{U}_{\varrho_n, u'_n}$ as in Proposition 2.5 and set

$$\begin{aligned} t_n &= \sup\{t \in [0, \infty[\mid \mathcal{U}_n(\tau) \in B_L(u,1)\ \forall \tau \in [0,t],\ f_{\varrho_n}(\mathcal{U}_n(t)) \geq c\}, \\ u_n &= \mathcal{U}_n(t_n) \end{aligned}$$

(t_n is taken to be zero if the above set is empty). It is simple to check that $t_n \in [0, (f_{\varrho_n}(u'_n) - c) \vee 0)/\sigma]$, and that either $u_n \in \partial B_L(u, 1)$ or $f_{\varrho_n}(u_n) \leq c$.

Then $t_n \to 0$ so $u_n \to u$ and eventually only the latter situation occurs. This proves that $(u_n)_n$ has the required properties.

(c) Let $\varepsilon, \delta, \sigma$ be as in (a) and let $\varrho \in B_P(\varrho_0, \delta)$. We consider a sequence of functions $(\chi_n)_n$, where $\chi_n : \mathbb{R} \to [0, \infty]$, χ_n is increasing, convex, twice differentiable (hence finite) in $]-\infty, c+\varepsilon[$, $\chi_n(t) = 0$ for $t \leq 0$, $\chi(t) = \infty$ for $t \geq c+\varepsilon$ and $\chi_n(t) \leq \chi_{n+1}(t) \to \infty$ as $n \to \infty$ for $t > 0$. We set $g_n(u) = \chi_n(f_\varrho(u))$. It is easy to see that for all n:

- $\mathcal{D}(g_n) = \{u \mid f_\varrho(u) < c+\varepsilon\}$;
- for all u in $\mathcal{D}(g_n)$,

$$\beta \in \partial^- g_n(u) \Leftrightarrow \exists \alpha \in \partial^- f_\varrho(u) \text{ such that } \beta = \chi_n'(f_\varrho(u))\alpha;$$

- for all v in $\mathcal{D}(g_n)$, $u \in \mathcal{D}(g_n) \cap B_L(u_0, R)$ and $\beta \in \partial^- g_n(u)$,

$$\begin{aligned} g_n(v) &\geq g_n(u) + \langle \beta, v-u \rangle - (p(\varrho, u)\|\beta\| + \chi_n'(f_\varrho(u))q(\varrho, u))\|v-u\|^2 \\ &\geq g_n(u) + \langle \beta, v-u \rangle - (p(\varrho, u)\|\beta\| + q(\varrho, u)/\sigma)\|v-u\|^2. \end{aligned}$$

This shows that $(g_n)_n$ are equi-$\mathcal{C}(p,q)$ (since the above inequality is independent of n); they are also equi-locally-coercive, as can be easily deduced from the corresponding property of f_ϱ. Moreover, it is easy to see that g_n Γ-converge to $I_{X_\varrho^{(c)}}$. Then, by (2.4), we have

$$I_{X_\varrho^{(c)}}(v) \geq I_{X_\varrho^{(c)}}(u) + \langle \nu, v-u \rangle - \overline{p}(\varrho, u)\|\nu\|\|v-u\|^2$$

for all $u, v \in B_L(u_0, R) \cap X_\varrho^{(c)}$ and $\nu \in \partial^- I_{X_\varrho^{(c)}}(u)$, where $\overline{p}(\varrho, u) = p(\varrho, u) + q(\varrho, u)/\sigma$. This means that $(X_\varrho^{(c)})_\varrho$ are equi-p-convex.

3. Setting of the bifurcation problem

In this section, following [11], we recall the notion of bifurcation for a variational inequality and provide a nonsmooth variational setting for this problem. The main result is Theorem 3.10 that states that bifurcation occurs at eigenvalues with nontrivial Conley index.

Let H and L be two Hilbert spaces such that $H \subset L$ and the embedding of H into L is compact. We consider two symmetric bilinear forms $a : H \times H \to \mathbb{R}$ and $b : L \times L \to \mathbb{R}$ such that

$$|a(u,v)| \leq \|a\|_{H,H}\|u\|_H\|v\|_H \quad \forall u, v \in H, \tag{3.1}$$

$$|b(u,v)| \leq \|b\|_{L,L}\|u\|_L\|v\|_L \quad \forall u, v \in L, \tag{3.2}$$

$$a(u,u) \geq \nu\|u\|_H^2 - \mu|b(u,u)| \quad \forall u \in H, \tag{3.3}$$

where $\|a\|_{H,H}, \|b\|_{L,L}, \mu, \nu \in \mathbb{R}$ and $\nu > 0$. We write in the following $\alpha(u) = a(u,u)$, $\beta(u) = b(u,u)$ and for λ in $\mathbb{R}$, $a_\lambda(u,v) = a(u,v) - \lambda b(u,v)$, $\alpha_\lambda(u) =$

$a_\lambda(u,u)$. We set

$$\mathcal{S}^+ = \{u \in L \mid \beta(u) = 1\}, \quad \mathcal{S}^- = \{u \in L \mid \beta(u) = -1\}, \quad \mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^-.$$

Furthermore, we consider a differentiable map $P : L \to \mathbb{R}$ such that $P' : L \to L$ is Lipschitz continuous, P is twice differentiable at zero and

$$P(0) = 0, \quad P'(0) = 0, \quad P''(0) = 0. \tag{3.4}$$

Finally, we consider a convex set $\mathbb{K} \subset H$ closed in H such that $0 \in \mathbb{K}$, and set

$$\mathbb{K}_0 = H\text{-closure of } \bigcup_{\varrho>0} \varrho\mathbb{K},$$

which is a convex cone closed in H.

We are interested in finding solutions (u,λ) of the following variational inequality:

$$\begin{cases} (u,\lambda) \in \mathbb{K} \times \mathbb{R}, \quad \beta(u) \neq 0, \\ a(u, v-u) + \langle P'(u), v-u\rangle_L \geq \lambda b(u, v-u) \quad \forall v \in \mathbb{K}. \end{cases} \tag{3.5}$$

Definition 3.1. We say that λ is a *bifurcation point* for the variational inequality (3.5) if there exists a sequence $((u_n,\lambda_n))_{n\in\mathbb{N}}$ of solutions of (3.5) with $u_n \overset{H}{\to} 0$ and $\lambda_n \to \lambda$.

For the proof of the following result see Theorem 3.14 of [11].

Proposition 3.2. *If λ is a bifurcation point for* (3.5)*, then there exists u in H such that (u,λ) is a solution of*

$$\begin{cases} (u,\lambda) \in \mathbb{K}_0 \times \mathbb{R}, \quad u \in \mathcal{S}, \\ a(u, v-u) \geq \lambda b(u, v-u) \quad \forall v \in \mathbb{K}_0. \end{cases} \tag{3.6}$$

Definition 3.3. If $\lambda \in \mathbb{R}$ and there exists u such that (u,λ) satisfies (3.5), we say that λ is an *eigenvalue* of the variational inequality (3.6).

Proposition 3.2 states that any bifurcation point for (3.5) is an eigenvalue of (3.6). The converse not true is in general, as shown for instance in [22], [11]. The main result in this paper concerns the existence of eigenvalues of (3.6) which, due to some kind of essentiality, are bifurcation points for (3.5). For this we first associate the solutions of (3.5) and (3.6) with the critical points of suitable functionals, and then use the results of Section 2 to prove that critical points with nontrivial index (relative to the associated flow) correspond to bifurcation points.

Let $\varrho \in \,]0,1]$. We set

$$\mathbb{K}_\varrho = \{u \in H \mid \varrho u \in \mathbb{K}\}, \quad P_\varrho(u) = \frac{1}{\varrho^2} P(\varrho u) \quad \text{for } u \text{ in } L.$$

Remark 3.4. Assume that $u \in \mathbb{K}_0$ and $(\varrho_n)_n$ is a sequence converging to 0 in $\mathbb{R}$. Then there exists a sequence $(u_n)_n$ converging to u in H such that $u_n \in \mathbb{K}_{\varrho_n}$ for all n.

Proof. Using the definition of $\mathbb{K}_0$ one can find a sequence $(t_n)_n$ in $]0,\infty[$ and a sequence $(v_n)_n$ converging to u in H such that $v_n \in \mathbb{K}_{t_n}$ for all n. For all n we can take an integer k_n such that $\varrho_k \le t_n$ for all $k \ge k_n$. We set

$$u_k = v_n \quad \text{for } k_{n-1} \le k < k_n$$

(assuming $k_0 = 0$). It is easy to check that the conclusion holds for $(u_k)_k$.

Now we introduce the functionals $f_\varrho, f_\varrho^+, f_\varrho^- : L \to \mathbb{R} \cup \{\infty\}$ defined by

$$f_\varrho(u) = \begin{cases} \dfrac{1}{2}\alpha(u) + P_\varrho(u) & \text{if } u \in \mathbb{K}_\varrho, \\ \infty & \text{otherwise,} \end{cases} \qquad f_\varrho^+ = f_\varrho + I_{\mathcal{S}^+}, \quad f_\varrho^- = f_\varrho + I_{\mathcal{S}^-}$$

if $\varrho \in]0,1]$, and

$$f_0(u) = \begin{cases} \dfrac{1}{2}\alpha(u) & \text{if } u \in \mathbb{K}_0, \\ \infty & \text{otherwise,} \end{cases} \qquad f_0^+ = f_0 + I_{\mathcal{S}^+}, \quad f_0^- = f_0 + I_{\mathcal{S}^-}.$$

From now on we concentrate our attention on f_ϱ^+ (and find solutions of (3.5) with $\beta(u) > 0$). Everything can be repeated for f_ϱ^-, making obvious arrangements.

We recall the following definition (see e.g. [7], [11]):

Definition 3.5. Two sets V_1, V_2 in X are said to be (*externally*) *tangent* at a point u in $V_1 \cap V_2$ if there exists ν in L with $\nu \ne 0$, $\nu \in \partial^- I_{V_1}(u)$ and $-\nu \in \partial^- I_{V_2}(u)$.

Lemma 3.6. *Let $\varrho \in]0,1]$.*

(a) *If (u,λ) is a solution of*

$$\begin{cases} (u,\lambda) \in \mathbb{K}_\varrho \times \mathbb{R}, \quad u \in \mathcal{S}^+, \\ a(u, v-u) + \langle P_\varrho'(u), v-u\rangle_L \ge \lambda b(u, v-u) \quad \forall v \in \mathbb{K}_\varrho, \end{cases} \tag{3.7}$$

then u is a lower critical point for f_ϱ^+ and $\lambda = \alpha(u)$.

(b) *Conversely, if u in $\mathbb{K}_\varrho \cap \mathcal{S}^+$ is a lower critical point for f_ϱ^+ and if, in addition, $\mathbb{K}_\varrho$ and $\mathcal{S}^+$ are not tangent at u, then, setting $\lambda = \alpha(u)$, (u,λ) is a solution of (3.7).*

Proof. The proof is standard in this context (see e.g. [7], [11]). We sketch the main steps. First we point out that given u_0 in $\mathbb{K}_\varrho$ and α_0 in L,

$$\alpha_0 \in f_\varrho(u) \Leftrightarrow a(u_0, v-u_0) + \langle P_\varrho'(u_0), v-u_0\rangle_L \ge \langle \alpha_0, v-u_0\rangle_L \ \forall v \in \mathbb{K}_\varrho$$

Moreover, notice that $\mathcal{S}^+$ is a smooth surface in L and that the normal space to $\mathcal{S}^+$ at u_0 is specified by

$$\nu \in N_{u_0}(\mathcal{S}^+) \Leftrightarrow \exists \lambda \in \mathbb{R}, \langle \nu, v\rangle_L = \lambda b(u_0, v)\ \forall v \in L.$$

Then the conclusion (which is a Lagrange multiplier like result) follows from Theorem (1.13) and Remark (1.12b) of [8] (cf. [11]).

Remark 3.7. Let $(\varrho_n)_n$ be a sequence in $]0,1]$ converging to zero and for all n let (u_n, λ_n) be solutions of (3.7), with $\varrho = \varrho_n$, such that the u_n converge in L to a point u and the λ_n are bounded. Also assume that $\mathbb{K}_\varrho$ and $\mathcal{S}^+$ are not tangent at u. Then $u \in H$, u_n converges to u in H and $\lambda = a(u,u)$ is a bifurcation point for (3.5).

Proof. Using Lemma 3.6(a) and Lemma 2.4 we deduce that u is a critical point for f_ϱ and that $f_{\varrho_n}(u_n)$ converges to $f_\varrho(u)$, hence $\lambda_n \to \lambda$. Since $\mathbb{K}_\varrho$ and $\mathcal{S}^+$ are not tangent at u by Lemma 3.6(b), (u,λ) is a solution of (3.7). The convergence of $f_{\varrho_n}(u_n)$ to $f_\varrho(u)$ implies that $\alpha(u_n) \to \alpha(u)$ so $\alpha_{-\mu}(u_n) \to \alpha_{-\mu}(u)$ and then $u_n \xrightarrow{H} u$ since $a_{-\mu}(\,\cdot\,,\,\cdot\,)$ is an equivalent inner product in H. Finally, a simple rescaling argument shows that $((\varrho_n u_n, \lambda_n))_n$ is a sequence of solutions of (3.5) such that $\varrho_n u_n \to 0$, that is, λ is a bifurcation point for (3.5).

The following remark is a consequence of Proposition 3.11 of [11].

Remark 3.8. $\mathbb{K}_0$ and $\mathcal{S}$ (hence $\mathcal{S}^+$) are not tangent at any u of their intersection. This implies that for any u_0 in $\mathbb{K}_0 \cap \mathcal{S}^+$ there exist $R, \delta > 0$ such that for all ϱ in $[0,\delta[$ and u in $\mathbb{K}_\varrho \cap \mathcal{S}^+$ with $\|u - u_0\|_L < R$, $\mathbb{K}_\varrho$ and $\mathcal{S}^+$ are not tangent at u.

Lemma 3.9. *The following facts are true.*

(a) *The functionals $(f_\varrho)_{\varrho\in[0,1]}$ are equi-$\mathcal{C}(p,q)$, with $p \equiv 0$, equi-locally coercive and Γ-continuous on H, according to the definitions of Section* 2.

(b) *Given u_0 in $\mathbb{K}_0 \cap \mathcal{S}^+$ there exist $\delta, R > 0$ such that the functionals $(f^+_\varrho)_{\varrho\in[0,\delta]}$ are equi-$\mathcal{C}(0,q)$, equi-locally coercive and Γ-continuous on $B_L(u_0,R)$.*

Proof. (a) is a simple consequence of assumptions (3.3) and (3.4). To prove (b) we can take δ, R as in Remark 3.8 and apply Theorem 3.14 of [11] in $B_L(u_0,R)$.

Theorem 3.10. *Let (u,λ) be a solution of* (3.6) *with $\beta(u) = 1$, that is, u is a lower critical point for f^+_ϱ and $\lambda = \alpha(u) = f^+_0(u)/2$. Assume that there exists $c > \lambda/2$ which is not critical for f^+_0, so that we can consider the flow associated with f^+_0 on $f^c_0 = \{f_0(v) \le c\}$, as in Remark* 2.6, *and assume that u*

is an isolated critical point in L with index different from $\overline{0}$ (the trivial pointed space $(\{p\},\{p\})$). Then λ is a bifurcation point for (3.5).

Proof. First of all, using Lemma 3.9, we take $\delta, R > 0$ such that $(f^+_\varrho)_{\varrho\in[0,\delta]}$ are equi-$\mathcal{C}(0,q)$, equi-locally coercive and Γ-continuous on $B_L(u,R)$. Using Proposition 2.10(a) we can also suppose that c is not a critical value for f^+_ϱ for all ϱ in $[0,\delta]$. Then we can consider on $B(u,R)$ the family of the local flows associated with $(f^+_\varrho)_{\varrho\in[0,\delta]}$ as in Remark 2.6, which by Proposition 1.9 and Theorem 2.6 of [27] have the continuation property. So if u is an isolated rest point in the zero flow then it continues to an isolated invariant set S_ϱ in the ϱ-flow, for ϱ small, with the same index. But by Remark 3.7 of [27], S_ϱ must contain a rest point u_ϱ, that is, a lower critical point for f^+_ϱ. Since this argument can be repeated in any ball $B_L(u,R')$ with $R' < R$, we find a sequence $(\varrho_n)_n$ converging to zero and a sequence u_n converging to u such that u_n is critical for $f^+_{\varrho_n}$. Using Lemma 3.6(b) we see that (u_n,λ_n) satisfies (3.7) with $\varrho = \varrho_n$, where $\lambda_n = \alpha(u_n)$ and $\lambda_n \le 2c$. By Remark 3.7 this implies that λ is a bifurcation point for (3.5).

4. Existence of essential eigenvalues

The main result of this section is Theorem 4.7 in which the existence of two bifurcation branches for problem (3.5) is proven. Let H, L, a, b, α, β, P, $\mathbb{K}$, $\mathcal{S}^+$, $\mathcal{S}^-$ be as in Section 3.

For n in $\mathbb{N}$ we consider

$$\lambda^+_n = \min_{F \text{ linear space}, \dim(F)=n} \ \max_{u\in F\cap\mathcal{S}^+} \alpha(u), \tag{4.1}$$

$$\lambda^-_n = \min_{F \text{ linear space}, \dim(F)=n} \ \max_{u\in F\cap\mathcal{S}^-} \alpha(u). \tag{4.2}$$

It is easy to check that, if finite, λ^+_n and λ^-_n are nontrivial eigenvalues of (a,b), that is, for all n there exist $e^+_n \in \mathcal{S}^+$ and $e^-_n \in \mathcal{S}^-$ (eigenvectors) such that

$$a(e^+_n, v) = \lambda^+_n b(e^+_n, v) \quad \forall v \in H, \qquad a(e^-_n, v) = \lambda^-_n b(e^-_n, v) \quad \forall v \in H,$$

and H is generated by $(e^+_n), (e^-_n)$ and $H_0 = \{v \in H \mid b(u,v) = 0 \ \forall v \in H\}$.

Remark 4.1. Let F be a linear space and suppose that there exists a real constant M such that

$$\alpha(u) \le M\beta(u) \quad \forall u \in F. \tag{4.3}$$

Then $\beta(u)$ has constant sign on F and $\beta(u) = 0$ only if $u = 0$.

Proof. It suffices to show that $\beta(u) = 0$ only for $u = 0$ (then use a connectedness argument). Actually if $\beta(u) = 0$ then by (3.3),

$$0 = M\beta(u) \ge \alpha(u) = \alpha(u) + \mu\beta(u) \ge \nu\|u\|_H,$$

which implies $u = 0$.

Lemma 4.2. *Let F be a linear space contained in $\mathbb{K}_0$, M such that* (4.3) *holds and suppose that $\beta(u) > 0$ for all u in $F \setminus \{0\}$* (*by Remark* 4.1 *it suffices that there exists just one such u*). *Furthermore, let $(\overline{u}, \overline{\lambda})$ be a solution of* (3.6) *with $\beta(\overline{u}) > 0$ and $\overline{\lambda} \geq M$. Set $\overline{F} = F \oplus \operatorname{span}(\overline{u})$. Then*

$$\beta(u) > 0 \quad \forall u \in \overline{F} \setminus \{0\}, \qquad \alpha(u) \leq \overline{\lambda}\beta(u) \quad \forall u \in \overline{F}.$$

Proof. Since $F \subset \mathbb{K}_0$, we have

$$a(\overline{u}, v) = \overline{\lambda} b(\overline{u}, v) \quad \forall v \in F$$

(using $\overline{u} + v$ and $\overline{u} - v$ in (3.6)) and since $0, 2\overline{u} \in \mathbb{K}_0$,

$$a(\overline{u}, \overline{u}) = \overline{\lambda} b(\overline{u}, \overline{u}).$$

Now let $u_0 \in F$ and $t \in \mathbb{R}$; we have

$$\begin{aligned} \alpha_{\overline{\lambda}}(u_0 + t\overline{u}) &= \alpha_{\overline{\lambda}}(u_0) + 2t a_{\overline{\lambda}}(u_0, \overline{u}) + t^2 \alpha_{\overline{\lambda}}(\overline{u}) \\ &= \alpha_M(u_0) + (M - \overline{\lambda})\beta(u_0) \leq 0, \end{aligned}$$

hence the second inequality. The first one follows from Remark 4.1.

Corollary 4.3. *Let F be a finite-dimensional space of dimension k. Assume that $F \subset \mathbb{K}_0$ and*

$$\widetilde{\lambda} = \max_{u \in F \cap \mathcal{S}^+} \alpha(u) < \lambda_{k+1}^+.$$

Then the functional f_0^+ has no critical points u with $f_0^+(u) \in \,]\widetilde{\lambda}/2, \lambda_{k+1}^+/2[$.

Proof. Such a critical point $\overline{u}$ would provide a solution $(\overline{u}, \overline{\lambda})$ of (3.6) with $\widetilde{\lambda} < \overline{\lambda} < \lambda_{k+1}^+$. By Lemma 4.2, $\alpha(u) \leq \overline{\lambda}\beta(u)$ for all u in $\overline{F} = F \oplus \operatorname{span}(\overline{u})$. Since $\alpha(\overline{u}) = \overline{\lambda}$, we get $u \notin F$, hence $\dim(\overline{F}) = k + 1$. But the definition of λ_{k+1}^+ implies $\overline{\lambda} \geq \lambda_{k+1}^+$, which gives a contradiction.

Remark 4.4. Let F, $\widetilde{\lambda}$ be as in the previous corollary and assume that there exists a point u in $\mathbb{K}_0$ such that $\beta(u) > 0$ and $\alpha(u) \geq \lambda_{k+1}\beta(u)$. Denote by B the set $\{u \in F \mid \beta(u) \leq 1\}$ and by S the set $\{u \in F \mid \beta(4) = 1\}$. Set

$$\begin{aligned} \mathcal{A}_1 = \{&\eta(S, 1) \mid \eta : S \times [0, 1] \to \mathcal{S}^+ \setminus \operatorname{span}(e_{k+1}, \dots) \text{ continuous}, \\ &\eta(u, 0) = u \ \forall u, t\}, \\ \mathcal{A}_2 = \{&\varphi(B) \mid \varphi : B \to \mathcal{S}^+ \text{ continuous}, \ \varphi(u) = u \ \forall u \in S\} \end{aligned}$$

and

$$\lambda' = \inf_{T \in \mathcal{A}_1} \sup_{u \in T} \alpha(u), \qquad \lambda'' = \inf_{T \in \mathcal{A}_2} \sup_{u \in T} \alpha(u).$$

Then $\lambda' \leq \widetilde{\lambda} < \lambda_{k+1} \leq \lambda'' < \infty$ (so $\lambda' \neq \lambda''$) and λ', λ'' are eigenvalues of (3.6), which are bifurcation points for (3.5).

PROOF. See Lemma 3.2 of [28]. For the existence of λ' see also [24, 29, 30]; for λ'' it should also be possible to use the notion of mountain pass type critical level in [30].

PROPOSITION 4.5. *Let F be a finite-dimensional space of dimension k such that $F \subset \mathbb{K}_0$ and assume that*

$$\underline{\lambda} = \max_{u \in F \cap \mathcal{S}^+} \alpha(u) < \lambda^+_{k+1}.$$

Then for all $\overline{\lambda}$ in $]\lambda_{k+1}, \lambda_{k+2}[$ there exists $R > 0$ such that for any pair (u_1, λ_1), (u_2, λ_2) of solutions of (3.6) in $\mathcal{S}^+ \times [\lambda_{k+1}, \overline{\lambda}]$ either $(u_1, \lambda_1) = (u_2, \lambda_2)$ or $\|u_1 - u_2\|_L \geq R$.

PROOF. Assume by contradiction that the conclusion is not true. Then we can find two sequences $((u'_n, \lambda'_n))_n$ $((u''_n, \lambda''_n))_n$ of solutions of (3.6) with $\lambda'_n, \lambda''_n \in [\lambda_{n+1}, \overline{\lambda}]$, $u'_n \neq u''_n$ and $\|u'_n - u''_n\|_L \to 0$. Then using the variational characterization 3.6 of the solutions and Lemma 2.4 (with a single functional; see also Remark 4.6) it is easy to see that, up to a subsequence, $u'_n \xrightarrow{H} u$, $u''_n \xrightarrow{H} u$, $\lambda'_n \to \lambda$ and $\lambda''_n \to \lambda$ where (u, λ) in $\mathcal{S}^+ \times [\lambda_{n+1}, \overline{\lambda}]$ is a solution of (3.6) (the H convergence is a consequence of the L convergence and of the convergence of the function, by arguing as in 2.4). We set

$$F'_n = F \oplus \operatorname{span}(u'_n), \quad F''_n = F \oplus \operatorname{span}(u''_n), \quad \overline{F}_n = F \oplus \operatorname{span}(u'_n, u''_n).$$

We claim that eventually $\dim(\overline{F}_n) = k + 2$. First of all it is clear that $u'_n \notin F$ and $u''_n \notin F$, since $\alpha(u) \leq \underline{\lambda}\beta(u)$ for $u \in F$ and $\alpha(u'_n) = \lambda'_n\beta(u'_n)$, $\alpha(u'_n) = \lambda'_n\beta(u'_n)$ (and the same for u''_n), so $\dim(F'_n) = \dim(F''_n) = k + 1$. Now two cases are possible: either $\lambda'_n = \lambda''_n$ or $\lambda'_n \neq \lambda''_n$. In the latter case, let for instance $\lambda'_n < \lambda''_n$. By Lemma 4.2 we have $\alpha(u) \leq \lambda'_n\beta(u)$ for $u \in F'_n$ and $\alpha(u''_n) = \lambda''_n\beta(u''_n)$ so $u''_n \notin F'_n$, hence $\dim(\overline{F}_n) = k + 2$. If conversely $\lambda'_n = \lambda''_n$ let u_0 in F and c', c'' in $\mathbb{R}$ be such that $u_0 + c'u'_n + c''u''_n = 0$. Then, by trivial computations,

$$\begin{aligned} 0 &= a_{\lambda'_n}(u_0 + c'u'_n + c''u''_n, u_0) = \alpha_{\lambda_{n'}}(u_0) \\ &= \alpha_{\underline{\lambda}}(u_0) + (\underline{\lambda} - \lambda'_n)\beta(u_0) \leq (\underline{\lambda} - \lambda'_n)\beta(u_0), \end{aligned}$$

which implies $u_0 = 0$. If $(c', c'') \neq (0, 0)$, then since $u'_n, u''_n \in \mathcal{S}^+$ it follows that $u'_n = \pm u''_n$; but equality is excluded by the assumptions and $u'_n = -u''_n$ is impossible, at least eventually, since $u'_n - u''_n \xrightarrow{L} 0$. Then $c' = c'' = 0$, that is, u_0, u'_n, u''_n are linearly indipendent, so $\dim(\overline{F}_n) = k + 2$.

Now the definition of λ_{k+2} implies that there must be a point $\overline{u}_n$ in $\overline{F}_n$ with $\beta(\overline{u}) = 1$ and $\alpha(\overline{u}_n) \geq \lambda_{k+2}$. Let $\overline{u}_n = u_{0,n} + c'_n u'_n + c''_n u''_n$. Then, assuming $\lambda'_n \leq \lambda''_n$,

$$
\begin{aligned}
0 \le \alpha_{\lambda_{k+2}}(\overline{u}) &= \alpha_{\lambda_{k+2}}(u_{0,n}) + c_n'^2 \alpha_{\lambda_{k+2}}(u_n') + c_n''^2 \alpha_{\lambda_{k+2}}(u_n'') \\
&\quad + 2c_n' a_{\lambda_{k+2}}(u_{0,n}, u_n') + 2c_n'' a_{\lambda_{k+2}}(u_{0,n}, u_n'') + 2c_n' c_n'' a_{\lambda_{k+2}}(u_n', u_n'') \\
&= \alpha_{\underline{\lambda}}(u_{0,n}) + (\lambda_n' - \underline{\lambda})\beta(u_{0,n}) + (\lambda_n'' - \lambda_n')\beta(u_{0,n}) + (\lambda_{k+2} - \lambda_n'')\beta(u_{0,n}) \\
&\quad + c_n'^2 \alpha_{\lambda_n'}(u_n') + c_n'^2(\lambda_n'' - \lambda_n')\beta(u_n') + (\lambda_{k+2} - \lambda_n'')\beta(u_n') + c_n''^2 \alpha_{\lambda_n''}(u_n'') \\
&\quad + c_n''^2(\lambda_{k+2} - \lambda_n'')\beta(u_n'') + 2c_n' a_{\lambda_n'}(u_n', u_{0,n}) + 2c_n'(\lambda_n'' - \lambda_n')b(u_n', u_{0,n}) \\
&\quad + 2c_n'(\lambda_{k+2} - \lambda_n'')b(u_n', u_{0,n}) + 2c_n'' a_{\lambda_n''}(u_n'', u_{0,n}) + 2c_n''(\lambda_{k+2} - \lambda_n'')b(u_n'', u_{0,n}) \\
&\quad + 2c_n' c_n'' a_{\lambda_n''}(u_n', u_n'') + 2c_n' c_n''(\lambda_{k+2} - \lambda_n'')b(u_n', u_n'') \\
&\le (\lambda_n' - \underline{\lambda})\beta(u_{0,n}) + (\lambda_n'' - \lambda_n')\beta(u_{0,n} + c_n' u_n') + (\lambda_{k+2} - \lambda_n'')\beta(\overline{u}_n) \\
&\quad + 2c_n' c_n'' a_{\lambda_n''}(u_n', u_n'') \le 2c_n' c_n'' a_{\lambda_n''}(u_n', u_n''),
\end{aligned}
$$

which implies $c_n' c_n'' \ge 0$, since $a_{\lambda_n''}(u_n', u_n'') \ge 0$. We can now take $w_{0,n}$ in F and $c_{0,n}$ in $\mathbb{R}$ such that $u_{0,n} = c_{0,n} w_{0,n}$ and $\beta(w_{0,n}) = 1$. Since F has finite dimension and β is a norm on F, we can suppose that $w_{0,n} \to w_0$ in F for a suitable w_0 in F with $\beta(w_0) = 1$. Since $u \notin F$ (because $\alpha(u) = \lambda = \lambda\beta(u)$), we get $b(w_0, u) > -1$ (notice that $b(\,\cdot\,,\,\cdot\,)$ is an inner product on $F \oplus \operatorname{span}(u)$, because $\beta > 0$ on $F \oplus \operatorname{span}(u) \setminus \{0\}$, by Lemma 4.2). Let $\varepsilon = (1 + b(w_0, u))/2 > 0$. Then eventually $b(w_{0,n}, u_n') \ge -1 + \varepsilon$, $b(w_{0,n}, u_n'') \ge -1 + \varepsilon$, $b(u_n', u_n'') \ge 1 - \varepsilon$ and

$$
\begin{aligned}
1 &= \beta(c_{0,n} w_{0,n} + c_n' u_n' + c_n'' u_n'') \\
&= c_{0,n}^2 + c_n'^2 + c_n''^2 + 2c_{0,n} c_n' b(w_{0,n}, u_n') \\
&\quad + 2c_{0,n} c_n'' b(w_{0,n}, u_n'') + 2c_n' c_n'' b(u_n', u_n'') \\
&\ge c_{0,n}^2 + c_n'^2 + c_n''^2 - 2(1-\varepsilon)c_{0,n} c_n' - 2(1-\varepsilon)c_{0,n} c_n'' + 2c_n' c_n''(1-\varepsilon) \\
&\ge \varepsilon(c_{0,n}^2 + c_n'^2 + c_n''^2)
\end{aligned}
$$

(we have used $c_n' c_n'' > 0$ and again the fact that b is an inner product on F_n' and on F_n''). Then everything is bounded, so $\overline{u}_n$ converges to a point $\overline{u}$ which must lie in $\mathcal{S}^+$ and satisfy $\alpha(\overline{u}) \ge \lambda_{k+2}$. Set $\overline{F} = F \oplus \operatorname{span}(u)$, using Lemma 4.2 it is easily seen that $\alpha(v) \le \lambda\beta(v)$ for all $v \in \overline{F}$, so we have a contradiction, since $\overline{u} \in \overline{F}$.

Remark 4.6. Proposition 4.5 can be proved in a more general way: for $\sigma \in [0,1]$ let a_σ, b_σ be linear forms and $\mathbb{K}_\sigma$ be closed convex cones such that, setting $f_\sigma(u) = a_\sigma(u)/2 + I_{K_\sigma \cup \mathcal{S}_\sigma^+}$ (with the obvious notations), $(f_\sigma)_\sigma$ are equi-$\mathcal{C}(p,q)$, equi-locally-coercive and Γ-continuous. Then if there exist two linear spaces F_1, F_2 which satisfy the assumptions of Proposition 4.5 with $a = a_\sigma$, $b = b_\sigma$ and $\mathbb{K}_0 = \mathbb{K}_\sigma$ for all $\sigma \in [0,1]$, then $R > 0$ can be found such that all pairs (u', λ') and (u'', λ'') corresponding to critical points of $f_{\sigma'}, f_{\sigma''}$ with $\sigma', \sigma'' \in [0,1]$ and $\lambda', \lambda'' \in [\lambda_{k+1}, \overline{\lambda}]$, if distinct, are such that $\|u' - u''\|_L \ge R$.

Proof. Just repeat the same proof of Proposition 4.5, using, in the first step, Lemma 2.4 for a family of functions.

Theorem 4.7. *Assume that there exist two linear spaces F_1, F_2 with $\dim(F_1) = k+1$ and $\dim(F_2) = k+2$ for an integer k, and such that*

$$F_1 \subset F_2 \subset \mathbb{K}_0, \quad \underline{\lambda} = \max_{u \in F_1 \cap \mathcal{S}^+} \alpha(u) < \lambda_{k+1}, \quad \overline{\lambda} = \max_{u \in F_2 \cap \mathcal{S}^+} \alpha(u) < \lambda_{k+2}$$

(hence λ_1 must be a simple eigenvalue). Then there exist two distinct solutions (u_1, λ_1), (u_2, λ_2) of (3.6) which lie in $\mathcal{S}^+ \times [\lambda_{k+1}, \overline{\lambda}]$ and are both bifurcation points for (3.5). If $\lambda_1 = \lambda_2$ this means that there are two bifurcating sequences, that is, there are two sequences $((u_n', \lambda_n'))$, $((u_n'', \lambda_n''))$ of solutions of (3.5) with $\beta(u_n') = \beta(u_n'')$, $u_n', u_n'' \xrightarrow{H} 0$ and $\lambda_n', \lambda_n'' \to \lambda_1$.

Proof. Let $H_1 = \{u \in H \mid \langle u, v\rangle_H = 0 \ \forall v \in F_2\}$ and $P_1 : H \to H_1$ the orthogonal projection. For $\sigma \in \,]0,1]$ we set $a_\sigma(u,v) = a(u,v) + \langle P_1(u), P_1(v)\rangle_H$ and define

$$\widetilde{f}_\sigma(u) = \frac{1}{2} a_\sigma(u,u) + I_{\mathbb{K}_0 \cap \mathcal{S}^+}, \quad \widetilde{f}_0 = \frac{1}{2} a(u,u) + I_{F_2 \cap \mathcal{S}^+}.$$

It is easy to see that $(\widetilde{f}_\sigma)_\sigma$ is equi-coercive, since every a_σ satisfies (3.3), and Γ-continuous. It can also be easily proven that $\widetilde{f}_\sigma$ are equi-$\mathcal{C}(p,q)$: just repeat the proof of Theorem (1.13) of [8] using the fact that all the functions $a_\sigma(\,\cdot\,,\,\cdot\,)/2 + I_{\mathbb{K}_0}$ and $a_\sigma(\,\cdot\,,\,\cdot\,)/2 + I_{\mathbb{K}_0}$ are convex (hence "equi-convex").

Moreover, for all $\sigma \in \,]0,1]$, F_1, F_2, a_σ satisfy the same assumptions, so by Corollary 4.3 it is clear that $\widetilde{f}_\sigma$ has no critical values in $[\underline{\lambda}/2, \lambda_{k+1}/2] \cup [\overline{\lambda}/2, \lambda_{k+2}/2]$. In particular, if $c \in \,]\lambda_{k+1}, \overline{\lambda}[$ then c is regular for $\widetilde{f}_\sigma$ for all $\sigma \in [0,1]$ (the case $\sigma = 0$ follows from Lemma 2.4). So we can consider the flows associated with $(\widetilde{f}_\sigma)_\sigma$ on $X_\sigma = \{u \mid \widetilde{f}_\sigma(u) \le c\}$, as in Section 2, which by Proposition 2.10 satisfy the assumptions of the continuation result of Section 1 of [27].

Now we consider $\sigma = 0$; it is clear that $X_0 = F_2 \cap \mathcal{S}^+$ and that $\widetilde{f}_0$ has two critical points $\widetilde{u}$, $-\widetilde{u}$ in $\{u \mid f_0(u) \ge \lambda_{k+1}/2\}$, namely the eigenvectors of the problem $a(u,v) = \lambda b(u,v)$ on F_2 with $\lambda \ge \lambda_{k+1}$, which are simple and correspond to the pair of maxima of $a(u,u)$ for $u \in \mathcal{S}^+ \cap F_2$. This implies that the indices of $\{\widetilde{u}\}$ and $\{-\widetilde{u}\}$ are both $\mathcal{S}^{k+1}$, the $(k+1)$-dimensional pointed sphere (because F_2 has dimension $k+2$). We claim that for $\sigma \in [0,1]$ there exist u_σ' and u_σ'' which are critical for $\widetilde{f}_\sigma$, with $\widetilde{f}_\sigma(u_\sigma'), \widetilde{f}_\sigma(u_\sigma'') \in [\lambda_{k+1}, \overline{\lambda}]$ and such that $u_0' = \widetilde{u}$, $u_0'' = -\widetilde{u}$, u_σ', u_σ'' are isolated and $\{u_\sigma'\}$, $\{u_\sigma''\}$ are continuations, as isolated invariant sets, of $\{\widetilde{u}\}$, $\{-\widetilde{u}\}$ (see [6]), so they both have index $\mathcal{S}^{k+1}$. To see this assume that $\sigma_0 \in [0,1[$ and that u_σ' has been found on $[0,\sigma_0]$; since the index is nontrivial there exists $\delta > 0$ such that for $\sigma \in [\sigma_0, \sigma_0 + \delta]$, u_{σ_0}' can be continued to an invariant set S_σ. Possibly reducing δ, S_σ must consist of a single point, otherwise it would contain two critical points (see Remark 3.7 of [27]) which would collapse to a single one as $\sigma \to \sigma_0$, and this is impossible according to Remark 4.6. Furthermore, $\widetilde{f}_\sigma(u'_\sigma) \in [\lambda_{k+1}, \overline{\lambda}]$, since $\widetilde{f}_\sigma$ has no critical values in $]\underline{\lambda}, \lambda_{k+1}[\cup]\overline{\lambda}, \lambda_{k+2}[$. Using a connectedness argument one can find u'_σ for $\sigma \in [0,1]$, and the same is true for u''_σ.

In this way we have found two solutions of (3.6) with nontrivial index, which by Theorem 3.10 give rise to bifurcation.

5. Bifurcation for nonlinear elliptic obstacle problems

Let Ω be a bounded open subset of $\mathbb{R}^N$. For $i, j = 1, \ldots, N$ let a_{ij} be in $\mathrm{L}^\infty(\Omega)$ with $a_{ij} = a_{ji}$ and such that

$$\forall \xi \in \mathbb{R}^N \ \forall x \in \Omega, \quad \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \nu \sum_{i=1}^N \xi_i^2$$

for a constant $\nu > 0$. Also let $a_0 \in \mathrm{L}^p$ for a $p \geq 1$ with $p > N/2$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that $s \mapsto g(x,s)$ is of class C^1 for almost all x in Ω, $x \mapsto g(x,s)$ is measurable for all s in $\mathbb{R}$ and

$$\begin{cases} \text{there exists } M > 0 \text{ such that} \\ |g'_s(x,s)| \leq M \quad \forall x \in \Omega, \forall s \in \mathbb{R}; \end{cases} \tag{5.1}$$

$$g(x,0) = g'_s(x,0) = 0 \quad \forall x \in \Omega, \tag{5.2}$$

Finally, let $\varphi_1 : \Omega \to [-\infty, 0]$ and $\varphi_2 : \Omega \to [0, \infty]$ be two functions such that φ_1 is quasi-upper semicontinuous and φ_2 is quasi-lower semicontinuous (see [2]). We consider the convex set

$$\mathbb{K} = \{u \in \mathrm{W}_0^{1,2}(\Omega) \mid \varphi_1(x) \leq \widetilde{u}(x) \leq \varphi_2(x) \text{ for quasi-every } x \text{ in } \Omega\},$$

where for every u in $\mathrm{W}_0^{1,2}(\Omega)$, $\widetilde{u}$ is the quasi-everywhere continuous function defined quasi-everywhere by

$$\widetilde{u}(x) = \lim_{r\to 0} \frac{1}{\mathrm{meas}(B(x,r))} \int_{B(x,r)} u(\xi)\, d\xi$$

(see [35]); we also set

$$E^- = \{x \in \Omega \mid \phi_2(x) = 0\}, \quad E^+ = \{x \in \Omega \mid \phi_1(x) = 0\}, \tag{5.3}$$

and denote by $(\lambda_n)_n$ and $(\lambda'_n)_n$ the eigenvalues of the operator $u \mapsto D_j(a_{ij}D_iu) + a_0u$ in $\mathrm{W}_0^{2,2}(\Omega \setminus (E^+ \cap E^-))$ and $\mathrm{W}_0^{2,2}(\Omega \setminus (E^+ \cup E^-))$ respectively. Of course $\lambda_n \leq \lambda'_n$ for all n. For convenience we agree that $\lambda_0 = \lambda'_0 = -\infty$.

Remark 5.1. If $\mathbb{K}_0 = \mathrm{W}_0^{2,2}(\Omega \setminus (E^+ \cap E^-))$-closure of $\bigcup_{t>0} t\mathbb{K}$, then

$$\begin{aligned} \mathbb{K}_0 = \{u \in \mathrm{W}_0^{1,2}(\Omega) \mid \widetilde{u} \leq 0 \text{ quasi-everywhere on } F^-, \\ \widetilde{u} \geq 0 \text{ quasi-everywhere on } F^+\}. \end{aligned}$$

Proof. See Proposition 4.9 of [11].

Theorem 5.2. *Let a_{ij}, a_0, g, φ_1, φ_2, $\mathbb{K}$, $\mathbb{K}_0$, $(\lambda_n)_n$, $(\lambda'_n)_n$ be as above. Suppose there exist $k \in \mathbb{N}$ such that*

$$\lambda_k \le \lambda'_k < \lambda_{k+1} \le \lambda'_{k+1} < \lambda_{k+2} \le \lambda'_{k+2} < \infty \tag{5.4}$$

(of course the $\le$ inequalities above hold always, while the last $<$ simply means that $\Omega \setminus (E^+ \cup E^-)$ is nonempty—notice that (5.4) implies that λ_{k+1} is simple). Then there exist four distinct solutions $(u^{(i)}, \lambda^{(i)})$, $i = 1, 2, 3, 4$, of

$$\begin{cases} \int_\Omega (\sum_{i,j=1}^N a_{ij} D_i u D_j (v-u) + a_0 u(v-u))\, dx \\ \quad \ge \lambda \int_\Omega u(v-u)\, dx \quad \forall v \in \mathbb{K}_0, \\ u \in \mathbb{K}_0, \ \int_\Omega u^2\, dx = 1, \quad \lambda \in \mathbb{R}, \end{cases} \tag{5.5}$$

with $\lambda^{(1)} \le \lambda'_k$, $\lambda^{(2)}, \lambda^{(3)} \in [\lambda'_{k+1}, \lambda_{k+2}]$, $\lambda^{(4)} \ge \lambda'_{k+2}$ ($\lambda^{(i)}$ are eigenvalues of (5.5) with eigenvectors $u^{(i)}$, normalized in the L^2 norm). Each $\lambda^{(i)}$ is a bifurcation point for the problem

$$\begin{cases} \int_\Omega (\sum_{i,j=1}^N a_{ij} D_i u D_j (v-u) + (a_0 + g(\,\cdot\,, u)) u(v-u))\, dx \\ \quad \ge \lambda \int_\Omega u(v-u)\, dx \quad \forall v \in \mathbb{K}, \\ u \in \mathbb{K}, \quad \lambda \in \mathbb{R}, \end{cases} \tag{5.6}$$

that is, for each $i = 1, 2, 3, 4$ there exists a sequence $((u_n^{(i)}, \lambda_n^{(i)}))_n$ of solutions of (5.6) such that $u_n^{(i)} \ne 0$ and $(u_n^{(i)}, \lambda_n^{(i)})$ converges to $(0, \lambda^{(i)})$ in $\mathrm{W}_0^{1,2}(\Omega) \times \mathbb{R}$. If $\lambda^{(2)} = \lambda^{(3)}$ then $\lambda^{(2)}$ is the origin of two "branches", in the sense that the two sequences $((u_n^{(2)}, \lambda_n^{(2)}))_n$ and $((u_n^{(3)}, \lambda_n^{(3)}))_n$ of solutions of (5.6) satisfy $\int_\Omega (u_n^{(2)})^2\, dx = \int_\Omega (u_n^{(3)})^2\, dx \to 0$, $u_n^{(2)} \ne u_n^{(3)}$, $\lambda_n^{(2)} \to \lambda^{(2)}$ and $\lambda_n^{(3)} \to \lambda^{(2)}$.

Proof. Consider $H = \mathrm{W}_0^{2,2}(\Omega \setminus (E^+ \cap E^-))$, $L = \mathrm{L}^2(\Omega)$,

$$a(u,v) = \int_\Omega \bigg(\sum_{i,j=1}^N a_{ij} D_i u D_j v + a_0 uv \bigg)\, dx, \quad b(u,v) = \int_\Omega uv\, dx,$$

and $P(u) = G(\,\cdot\,, u)$, where $G(x,s) = \int_0^s g(x,\sigma)\, d\sigma$. It easy to check that H, L, $\mathbb{K}$, a, b, P fall within the framework of Sections 3 and 4. Now take

$$F_1 = \operatorname{span}(e'_1, \dots, e'_k), \quad F_2 = \operatorname{span}(e'_1, \dots, e'_{k+1}).$$

It is clear that the assumptions of Theorem 4.7 are fulfilled. Combining the theorem and Remark 4.4 we get the conclusion.

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