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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF THE BOUSSINESQ SYSTEM WITH NONLINEAR THERMAL DIFFUSION

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Dedicated to Professor O. A. Ladyzhenskaja

1. The model

The Boussinesq system of hydrodynamics equations [3], [26] arises from a zero order approximation to the coupling between the Navier–Stokes equations and the thermodynamic equation [25]. Presence of density gradients in a fluid allows the conversion of gravitational potential energy into motion through the action of buoyant forces. Density gradients are induced, for instance, by temperature differences arising from non-uniform heating of the fluid. In the Boussinesq approximation of a large class of flow problems, thermodynamical coefficients such as viscosity, specific heat and thermal conductivity may be assumed to be constants, leading to a coupled system of parabolic equations with linear second order operators, see, e.g. [11], [12], [17], [31]. However, there are some fluids such as lubrificants or some plasma flow for which this is not an accurate assumption [16], [29] and a quasilinear parabolic system has to be considered. In this paper we present some results on existence and uniqueness of weak solutions for this kind of models. Results on some qualitative properties related with spatial and time localization of the support of solutions will be published elsewhere, see

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also [6]. We start considering the system derived in [25]:

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}(\mu(\theta)D(\mathbf{u})) + \nabla p = \mathbf{F}(\theta), \\ \operatorname{div}\mathbf{u} = 0, \\ \mathcal{C}(\theta)_t + \mathbf{u}\cdot\nabla\mathcal{C}(\theta) - \Delta\varphi(\theta) = 0, \end{cases} \tag{1}$$

with $\mathbf{u}$ the velocity field of the fluid, θ its temperature, p pressure, $\mu(\theta)$ viscosity of the fluid, $\mathbf{F}(\theta)$ buoyancy force, $D(\mathbf{u}) := \nabla\mathbf{u} + \nabla\mathbf{u}^T$,

$$\mathcal{C}(\theta) := \int_{\theta_0}^{\theta} C(s)\,ds \quad \text{and} \quad \varphi(\theta) := \int_{\theta_0}^{\theta} \kappa(s)\,ds,$$

with $C(\tau)$ and $\kappa(\tau)$ specific heat and thermal conductivity of the fluid, respectively, and with θ_0 a reference temperature. Assuming, as usual, $C > 0$ then $\mathcal{C}$ is inversible, and so $\theta = \mathcal{C}^{-1}(\overline{\theta})$ for some real argument $\overline{\theta}$. Then we can define the functions

$$\overline{\varphi}(\overline{\theta}) := \varphi\circ\mathcal{C}^{-1}(\overline{\theta}), \quad \overline{\mathbf{F}}(\overline{\theta}) := \mathbf{F}\circ\mathcal{C}^{-1}(\overline{\theta}), \quad \overline{\mu}(\overline{\theta}) := \mu\circ\mathcal{C}^{-1}(\overline{\theta}).$$

Substituting these expressions in (1) and omitting the bars we obtain the following formulation of the Boussinesq system

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}(\mu(\theta)D(\mathbf{u})) + \nabla p = \mathbf{F}(\theta), \\ \operatorname{div}\mathbf{u} = 0, \\ \theta_t + \mathbf{u}\cdot\nabla\theta - \Delta\varphi(\theta) = 0. \end{cases} \tag{2}$$

We briefly comment some interesting features which characterize this model. There are two paradigmatic situations: the *fast* and the *slow* heat diffusion. These cases may mathematically correspond to the singular or degenerate character of the heat equation which occur according to the relative behavior of C and κ. A simple example illustrating these phenomena is the following: suppose that a perturbation from a constant temperature θ_0 occurs in a region, producing a small gradient of temperature between the boundary (higher temperature, say) and the interior (lower temperature). Assume that the behavior of $\mathcal{C}$ and φ may be approximated by

$$\mathcal{C}(s) = c_1(s-\theta_0) + c_2(s-\theta_0)^p, \quad \varphi(s) = k_1(s-\theta_0) + k_2(s-\theta_0)^q,$$

for $s > \theta_0$, with $p, q > 0$. Then, we have

$$\overline{\varphi}'(\mathcal{C}(s)) = \varphi'(s)(\mathcal{C}^{-1})'(\mathcal{C}(s)) = \frac{k_1 + k_2 q(s-\theta_0)^{q-1}}{c_1 + c_2 p(s-\theta_0)^{p-1}}.$$

When $s \to \theta_0$, and therefore $\mathcal{C}(s) \to 0$, we obtain one of the following behaviors of $\overline{\varphi}'$ close to zero:

(i) if $p, q > 1$ then $\overline{\varphi}'(0) = k_1/c_1$,
(ii) if $1 > q > p$ either $q > 1 > p$ then $\lim_{\mathcal{C}(s)\to 0}\overline{\varphi}'(\mathcal{C}(s)) = 0$,
(iii) if $p > 1 > q$ either $1 > p > q$ then $\lim_{\mathcal{C}(s)\to 0}\overline{\varphi}'(\mathcal{C}(s)) = \infty$.

In (i) linear parts dominate: this case arises, for instance, when conductivity and specific heat are considered constants, leading to the heat equation with linear diffusion. In (ii) and (iii) nonlinear parts dominate leading to two different behaviors:

(a) if $p < q$ specific heat dominates over conductivity, i.e., when temperature approaches θ_0 heat is more easily stored by the fluid but worstly conduced. Under suitable assumptions, it was proven in [6] that a front of temperature $\theta = \theta_0$ arises. This type of phenomenon is known as *slow diffusion*: heat spends a positive time to spread over the neighbourhood,

(b) if $p > q$ the opposite effect arises: conductivity dominates over specific heat. In this case the phenomenon is called *fast diffusion*. Under suitable assumptions, it was proven in [6] the stabilization in finite time of θ towards the value on the boundary of the neighbourhood.

The outline of the paper is the following. In Section 2 we state the main assumptions on the data that will hold through the article and introduce the usual Navier–Stokes functional setting consisting of the variational formulation introduced by Leray [23] under the framework of free divergence functional spaces. We also define the notion of weak solution for the heat equation. In Section 3 we prove existence of solutions by introducing an iterative scheme to uncouple the system. We then adapt some results for Navier–Stokes equations [24] and use a regularization technique together with results in [1] to prove existence of weak solutions for the uncoupled problems. Finally we pass to the limit in the iterative scheme to find a weak solution of the coupled system. This result improves the one of [9]. In Section 4 we present two results on uniqueness of solutions in spatial dimension $N = 2$ corresponding to singular and degenerate diffusion. Proofs of both results are based in duality techniques involving coupled linear systems in non-divergence form.

2. Functional setting

We consider the system of equations given by (2) holding in $Q_T := \Omega \times (0,T)$ and satisfying the following auxiliary conditions:

$$\begin{cases} \mathbf{u} = \mathbf{0} \text{ and } \varphi(\theta) = \phi_D & \text{on } \Sigma_T, \\ \mathbf{u}(\,\cdot\,,0) = \mathbf{u}_0 \text{ and } \theta(\,\cdot\,,0) = \theta_0 & \text{in } \Omega, \end{cases}$$

with $\Sigma_T := \partial\Omega \times (0,T)$. We introduce the usual Navier–Stokes functional setting [23], [21], [24], [32], considering the functional spaces

$$\begin{aligned} \mathcal{C}^\infty_\sigma(\Omega) &:= \{\mathbf{u} \in \mathcal{C}^\infty_0(\Omega;\mathbb{R}^N) : \operatorname{div}\mathbf{u} = 0\}, \\ L^p_\sigma(\Omega) &:= \text{closure of } \mathcal{C}^\infty_\sigma(\Omega) \text{ in } L^p(\Omega;\mathbb{R}^N), \end{aligned}$$

$$W^{q,p}_\sigma(\Omega) := \text{closure of } \mathcal{C}^\infty_\sigma(\Omega) \text{ in } W^{q,p}(\Omega;\mathbb{R}^N),$$
$$L^p_\sigma(Q_T) := L^p(0,T;L^p_\sigma(\Omega)).$$

The following set of assumptions will be recalled along the paper:

Assumptions on the data.

(H_1) $\Omega\subset\mathbb{R}^N$, $N=2,3$ denotes an open, bounded and connected set, with boundary $\partial\Omega$ of class $\mathcal{C}^1$. We suppose $T>0$ arbitrarily chosen.

(H_2) We assume

$$\varphi\in\mathcal{C}([0,\infty))\cap\mathcal{C}^1((0,\infty)),\quad \varphi(0)=0,\quad \varphi \text{ non-decreasing}$$
$$\mathbf{F}\in\mathcal{C}^{0,1}_{\text{loc}}(([0,\infty);\mathbb{R}^N),$$
$$\mu\in\mathcal{C}^{0,1}_{\text{loc}}([0,\infty))\quad\text{with } m_0\le\mu(s)\le m_1\quad\forall s\in[0,\infty),$$

for some constants $m_1\ge m_0>0$.

(H_3) $\mathbf{u}_0\in L^2_\sigma(\Omega)$, $\theta_0\in L^\infty(\Omega)$ and $\theta_0\ge 0$, $\phi_D\in L^2(0,T;\ H^1(\Omega))\cap H^1(0,T;L^2(\Omega))\cap L^\infty(Q_T)$.

(H_4) If $\mu'\neq 0$ or $\mathbf{F}'\neq 0$ we assume that φ^{-1} is Hölder continuous of exponent $\alpha\in(0,1)$.

Note that functions φ, $\mathbf{F}$ and μ apply on θ. We shall later show that $\theta\in L^\infty(Q_T)$ and, therefore, local and global Lipschitz continuity will be equivalent. Let us consider the orthogonal projection

$$P_\sigma : L^2(\Omega;\mathbb{R}^N)\to L^2_\sigma(\Omega).$$

Applying P_σ to the first equation of (2) and taking into account $P_\sigma\nabla p\equiv 0$ and $\mathbf{u}=P_\sigma\mathbf{u}$ due to $\operatorname{div}\mathbf{u}=0$ we obtain

$$\text{(3)}\qquad\begin{cases}\mathbf{u}_t+P_\sigma(\mathbf{u}\cdot\nabla)\mathbf{u}-P_\sigma\operatorname{div}(\mu(\theta)D(\mathbf{u}))=P_\sigma\mathbf{F}(\theta) & \text{in } Q_T,\\ \theta_t+\mathbf{u}\cdot\nabla\theta-\Delta\varphi(\theta)=0 & \text{in } Q_T,\\ \mathbf{u}=\mathbf{0} \text{ and } \varphi(\theta)=\phi_D & \text{on } \Sigma_T,\\ \mathbf{u}(\,\cdot\,,0)=\mathbf{u}_0 \text{ and } \theta(\,\cdot\,,0)=\theta_0 & \text{in } \Omega,\end{cases}$$

which is the final differential formulation of the problem we shall study. To introduce the weak formulation of Problem (3) we consider the usual bilinear and trilinear forms defined by

$$a_\theta(\mathbf{u},\mathbf{v}) := \frac{1}{2}\sum_{i,j=1}^N\int_\Omega\mu(\theta)\bigg(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\bigg)\frac{\partial v_j}{\partial x_i} := \int_\Omega\mu(\theta)D(\mathbf{u}):\nabla\mathbf{v},$$

for all $\mathbf{u},\mathbf{v}\in W^{1,2}_\sigma(\Omega)$ and with given $\theta\in L^\infty(Q_T)$, and

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) := \sum_{i,j=1}^N\int_\Omega u_i\frac{\partial v_j}{\partial x_i}w_j := \int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w},$$

for all $\mathbf{u}, \mathbf{v} \in W^{1,2}_\sigma(\Omega)$, $\mathbf{w} \in W^{1,2}_\sigma(\Omega) \cap L^N_\sigma(\Omega)$. It is well known that a_θ is continuous and coercive in $W^{1,2}_\sigma(\Omega) \times W^{1,2}_\sigma(\Omega)$ for a.e. $t \in [0,T]$ and that b is anti-symmetric and continuous in $W^{1,2}_\sigma(\Omega) \times W^{1,2}_\sigma(\Omega) \times (W^{1,2}_\sigma(\Omega) \cap L^N_\sigma(\Omega))$. We shall denote a duality product by $\langle \cdot\,, \cdot \rangle_{V,V'}$, or $\langle \cdot\,, \cdot \rangle_V$ if V is reflexive.

Remark 1. The main advantage of the formulation of the Navier–Stokes equations in free divergence spaces is that the pressure p is *eliminated* from the system. As it is well known, De Rham's Lemma [28] allows to recover this unknown due to the following property: *if* $\langle \mathbf{q}, \mathbf{w} \rangle = 0$ *for all* $\mathbf{w} \in W^{1,2}_\sigma(\Omega)$ *then there exists* $p \in L^2(\Omega)$ *such that* $\mathbf{q} = -\nabla p$.

We consider the following notion of solution, see [23], [1]:

Definition 1. Assume (H_3). Then, the pair $(\mathbf{u}, \theta)$ is a *weak solution* of (3) if:

(i) $\mathbf{u} \in L^2(0,T; W^{1,2}_\sigma(\Omega)) \cap L^\infty(0,T; L^2_\sigma(\Omega))$, $\varphi(\theta) \in \phi_D + L^2(0,T; H^1_0(\Omega))$, $\theta \in L^\infty(Q_T)$,

(ii) $\mathbf{u}(\,\cdot\,, 0) = \mathbf{u}_0$ a.e. in Ω, and for a.e. $t \in (0,T)$, we have

$$\langle \mathbf{u}_t, \mathbf{w} \rangle_{L^2_\sigma(\Omega)} + a_\theta(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{w}) = \langle \mathbf{F}(\theta), \mathbf{w} \rangle_{L^2_\sigma(\Omega)}$$

for any $\mathbf{w} \in W^{1,2}_\sigma(\Omega) \cap L^N_\sigma(\Omega)$,

(iii) $\theta_t \in L^2(0,T; H^{-1}(\Omega))$,

$$\int_0^T \langle \theta_t, \zeta \rangle_{H^{-1},H^1_0} + \int_0^T \int_\Omega (\nabla \varphi(\theta) - \theta \mathbf{u}) \cdot \nabla \zeta = 0,$$

for any $\zeta \in L^2(0,T; H^1_0(\Omega))$ and

$$\int_0^T \langle \theta_t, \psi \rangle_{H^{-1},H^1_0} + \int_0^T \int_\Omega (\theta - \theta_0)\psi_t = 0,$$

for any $\psi \in L^2(0,T; H^1_0(\Omega)) \cap W^{1,1}(0,T; L^2(\Omega))$ with $\psi(T) = 0$.

3. Existence of solutions

Existence of solutions of Problem (3) is obtained by using results on Navier–Stokes and non linear–diffusion equations. We shall give a proof based on Galerkin's method although other strategies are also possible, see [29].

Theorem 1. *Assume* (H_1)–(H_4). *Then, there exists a weak solution of Problem* (3) *such that*

$$\mathbf{u} \in \mathcal{C}([0,T]; W^{-1,2}_\sigma(\Omega)) \quad \textit{and} \quad \theta \in \mathcal{C}([0,T]; H^{-1}(\Omega)). \tag{4}$$

Moreover, if there exist non-negative constants $k, m, \lambda_0, \lambda_1$ *such that*

$$k \geq \theta_0 \geq m \geq 0 \quad \textit{and} \quad \varphi(k e^{\lambda_0 t}) \geq \phi_D(\,\cdot\,, t) \geq \varphi(m e^{-\lambda_1 t}) \geq 0$$

a.e. in Ω and Σ_T, respectively, then there exists a constant $\lambda \geq 0$ independent of φ such that

$$ke^{\lambda t} \geq \theta(\,\cdot\,, t) \geq me^{-\lambda t} \geq 0 \tag{5}$$

for $t \in (0, T)$ and a.e. in Ω.

Proof. We start introducing the following iterative scheme to uncouple Problem (3): for each $n \in \mathbb{N}$ we set

$$\begin{cases} \mathbf{u}_{nt} + (\mathbf{u}_n \cdot \nabla)\mathbf{u}_n - \operatorname{div}(\mu(\theta_{n-1})D(\mathbf{u}_n)) = \mathbf{F}(\theta_{n-1}) & \text{in } Q_T, \\ \theta_{nt} + \mathbf{u}_{n-1} \cdot \nabla\theta_n - \Delta\varphi(\theta_n) = 0 & \text{in } Q_T, \\ \mathbf{u}_n = \mathbf{0} \text{ and } \varphi(\theta_n) = \phi_D & \text{on } \Sigma_T, \\ \mathbf{u}_n(\,\cdot\,, 0) = \mathbf{u}_0 \text{ and } \theta_n(\,\cdot\,, 0) = \theta_0 & \text{in } \Omega, \end{cases} \tag{6}$$

with $\theta_0(\,\cdot\,, t) = \theta_0$ and $\mathbf{u}_0(\,\cdot\,, t) = \mathbf{u}_0$. In Problem (6) and in the sequel we drop the symbol P_σ making reference to the projection on free divergence spaces.

3.1. On the Navier–Stokes problem with non constant viscosity. Let us consider the problem

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \operatorname{div}(\mu(\widehat{\theta})D(\mathbf{u})) = \mathbf{F}(\widehat{\theta}) & \text{in } Q_T, \\ \mathbf{u} = \mathbf{0} & \text{on } \Sigma_T, \\ \mathbf{u}(\,\cdot\,, 0) = \mathbf{u}_0 & \text{in } \Omega, \end{cases} \tag{7}$$

with the change in notation from $\mathbf{u}_n$, θ_{n-1} to $\mathbf{u}$, $\widehat{\theta}$. Problem (7) corresponds to the usual Navier–Stokes problem but with viscosity depending upon the spatial and time variables.

Lemma 2. *Assume* (H_1)–(H_3) *and $\widehat{\theta} \in L^\infty(Q_T)$. Then, there exists a weak solution of Problem* (7) *such that*

$$\mathbf{u} \in \mathcal{C}([0, T]; W_\sigma^{-1,2}(\Omega)).$$

Remark 2. The following result is a consequence of Sobolev's Theorem: the imbedding $L^r(Q_T) \subset L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ is continuous for

$$r := 4\left(1 - \frac{1}{2^*}\right) = \begin{cases} 4 - \varepsilon & \text{if } N = 2, \text{ for all } \varepsilon > 0, \\ 10/3 & \text{if } N = 3. \end{cases} \tag{8}$$

Proof of Lemma 2. We closely follows the proof given in [24], which we divide in two steps.

Step 1. We consider the basis $\{\mathbf{v}^j\}_{j \in \mathbb{N}}$ of $W_\sigma^{s,2}(\Omega)$, $s = N/2$, given by the solutions of the spectral problem

$$\langle \mathbf{v}, \mathbf{w} \rangle_{W_\sigma^{s,2}(\Omega)} = \lambda \langle \mathbf{v}, \mathbf{w} \rangle_{L_\sigma^2(\Omega)}, \quad \text{for all } \mathbf{w} \in W_\sigma^{s,2}(\Omega),$$

$\lambda > 0$, and the corresponding m-dimensional space $\mathcal{V}^m$ spanned by $\mathbf{v}^1, \dots, \mathbf{v}^m$. We define

$$\mathbf{u}_m(t) := \sum_{j=1}^{m} h_j(t)\mathbf{v}^j, \tag{9}$$

with h_j to be determined, and set the problem

$$\begin{cases} \langle \mathbf{u}_{mt}, \mathbf{w}\rangle_{L^2_\sigma(\Omega)} + a_{\widehat{\theta}}(\mathbf{u}_m, \mathbf{w}) + b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{w}) = \langle \mathbf{F}(\widehat{\theta}), \mathbf{w}\rangle_{L^2_\sigma(\Omega)}, \\ \mathbf{u}_m(0) = \mathbf{u}_{m0} := \sum_{j=1}^m \langle \mathbf{u}_0, \mathbf{v}^j\rangle_{L^2_\sigma(\Omega)} \mathbf{v}^j, \end{cases} \tag{10}$$

a.e. in $(0,T)$, for any $\mathbf{w} \in \mathcal{V}^m$. We have

$$\lim_{m\to\infty} \|\mathbf{u}_{m0} - \mathbf{u}_0\|_{L^2_\sigma(\Omega)} = 0. \tag{11}$$

Introducing in (10) the expression of $\mathbf{u}_m$ given in (9) and taking $\mathbf{w} = \mathbf{v}^k$, $k = 1, \dots, m$, we obtain the following system of ordinary differential equations:

$$\begin{aligned} \sum_{j=1}^{m} \langle \mathbf{v}^j, \mathbf{v}^k\rangle_{L^2_\sigma(\Omega)} h_j'(t) \;+\; &\sum_{i,j=1}^{m} b(\mathbf{v}^j, \mathbf{v}^i, \mathbf{v}^k) h_i(t) h_j(t) \\ &- \sum_{j=1}^{m} a_{\widehat{\theta}}(\mathbf{v}^j, \mathbf{v}^k)(t) h_j(t) = f(\mathbf{v}^k)(t), \end{aligned} \tag{12}$$

with $f(\mathbf{v}^k)(t) := \langle \mathbf{F}(\widehat{\theta}(t)), \mathbf{v}^k\rangle_{L^2_\sigma(\Omega)}$, to which we impose the initial condition $h_k(0) = \langle \mathbf{u}_0, \mathbf{v}^k\rangle_{L^2_\sigma(\Omega)}$. Since $\widehat{\theta} \in L^\infty(Q_T)$ and both μ and $\mathbf{F}$ are locally Lipschitz continuous we deduce $a_{\widehat{\theta}}(\mathbf{v}^j, \mathbf{v}^k), f(\mathbf{v}^k) \in L^\infty(0,T)$ for all j, k, with $1 \le j, k \le m$. Therefore, we can express (12) in the form

$$\begin{cases} \mathbf{h}'(t) = \mathbf{g}(t, \mathbf{h}(t)) \quad t \in (0,T), \\ \mathbf{h}(0) = \mathbf{h}_0, \end{cases} \tag{13}$$

with $\mathbf{g}(t, \mathbf{y})$ measurable in the first variable and Lipschitz continuous in the second. By well-known results, we can ensure the existence and uniqueness of a continuous solution of (13) in a maximal interval $(0, T_m)$ with $T_m > 0$.

Step 2. In the second part of the proof we show that from a priori estimates on the approximate problems we can deduce $T_m = T$ for all $m \in \mathbb{N}$ and that the passing to the limit defines $\lim_{m\to\infty} \mathbf{u}_m$ as a weak solution of Problem (7). Since $\mathbf{u}_m(t) \in \mathcal{V}^m$ we may take $\mathbf{w} = \mathbf{u}_m(t)$ in (10), obtaining, due to the anti-symmetry of b,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_m\|^2_{L^2_\sigma(\Omega)} + a_{\widehat{\theta}}(\mathbf{u}_m, \mathbf{u}_m) = \langle \mathbf{F}(\widehat{\theta}), \mathbf{u}_m\rangle_{L^2_\sigma(\Omega)}$$

in $(0, T_m)$. Applying Hölder and Young's inequalities we deduce

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_m\|^2_{L^2_\sigma(\Omega)} + m_0\|\mathbf{u}_m\|^2_{W^{1,2}_\sigma(\Omega)} \le c\|\mathbf{F}(\widehat{\theta}(t))\|^2_{W^{-1,2}_\sigma(\Omega)}.$$

Integrating in $(0,t)$ we obtain

$$\|\mathbf{u}_m\|_{L^\infty(0,T;L^2_\sigma(\Omega))} + \|\mathbf{u}_m\|_{L^2(0,T;W^{1,2}_\sigma(\Omega))} \leq c(\|\mathbf{F}(\widehat{\theta})\|_{L^2(0,T;W^{-1,2}_\sigma(\Omega))} + \|\mathbf{u}_{m0}\|_{L^2_\sigma(\Omega)}), \tag{14}$$

with c independent of m. Using the Lipschitz continuity of $\mathbf{F}$, $\widehat{\theta} \in L^\infty(Q_T)$ and (11) we deduce $T_m = T$ and

$$\mathbf{u}_m \text{ bounded in } L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;W^{1,2}_\sigma(\Omega)).$$

On the other hand, if we denote the orthogonal projection of $L^2_\sigma(\Omega)$ into $\mathcal{V}^m$ by P_m, we obtain from the equation of Problem (7)

$$\mathbf{u}_{mt} = -P_m((\mathbf{u}_m \cdot \nabla)\mathbf{u}_m) - P_m \operatorname{div}(\mu(\widehat{\theta})D(\mathbf{u}_m)) + P_m\mathbf{F}.$$

Using (14) and the special choice of the basis of $\mathcal{V}^m$ we deduce that the sequences $P_m((\mathbf{u}_m \cdot \nabla)\mathbf{u}_m)$ and $P_m \operatorname{div}(\mu(\widehat{\theta})D(\mathbf{u}_m))$ remain bounded in $L^2(0,T;W^{-s,2}_\sigma(\Omega))$, as clearly also happens with $P_m\mathbf{F}$. Then we have

$$\mathbf{u}_{mt} \quad \text{bounded in} \quad L^2(0,T;W^{-s,2}_\sigma(\Omega)).$$

This property together with (14) allows us to use [24, Theorem 5.1], to deduce the existence of a subsequence, again denoted by $\mathbf{u}_m$, such that

$$\mathbf{u}_m \to \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,2}_\sigma(\Omega)), \tag{15}$$

$$\mathbf{u}_m \to \mathbf{u} \text{ weakly } * \text{ in } L^\infty(0,T;L^2_\sigma(\Omega)),$$

$$\mathbf{u}_m \to \mathbf{u} \text{ strongly in } L^2_\sigma(Q_T)), \text{ and a.e. in } Q_T, \tag{16}$$

$$\mathbf{u}_{mt} \to \mathbf{u}_t \text{ weakly in } L^2(0,T;W^{-s,2}_\sigma(\Omega)). \tag{17}$$

From (15) and (17) we deduce $\mathbf{u}_m(0) \to \mathbf{u}(0)$ weakly in $W^{-s,2}_\sigma(\Omega)$ and, therefore, $\mathbf{u}(\,\cdot\,,0) = \mathbf{u}_0$ a.e. in Ω. Thanks to Remark 2 the product $(u_i)_m(u_j)_m$ is bounded in $L^{r/2}(Q_T)$, so there exists $v_{ij} \in L^{r/2}(Q_T)$ such that

$$(u_i)_m(u_j)_m \to v_{ij} \text{ weakly in } L^{r/2}(Q_T).$$

Due to (16), $v_{ij} = u_i u_j$, so we deduce

$$(\mathbf{u}_m \cdot \nabla)\mathbf{u}_m \to (\mathbf{u} \cdot \nabla)\mathbf{u} \text{ weakly in } L^{r/2}(Q_T).$$

From (15) and $\widehat{\theta} \in L^\infty(Q_T)$ we obtain

$$\operatorname{div}(\mu(\widehat{\theta})D(\mathbf{u}_m)) \to \operatorname{div}(\mu(\widehat{\theta})D(\mathbf{u})) \quad \text{weakly in } L^2(0,T;W^{-1,2}_\sigma(\Omega)).$$

Hence, $\mathbf{u}$ satisfies Problem (7) in the weak sense. Finally, by [32, Lemma 3.1], $(\mathbf{u} \cdot \nabla)\mathbf{u} \in L^1(0,T;W^{-1,2}_\sigma(\Omega))$ and then we deduce $\mathbf{u}_t \in L^1(0,T;W^{-1,2}_\sigma(\Omega))$. Hence,

$$\mathbf{u} \in \mathcal{C}([0,T],W^{-1,2}_\sigma(\Omega)). \qquad \square$$

3.2. The non linear diffusion equation with prescribed convection. We pass to analyze the second problem arising from Problem (6). We again use the notation $(\widehat{\mathbf{u}}, \theta)$ instead of $(\mathbf{u}_{n-1}, \theta_n)$. Let us consider the problem

$$\begin{cases} \theta_t + \widehat{\mathbf{u}} \cdot \nabla\theta - \Delta\varphi(\theta) = 0 & \text{in } Q_T, \\ \varphi(\theta) = \phi_D & \text{on } \Sigma_T, \\ \theta(\,\cdot\,, 0) = \theta_0 & \text{in } \Omega. \end{cases} \tag{18}$$

Lemma 3. *Assume* (H_1)–(H_3) *and* $\widehat{\mathbf{u}} \in L^r_\sigma(Q_T)$, *with* r *given by* (8). *Then, there exists a weak solution of Problem* (18) *such that*

$$\theta \in \mathcal{C}([0,T]; H^{-1}(\Omega)). \tag{19}$$

Moreover, if there exist non-negative constants $k, m, \lambda_0, \lambda_1$ *such that*

$$k \geq \theta_0 \geq m \geq 0 \quad \textit{and} \quad \varphi(ke^{\lambda_0 t}) \geq \phi_D(\,\cdot\,, t) \geq \varphi(me^{-\lambda_1 t}) \geq 0$$

a.e. in Ω *and* Σ_T, *respectively, then there exists a constant* $\lambda \geq 0$ *independent of* φ *such that*

$$ke^{\lambda t} \geq \theta(\,\cdot\,, t) \geq me^{-\lambda t} \geq 0 \tag{20}$$

for $t \in (0,T)$ *and a.e. in* Ω.

Proof of Lemma 3. We proceed by approximation. Consider the problem

$$\begin{cases} \theta_t + \widehat{\mathbf{u}}_j \cdot \nabla\theta - \Delta\varphi(\theta) = 0 & \text{in } Q_T, \\ \varphi(\theta) = \phi_D & \text{on } \Sigma_T, \\ \theta(\,\cdot\,, 0) = \theta_0 & \text{in } \Omega, \end{cases} \tag{21}$$

with $\widehat{\mathbf{u}}_j \in L^r_\sigma(Q_T)$ satisfying

$$\|\widehat{\mathbf{u}}_j\|_{L^\infty_\sigma(Q_T)} \leq j \quad \text{and} \quad \widehat{\mathbf{u}}_j \to \widehat{\mathbf{u}} \quad \text{in } L^r_\sigma(Q_T) \quad \text{as } j \to \infty. \tag{22}$$

The existence of a weak solution, θ_j, of Problem (21) satisfying (19) and (20) is a well known result, see e.g. [1]. Using $\varphi(\theta) - \phi_D$ as a test function we obtain the uniform estimate

$$\|\Phi(\theta_j)\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla\varphi(\theta_j)\|_{L^2(Q_T)} \leq \Lambda, \tag{23}$$

with $\Phi(s) = \int_0^s \varphi(\sigma)\,d\sigma$ and Λ a constant independent of j. Since $r > 2$, from (22) we also obtain a uniform estimate for $\widehat{\mathbf{u}}_j$ in $L^2(Q_T)$. Using the equation of Problem (21) and (23) we may estimate $\|\theta_{jt}\|_{L^2(0,T;H^{-1}(\Omega))}$ uniformly in j. Hence we can extract some subsequences $\varphi(\theta_j)$ and θ_j such that

$$\begin{aligned} &\theta_j \to \theta \text{ weakly } * \text{ in } L^\infty(Q_T), \\ &\varphi(\theta_j) \to \psi \text{ weakly in } L^2(0,T; H^1_0(\Omega)), \\ &\theta_{jt} \to \theta_t \text{ weakly in } L^2(0,T; H^{-1}(\Omega)). \end{aligned}$$

From the compact imbedding $L^\infty(\Omega) \subset H^{-1}(\Omega)$ and [30, Corollary 4, p. 85], we deduce

$$\theta_j \to \theta \quad \text{in } \mathcal{C}([0,T], H^{-1}(\Omega)).$$

Moreover, using that the formal operator $-\Delta\varphi(\,\cdot\,)$ is a maximal monotone operator in $L^2(0,T;H^{-1}(\Omega))$ and therefore it is strongly-weakly closed we deduce that necessarily $\varphi(\theta) = \psi$, see [4]. Finally, since $\widehat{\mathbf{u}}_j \to \widehat{\mathbf{u}}$ in $L^r_\sigma(Q_T)$ and $r > 2$ we obtain

$$\int_{Q_T} \theta_j \widehat{\mathbf{u}}_j \cdot \nabla\zeta \to \int_{Q_T} \theta\widehat{\mathbf{u}} \cdot \nabla\zeta \quad \text{for any } \zeta \in L^2(0,T;H^1_0(\Omega)),$$

so the limit is identified as a solution of Problem (18). To finish, we note that since θ_j satisfies (20) for all j, this property also holds in the limit. □

End of proof of Theorem 1. We again consider Problem (6). Thanks to Lemmas 2 and 3 we have that, for each $n \in \mathbb{N}$, there exist functions $\mathbf{u}_n$, θ_n, solutions of Problem (6), such that

$$\begin{aligned}
&\mathbf{u}_n \text{ is bounded in } L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;W^{1,2}_\sigma(\Omega)),\\
&\mathbf{u}_{nt} \text{ is bounded in } L^2(0,T;W^{-s,2}_\sigma(\Omega)),\\
&\theta_n \text{ is bounded in } L^\infty(Q_T),\\
&\varphi(\theta_n) \text{ is bounded in } L^\infty(Q_T) \cap L^2(0,T;H^1_0(\Omega)),
\end{aligned}$$

uniformly with respect to n. We may, then, extract subsequences such that

$$\begin{aligned}
&\mathbf{u}_n \to \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,2}_\sigma(\Omega)),\\
&\mathbf{u}_n \to \mathbf{u} \text{ strongly in } L^2_\sigma(Q_T)) \text{ and a.e. in } Q_T,\\
&\mathbf{u}_{nt} \to \mathbf{u}_t \text{ weakly in } L^2(0,T;W^{-s,2}_\sigma(\Omega))
\end{aligned}$$

and

$$\begin{aligned}
&\theta_n \to \theta \text{ weakly } * \text{ in } L^\infty(Q_T),\\
&\varphi(\theta_n) \to \psi \text{ weakly in } L^2(0,T;H^1_0(\Omega)),\\
&\theta_{nt} \to \theta_t \text{ weakly in } L^2(0,T;H^{-1}(\Omega)).
\end{aligned}$$

As in the proof of Lemma 3 we deduce $\theta_n \to \theta$ in $\mathcal{C}([0,T];H^{-1}(\Omega))$ and $\varphi(\theta) = \psi$.

We now turn to study the passing to the limit of the coupling terms of Problem (3). First, since $\mathbf{u}_n \to \mathbf{u}$ strongly in $L^2_\sigma(Q_T)$ and $\theta_n \to \theta$ weakly $*$ in $L^\infty(Q_T)$ we deduce

$$\int_{Q_T} \theta_n \mathbf{u}_n \cdot \nabla\zeta \to \int_{Q_T} \theta\mathbf{u} \cdot \nabla\zeta \quad \text{for any } \zeta \in L^2(0,T;H^1_0(\Omega)).$$

Next we consider the alternative in Assumption (H_4). In the trivial case $\mu' \equiv \mathbf{F}' \equiv 0$ we directly deduce $\operatorname{div}(\mu D(\mathbf{u}_n)) = \mu\operatorname{div}(D(\mathbf{u}_n)) \to \mu\operatorname{div}(D(\mathbf{u}))$ weakly in $L^2(0,T;W^{-1,2}_\sigma(\Omega))$. Let us then assume $\mu' \neq 0$ either $\mathbf{F}' \neq 0$. To pass to the limit we shall prove $\theta_n \to \theta$ strongly in $L^p(Q_T)$ for all $p < \infty$ using a modification of the argument given in [10]. Consider $\alpha > 0$ given in (H_4) and the space

$$\mathcal{H} = \{\theta \in L^{2/\alpha}(0,T;W^{\alpha,2/\alpha}(\Omega)) : \theta_t \in L^2(0,T;H^{-1}(\Omega))\}.$$

Since φ^{-1} is Hölder continuous of exponent α, we have that θ_n is uniformly bounded in $\mathcal{H}$, see [13]. Then, from the compact imbedding $\mathcal{H} \subset L^{2/\alpha}(Q_T)$ we deduce the existence of a subsequence of θ_n such that

$$\theta_n \to \theta \text{ strongly in } L^{2/\alpha}(Q_T) \text{ and a.e. in } Q_T.$$

This fact together with the weak $*$ convergence of θ_n to θ in $L^\infty(Q_T)$ implies that $\theta_n \to \theta$ strongly in $L^p(Q_T)$ for all $p < \infty$. Then, since $\mathbf{F}$ and μ are locally Lipschitz continuous we deduce both

$$\mathbf{F}(\theta_n) \to \mathbf{F}(\theta) \text{ strongly in } L^p(Q_T)$$

and

$$\mu(\theta_n) \to \mu(\theta) \text{ strongly in } L^p(Q_T)$$

for all $p < \infty$. Then

$$\int_{Q_T} \mu(\theta_n) D(\mathbf{u}_n) : \nabla \mathbf{w} \to \int_{Q_T} \mu(\theta) D(\mathbf{u}) : \nabla \mathbf{w},$$

for any $\mathbf{w} \in L^2(0,T;W^{1,2}_\sigma(\Omega))$. Finally, properties (4) and (5) are deduced as in Lemmas 2 and 3. □

Remark 3. (i) If $N = 2$ and $\mu \equiv$ const., it is possible to deduce further regularity on the velocity field. In particular, using formally $\Delta\mathbf{u}$ as test function we obtain

$$\mathbf{u} \in L^\infty(0,T;W^{1,2}_\sigma(\Omega)) \cap L^2(0,T;W^{2,2}_\sigma(\Omega)).$$

From Sobolev's Theorem we deduce $\mathbf{u} \in L^\infty(Q_T)$. It also holds, see [24], $\mathbf{u} \in \mathcal{C}([0,T],L^2_\sigma(\Omega))$

(ii) We point out that assumption (H_4) could be removed by using time discretization arguments as in [1].

4. Uniqueness of solutions

As it is well known, uniqueness of solutions of the Navier–Stokes equations in dimension $N = 3$ is an open problem. We shall, therefore, restrict ourselves to study the uniqueness of solutions of the Boussinesq system in dimension $N = 2$. When the diffusion term of the heat equation is linear, it has been proven that the solution found in the existence theorem is unique, see e.g. [12]. The proof relies in the fact that natural energy spaces for the velocity field and temperature are the same, L^2, making possible to combine their energy relations in a suitable way. However, when the diffusion of heat is not linear, and specially when it is degenerate, a proof of uniqueness of solutions is more involved: natural estimates for the heat equation are obtained in L^1 but proving the well possednes of the Navier–Stokes equations in this space seems to be a difficult task. The $L^2 - L^1$ character of the system makes complicate to apply L^1 techniques developed in the last years which have been successfully used to prove uniqueness for degenerate scalar equations as well as for certain systems of equations for which the comparison principle still holds, see [19], [5], [13], [8], [27]. In this paper we shall approach the problem from a duality technique, i.e., from the search of suitable test functions which allows to conclude the uniqueness property. It is worth strengthening here that no general comparison principle holds for the Boussinesq system, being this one of the main sources of complexity for the problem.

In this section we consider the case when the second order coupling in the viscosity term of the Navier–Stokes equations is not present. For the singular diffusion case this coupling does not involve important difficulties but for the degenerate diffusion case it does, making necessary the assumption of unrealistic conditions on the velocity field. Besides, the most interesting feature of the model is the nonlinear thermal diffusion, specially the degenerate diffusion. We shall therefore leave the problem introduced by the coupling in the viscosity term for future researching and shall study here the uniqueness of solutions of the following problem:

$$
\begin{cases}
\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \Delta\mathbf{u} = \mathbf{F}(\theta) & \text{in } Q_T,\\
\operatorname{div}\mathbf{u} = 0 & \text{in } Q_T,\\
\theta_t + \mathbf{u}\cdot\nabla\theta - \Delta\varphi(\theta) = 0 & \text{in } Q_T,\\
\mathbf{u} = \mathbf{0} \text{ and } \varphi(\theta) = \phi_D & \text{on } \Sigma_T,\\
\mathbf{u}(\,\cdot\,,0) = \mathbf{u}_0 \text{ and } \theta(\,\cdot\,,0) = \theta_0 & \text{in } \Omega.
\end{cases} \tag{24}
$$

We first present the result on singular diffusion.

Theorem 4. *Let $N = 2$ and $\mu \equiv 1$. Assume $\varphi^{-1} \in \mathcal{C}^{0,1}([0,\infty))$. Then, under conditions of Theorem* 1*, there exists a unique weak solution of Problem* (24)*.*

Remark 4. By Theorem 1 we have $\nabla\varphi(\theta) \in L^2(Q_T)$. Then, assumption $\varphi^{-1} \in \mathcal{C}^{0,1}([0,\infty))$ implies $\nabla\theta \in L^2(Q_T)$.

Proof. Suppose there exist two weak solutions $(\mathbf{u}_1,\theta_1)$, $(\mathbf{u}_2,\theta_2)$ and define $(\mathbf{u},\theta) := (\mathbf{u}_1 - \mathbf{u}_2, \theta_1 - \theta_2)$ and $\mathbf{F}_i := \mathbf{F}(\theta_i)$. Then $(\mathbf{u},\theta)$ satisfies:

$$
\begin{cases}
\mathbf{u}_t + (\mathbf{u}_1\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}_2 - \Delta\mathbf{u} = \mathbf{F}_1 - \mathbf{F}_2 & \text{in } Q_T,\\
\operatorname{div}\mathbf{u} = 0 & \text{in } Q_T,\\
\theta_t + \mathbf{u}_1\cdot\nabla\theta + \mathbf{u}\cdot\nabla\theta_2 - \Delta(\varphi(\theta_1) - \varphi(\theta_2)) = 0 & \text{in } Q_T,\\
\mathbf{u} = \mathbf{0} \text{ and } \varphi(\theta_1) - \varphi(\theta_2) = 0 & \text{on } \Sigma_T,\\
\mathbf{u}(\,\cdot\,,0) = \mathbf{0} \text{ and } \theta(\,\cdot\,,0) = 0 & \text{in } \Omega.
\end{cases}
$$

For a.e. $t \in (0,T)$ we consider smooth test functions $\mathbf{w}(\,\cdot\,,t)$ and ξ such that $\mathbf{w}(\,\cdot\,,T) = \mathbf{0}$ in Ω. Integrating by parts and adding the resulting integral identities we get

$$\begin{aligned}
\int_\Omega \theta(T)\xi(T) &= \int_{Q_T} \mathbf{u}\cdot\big(\mathbf{w}_t + (\mathbf{u}_1\cdot\nabla)\mathbf{w} + \Delta\mathbf{w}\big) + \int_{Q_T} \mathbf{u}_2\cdot(\mathbf{u}\cdot\nabla)\mathbf{w} \\
&\quad + \int_{Q_T} (\mathbf{F}_1 - \mathbf{F}_2)\cdot\mathbf{w} + \int_{Q_T} \theta(\xi_t + \mathbf{u}_1\cdot\nabla\xi) \\
&\quad - \int_{Q_T} \xi\mathbf{u}\cdot\nabla\theta_2 + \int_{Q_T} (\varphi(\theta_1) - \varphi(\theta_2))\Delta\xi.
\end{aligned} \tag{25}$$

We define the differential operator $\mathcal{L} : L^2(0,T;W^{1,2}_\sigma(\Omega)) \to L^2(Q_T)$ by

$$\mathbf{u}\cdot(\mathcal{L}\mathbf{w} : \mathbf{u}_2) := \mathbf{u}_2\cdot(\mathbf{u}\cdot\nabla)\mathbf{w} = \mathbf{u}\cdot\left(\frac{\partial\mathbf{w}}{\partial x}\cdot\mathbf{u}_2, \frac{\partial\mathbf{w}}{\partial y}\cdot\mathbf{u}_2\right), \tag{26}$$

with $\mathbf{x} := (x,y)$. It is straightforward to check that $\mathcal{L}$ is linear and continuous. Adding and substracting in (25) the term θh_m, with $h_m : Q_T \to [0,m]$ given by

$$h_m := \begin{cases} h & \text{if } h \le m, \\ m & \text{if } h > m, \end{cases}$$

with $m > 0$ a constant large enough ($m > h_0$, see (28) below), and $h : Q_T \to [0,\infty)$ given by

$$h := \begin{cases} (\varphi(\theta_1) - \varphi(\theta_2)/\theta & \text{if } \theta \neq 0, \\ 0 & \text{if } \theta = 0, \end{cases}$$

and using (26) we obtain from (25)

$$\begin{aligned}
\int_\Omega \theta(T)\xi(T) &= \int_{Q_T} \mathbf{u}\cdot(\mathbf{w}_t + (\mathbf{u}_1\cdot\nabla)\mathbf{w} + \mathcal{L}\mathbf{w} : \mathbf{u}_2 - \xi\nabla\theta_2 + \Delta\mathbf{w}) \\
&\quad + \int_{Q_T} \theta(\xi_t + \mathbf{u}_1\cdot\nabla\xi + \mathbf{f}\cdot\mathbf{w} + h_m\Delta\xi) \\
&\quad + \int_{Q_T} (h - h_m)\theta\Delta\xi,
\end{aligned} \tag{27}$$

with $\mathbf{f} : Q_T \to \mathbb{R}^2$ given by

$$\mathbf{f} := \begin{cases} \big(\mathbf{F}(\theta_1) - \mathbf{F}(\theta_2)\big)/\theta & \text{if } \theta \neq 0, \\ 0 & \text{if } \theta = 0. \end{cases}$$

Note that since $\mathbf{F}$ and φ^{-1} are Lipschitz continuous then $\mathbf{f} \in L^\infty_\sigma(Q_T)$ and there exists a constant $h_0 > 0$ such that

$$h > h_0 \quad \text{a.e. in } Q_T. \tag{28}$$

We set the following problem to choose the test functions:

$$
(29)\qquad \begin{cases} \mathbf{w}_t + (\mathbf{u}_1\cdot\nabla)\mathbf{w} + \mathcal{L}\mathbf{w} : \mathbf{u}_2 - \xi\nabla\theta_2 + \Delta\mathbf{w} = \mathbf{0} & \text{in } Q_T,\\ \operatorname{div}\mathbf{w} = 0 & \text{in } Q_T,\\ \xi_t + \mathbf{u}_1\cdot\nabla\xi + \mathbf{f}\cdot\mathbf{w} + h_m\Delta\xi + \theta = 0 & \text{in } Q_T,\\ \mathbf{w} = \mathbf{0} \quad \text{and} \quad \xi = 0 & \text{on } \Sigma_T,\\ \mathbf{w}(\,\cdot\,,T) = \mathbf{0} \text{ and } \xi(\,\cdot\,,T) = 0 & \text{in } \Omega. \end{cases}
$$

We state here the result on existence, uniqueness and regularity of solutions of Problem (29) and prove it at the end of this section.

Lemma 5. *Problem* (29) *has a unique weak solution with the regularity of test functions of* (24)*, see Definition* 1*. Moreover,*

$$
\begin{aligned} \mathbf{w} &\in H^1(0,T;L^2_\sigma(\Omega))\cap L^\infty(0,T;W^{1,2}_\sigma(\Omega))\cap L^2(0,T;W^{2,2}_\sigma(\Omega)),\\ \xi &\in H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;H^1_0(\Omega))\cap L^2(0,T;H^2(\Omega)), \end{aligned}
$$

and there exists a positive constant C independent of m such that

$$
(30)\qquad \int_{Q_T}|\Delta\xi|^2 \le C.
$$

End of proof of Theorem 4. Using the test functions provided by Lemma 5 we obtain from (27)

$$
\int_{Q_T}\theta^2 = \int_{Q_T}(h-h_m)\theta\Delta\xi.
$$

Now we pass to the limit $m\to\infty$. Since h_m converges pointwise to h and $|\theta(h-h_m)| \le 2|\varphi(\theta_1)+\varphi(\theta_2)| \le \text{const.}$, we obtain from Lebesgue's Theorem $\|(h-h_m)\theta\|_{L^2(Q_T)}\to 0$ as $m\to\infty$. Due to the uniform estimate (30) we deduce

$$
\int_{Q_T}\theta(h-h_m)\Delta\xi \to 0 \quad \text{as } m\to\infty,
$$

and therefore we obtain

$$
\int_{Q_T}\theta^2 = 0.
$$

Hence, $\theta_1=\theta_2$ a.e. in Q_T. Finally, standard arguments for the Navier–Stokes equations in space dimension N=2 imply $\mathbf{u}_1=\mathbf{u}_2$ a.e. in Q_T. □

Our next result is based on the technique introduced in [7] to study the uniqueness of solutions of a one dimensional scalar equation. The method of proof consists on making a comparison between the weak solution, $(\mathbf{u},\theta)$, constructed as the limit of a sequence of solutions, $(\mathbf{u}_\varepsilon,\theta_\varepsilon)$, of approximated problems and an arbitrary weak solution, $(\mathbf{u}_2,\theta_2)$. After regularizing the coeficients of the problem satisfied by $(\mathbf{u}_\varepsilon-\mathbf{u}_2,\theta_\varepsilon-\theta_2)$, see (37), we introduce an auxiliary problem to determine suitable test functions, see (41). The main properties needed for these test functions are an L^∞ uniform bound, see Lemma 7, and smoothness, see Lemma 8. Since the uniform bound is obtained locally in time, we need to restrict a priori the time domain, see (32). The smoothness does not involve any difficulty after the introduction of regularized coefficients above mentioned. In this situation we may pass to the limit in the approximating and regularizating parameters and, by using another auxiliary problem to estimate a singular boundary integral, see Lemma 8, we deduce the uniqueness of solutions locally in time. Finally, due to the uniform continuity of a function involved in the L^∞ bound of the test functions, see (61), we can use a continuation argument to deduce the global uniqueness result.

Theorem 6. *Let $N = 2$, $\mu \equiv 1$ and $\partial\Omega \in \mathcal{C}^2$. Let $\varphi \in \mathcal{C}^1([0,\infty)) \cap \mathcal{C}^2((0,\infty))$, with $\varphi(0) = \varphi'(0) = 0$ and assume the existence of a convex function $\nu \in \mathcal{C}^0([0,\infty)) \cap \mathcal{C}^2((0,\infty))$ such that $\nu(0) = 0$ and*

$$0 < \nu'(s) \le \varphi'(s) \quad \text{for } s > 0. \tag{31}$$

Then, under the conditions of Theorem 1, there exists a unique solution of Problem (24) in the class of weak solutions such that $\nabla\theta \in L^2(Q_T)$.

Proof. Consider the sequence of Problems $(24)_\varepsilon$ in which we approximate solutions of Problem (24) by modifying the initial and boundary data corresponding to θ in the following way:

$$\begin{cases} \varphi(\theta_\varepsilon) = \phi_D + \varphi(\varepsilon e^{-\lambda_1 t}) & \text{on } \Sigma_T, \\ \theta_{0\varepsilon} = \theta_0 + \varepsilon & \text{in } \Omega, \end{cases}$$

for some constant $\lambda_1 > 0$. Theorem 1 ensures that for each $\varepsilon > 0$ Problem $(24)_\varepsilon$ has a solution $(\mathbf{u}_\varepsilon, \theta_\varepsilon)$ satisfying $\theta_\varepsilon \ge \varepsilon e^{-\lambda t}$ a.e. in Q_T, with $\lambda > 0$ independent of φ and ε. Following the same scheme than in Section 3.2 it is possible to prove that $(\mathbf{u}_\varepsilon, \theta_\varepsilon) \to (\mathbf{u}, \theta)$ strongly en $L^2_\sigma(Q_T) \times L^2(Q_T)$ with $(\mathbf{u}, \theta)$ a weak solution of Problem (24). Now let us suppose that there exists another weak solution $(\mathbf{u}_2, \theta_2)$ of Problem (24) and consider the function $g : [0,T] \to [0,\infty)$ defined by

$$g(t) := \max\{c\|\nabla\theta_2\|_{L^2(Q_t)}, k_0|\Omega|t\}, \tag{32}$$

with c, k_0 positive constants. Since g is uniformly continuous in $[0,T]$, there exists a positive constant δ, independent of t, such that if $|t_1 - t_2| < \delta$ then $|g(t_1) - g(t_2)| < 1/2$ for all $t_1, t_2 \in [0,T]$. We consider now Problems (24), $(24)_\varepsilon$ and their corresponding solutions restricted to the domain $\Omega \times (0, T^*)$, with $T^* < \delta$. In the following discussion we shall prove uniqueness of solutions in such domain. Once this is done, the proof of uniqueness in Q_T, for any $T < \infty$, follows from a continuation argument due to the uniform continuity of g. To simplify the notation we shall write T insted of T^*.

The pair $(\mathbf{U}_\varepsilon, \Theta_\varepsilon) := (\mathbf{u}_\varepsilon - \mathbf{u}_2, \theta_\varepsilon - \theta_2)$ satisfies

$$(33)\quad \begin{cases} \mathbf{U}_{\varepsilon t} + (\mathbf{u}_\varepsilon \cdot \nabla)\mathbf{U}_\varepsilon + (\mathbf{U}_\varepsilon \cdot \nabla)\mathbf{u}_2 - \Delta \mathbf{U}_\varepsilon = \mathbf{F}(\theta_\varepsilon) - \mathbf{F}(\theta_2) & \text{in } Q_T, \\ \operatorname{div} \mathbf{U}_\varepsilon = 0 & \text{in } Q_T, \\ \Theta_{\varepsilon t} + \mathbf{u}_\varepsilon \cdot \nabla \Theta_\varepsilon + \mathbf{U}_\varepsilon \cdot \nabla \theta_2 - \Delta(\varphi(\theta_\varepsilon) - \varphi(\theta_2)) = 0 & \text{in } Q_T, \\ \mathbf{U}_\varepsilon = \mathbf{0} \text{ and } \varphi(\theta_\varepsilon) - \varphi(\theta_2) = \varphi(\varepsilon e^{-\lambda_1 t}) & \text{on } \Sigma_T, \\ \mathbf{U}_\varepsilon(\,\cdot\,, 0) = \mathbf{0} \text{ and } \Theta_\varepsilon(\,\cdot\,, 0) = \varepsilon & \text{in } \Omega. \end{cases}$$

Taking smooth test functions $\mathbf{w}(\,\cdot\,, t)$ and ξ with $\mathbf{w}(\,\cdot\,, T) = \mathbf{0}$, integrating by parts and adding the resulting integral identities we obtain

$$(34)\quad \begin{aligned} \int_\Omega \Theta_\varepsilon(T)\xi(T) = \int_{Q_T} &\mathbf{U}_\varepsilon \cdot [\mathbf{w}_t + (\mathbf{u}_\varepsilon \cdot \nabla)\mathbf{w} + \mathcal{L}\mathbf{w} : \mathbf{u}_2 - \xi \nabla \theta_2 + \Delta \mathbf{w}] \\ &+ \int_{Q_T} \Theta_\varepsilon(\xi_t + \mathbf{u}_\varepsilon \cdot \nabla \xi + \mathbf{f}_\varepsilon \cdot \mathbf{w} + h_\varepsilon \Delta \xi) \\ &- \int_{\Sigma_T} \varphi(\varepsilon e^{-\lambda_1 t}) \nabla \xi \cdot \nu + \varepsilon \int_\Omega \xi(0) \end{aligned}$$

with $\mathcal{L}$ defined by (26), $h_\varepsilon : Q_T \to [0, \infty)$ given by

$$h_\varepsilon := \begin{cases} \big(\varphi(\theta_\varepsilon) - \varphi(\theta_2)\big)/\Theta_\varepsilon & \text{if } \Theta_\varepsilon \neq 0, \\ 0 & \text{if } \Theta_\varepsilon = 0, \end{cases}$$

and $\mathbf{f}_\varepsilon : Q_T \to \mathbb{R}^2$ given by

$$\mathbf{f}_\varepsilon := \begin{cases} \big(\mathbf{F}(\theta_\varepsilon) - \mathbf{F}(\theta_2)\big)/\Theta_\varepsilon & \text{if } \Theta_\varepsilon \neq 0, \\ 0 & \text{if } \Theta_\varepsilon = 0. \end{cases}$$

Since $\mathbf{F}$ is Lipschitz continuous, φ is convex and $\theta_\varepsilon \geq \varepsilon e^{-\lambda T}$ there exist positive constants k_0 and

$$(35)\qquad k(\varepsilon) := \varepsilon^{-1} e^{\lambda T} \varphi(\varepsilon e^{-\lambda T})$$

such that

$$(36)\qquad 0 < k(\varepsilon) \leq h_\varepsilon \leq k_0 \quad \text{and} \quad |\mathbf{f}_\varepsilon| \leq k_0,$$

a.e. in Q_T. We consider regularizing sequences h_ε^n, θ_2^n, $\theta_\varepsilon^n \in \mathcal{C}^\infty(\overline{Q}_T)$ and $\mathbf{u}_\varepsilon^n, \mathbf{u}_2^n, \mathbf{f}_\varepsilon^n \in \mathcal{C}_\sigma^\infty(\overline{Q}_T)$ with the following properties:

$$(37)\quad \begin{aligned} &h_\varepsilon^n \to h_\varepsilon \text{ strongly in } L^2(Q_T), \\ &\theta_2^n \to \theta_2 \text{ and } \theta_\varepsilon^n \to \theta_\varepsilon \text{ strongly in } L^2(0, T; H_0^1(\Omega)), \\ &\mathbf{u}_\varepsilon^n \to \mathbf{u}_\varepsilon \text{ and } \mathbf{u}_2^n \to \mathbf{u}_2 \text{ strongly in } L_\sigma^2(Q_T), \\ &\mathbf{f}_\varepsilon^n \to \mathbf{f}_\varepsilon \text{ strongly in } L^2(0, T; L^2(\Omega)^N), \end{aligned}$$

with h_ε^n monotone increasing with respect to n. From (36), the regularity $\mathbf{u}, \mathbf{u}_2 \in L^\infty_\sigma(Q_T)$, $\theta, \theta_2 \in L^2(0,T;H^1_0(\Omega)) \cap L^\infty(Q_T)$ and the Lipschitz continuity of $\mathbf{F}$ we deduce, for a new constant k_0,

$$0 < k(\varepsilon) \le h_\varepsilon^n \le k_0, \tag{38}$$

$$\max\{\|\mathbf{f}_\varepsilon^n\|_{L^\infty(Q_T)}, \|\mathbf{u}_\varepsilon^n\|_{L^\infty_\sigma(Q_T)}, \|\mathbf{u}_2^n\|_{L^\infty_\sigma(Q_T)}, \|\theta_2^n\|_{L^\infty(Q_T)}, \\ \|\theta_\varepsilon^n\|_{L^\infty(Q_T)}, \|\theta_2^n\|_{L^2(0,T;H^1_0(\Omega))}, \|\theta_\varepsilon^n\|_{L^2(0,T;H^1_0(\Omega))}\} \le k_0, \tag{39}$$

with k_0 independent of n and ε. We rewrite (34) as

$$\begin{aligned} \int_\Omega \Theta_\varepsilon(T)\xi(T) = {} & \int_{Q_T} \mathbf{U}_\varepsilon \cdot [\mathbf{w}_t + (\mathbf{u}_\varepsilon^n \cdot \nabla)\mathbf{w} + \mathcal{L}\mathbf{w} : \mathbf{u}_2^n - \xi\nabla\theta_2^n + \Delta\mathbf{w}] \\ & + \int_{\Omega_T} \mathbf{U}_\varepsilon \cdot [((\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^n) \cdot \nabla)\mathbf{w} + \mathcal{L}\mathbf{w} : (\mathbf{u}_2 - \mathbf{u}_2^n) - \xi\nabla(\theta_2 - \theta_2^n)] \\ & + \int_{\Omega_T} \Theta_\varepsilon(\xi_t + \mathbf{u}_\varepsilon^n \cdot \nabla\xi + \mathbf{f}_\varepsilon^n \cdot \mathbf{w} + h_\varepsilon^n \Delta\xi) \\ & + \int_{Q_T} \Theta_\varepsilon((\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^n) \cdot \nabla\xi + (\mathbf{f}_\varepsilon - \mathbf{f}_\varepsilon^n) \cdot \mathbf{w} + (h_\varepsilon - h_\varepsilon^n)\Delta\xi) \\ & - \int_{\Sigma_T} \varphi(\varepsilon e^{-\lambda_1 t})\nabla\xi \cdot \nu + \varepsilon \int_\Omega \xi(0), \end{aligned} \tag{40}$$

and choose the test functions as solutions of

$$\begin{cases} \mathbf{w}_t + (\mathbf{u}_\varepsilon^n \cdot \nabla)\mathbf{w} + \mathcal{L}\ \mathbf{w} : \mathbf{u}_2^n - \xi\nabla\theta_2^n + \Delta\mathbf{w} = \mathbf{0} & \text{in } Q_T, \\ \xi_t + \mathbf{u}_\varepsilon^n \cdot \nabla\xi + \mathbf{f}_\varepsilon^n \cdot \mathbf{w} + h_\varepsilon^n \Delta\xi + \Theta_\varepsilon^n = 0 & \text{in } Q_T, \\ \operatorname{div} \mathbf{w} = 0 & \text{in } Q_T, \\ \mathbf{w} = \mathbf{0} \text{ and } \xi = 0 & \text{on } \Sigma_T, \\ \mathbf{w}(\,\cdot\,, T) = \mathbf{0} \text{ and } \xi(\,\cdot\,, T) = 0 & \text{in } \Omega. \end{cases} \tag{41}$$

Note the similarity between Problems (41) and (29). We again state the result on existence, uniqueness and regularity of Problem (41) and prove it at the end of this section, together with Lemma 5.

Lemma 7. *There exists a unique solution of Problem* (41) *with* $\mathbf{w} \in \mathcal{C}^{2,1}_\sigma(\overline{Q}_T)$ *and* $\xi \in \mathcal{C}^{2,1}(\overline{Q}_T)$. *Moreover, there exist constants* C_1*, independent of* n*, and* C_2*, independent of* n *and* ε *such that*

$$\max\{\|\mathbf{w}\|_{L^2(0,T;W^{1,2}_\sigma(\Omega))}, \|\xi\|_{L^2(0,T;H^2(\Omega))}\} \le C_1 \tag{42}$$

and

$$\max\{\|\mathbf{w}\|_{L^\infty_\sigma(Q_T)}, \|\xi\|_{L^\infty(Q_T)}\} \le C_2. \tag{43}$$

Continuation of proof of Theorem 6. Using the test functions provided by Lemma 7 we obtain from (40)

$$\begin{aligned}(44)\qquad \int_{Q_T}\Theta_\varepsilon\Theta_\varepsilon^n &= \varepsilon\int_\Omega \xi(0) - \int_{\Sigma_T}\varphi(\varepsilon e^{-\lambda_1 t})\nabla\xi\cdot\nu \\ &+ \int_{Q_T}\mathbf{U}_\varepsilon\cdot[((\mathbf{u}_\varepsilon-\mathbf{u}_\varepsilon^n)\cdot\nabla)\mathbf{w}+\mathcal{L}\mathbf{w}:(\mathbf{u}_2-\mathbf{u}_2^n)+(\theta_2-\theta_2^n)\nabla\xi] \\ &+ \int_{Q_T}\Theta_\varepsilon((\mathbf{u}_\varepsilon-\mathbf{u}_\varepsilon^n)\cdot\nabla\xi+(\mathbf{f}_\varepsilon-\mathbf{f}_\varepsilon^n)\cdot\mathbf{w}+(h_\varepsilon-h_\varepsilon^n)\Delta\xi).\end{aligned}$$

The following lemma, which we prove at the end of this section, allows us to estimate the boundary integral when $\varepsilon\to 0$.

Lemma 8. *Let A_ε, $g\in\mathcal{C}^\infty(\overline{Q}_T)$ with $A_\varepsilon > k(\varepsilon)$, $k(\varepsilon)$ given by* (35), *and $\mathbf{B}\in\mathcal{C}^\infty_\sigma(\overline{Q}_T)$. Let $\psi\in\mathcal{C}^{2,1}(\overline{Q}_T)$ be the solution of*

$$(45)\qquad \begin{cases}\psi_t+\mathbf{B}\cdot\nabla\psi+A_\varepsilon\Delta\psi+g=0 & \text{in } Q_T,\\ \psi=0 & \text{on } \Sigma_T,\\ \psi(\,\cdot\,,T)=0 & \text{in } \Omega.\end{cases}$$

Then, there exists a positive constant, C, depending only on the L^∞ norms of ψ, $\mathbf{B}$ and g, such that

$$\nabla\psi\cdot\nu\ge -C/k(\varepsilon)\qquad \text{on } \Sigma_T.$$

End of proof of Theorem 6. In the sequel we denote by C any constant independent of n and ε and possibly changing from one estimate to another. Applying Lemma 8 with $A_\varepsilon := h_\varepsilon^n$, $\mathbf{B} := \mathbf{u}_\varepsilon^n$ and $g := \mathbf{f}_\varepsilon^n\cdot\mathbf{w}+\Theta_\varepsilon^n$, and using (36) and the bounds for the coefficients and test functions given by (39) and (42)–(43), respectively, we deduce

$$\nabla\xi\cdot\nu\ge -\frac{C}{k(\varepsilon)}\qquad \text{on } \Sigma_T.$$

Letting $n\to\infty$ in (44) and taking into account the uniform estimates given by (42) we obtain

$$\int_{Q_T}\Theta_\varepsilon^2\le\varepsilon\int_\Omega\xi(0)+\frac{C}{k(\varepsilon)}\int_{\Sigma_T}\varphi(\varepsilon e^{-\lambda_1 t}).$$

Due to assumption (31) we have $\varphi(\varepsilon e^{-\lambda_1 t})/k(\varepsilon)\le C\varepsilon$. Therefore, letting $\varepsilon\to 0$ and using again (43) leads to

$$\int_{Q_T}\Theta^2\le 0,$$

and the assertion follows. □

Proof of Lemmas 5 and 7. The proofs of existence and uniqueness of solutions for Problems (29) and (41) are similar so we shall only give the proof for Problem (41). The main difference between Lemmas 5 and 7 are estimates (30) and (43) which we shall prove separately.

1. A priori estimates. We introduce the change of variable $\tau := T - t$ and the change of unknowns $\mathbf{v} := e^{-k\tau}\mathbf{w}$, $\quad \zeta := e^{-k\tau}\xi$, with $k > 0$ to be fixed. Then, if $(\mathbf{w}, \xi)$ is a solution of Problem (41) then $(\mathbf{v}, \zeta)$ is a solution of

$$(46)\quad \begin{cases} \mathcal{L}_1(\mathbf{v},\zeta) := \mathbf{v}_\tau + k\mathbf{v} - (\mathbf{u}_\varepsilon^n \cdot \nabla)\mathbf{v} - \mathcal{L}\ \mathbf{w} : \mathbf{u}_2^n + \zeta\nabla\theta_2^n - \Delta\mathbf{v} = \mathbf{0} & \text{in } Q_T,\\ \mathcal{L}_2(\mathbf{v},\zeta) := \zeta_\tau + k\zeta - \mathbf{u}_\varepsilon^n \cdot \nabla\zeta - \mathbf{f}_\varepsilon^n \cdot \mathbf{v} - h_\varepsilon^n \Delta\zeta - e^{-k\tau}\Theta_\varepsilon^n = 0 & \text{in } Q_T,\\ \operatorname{div}\mathbf{v} = 0 & \text{in } Q_T,\\ \mathbf{v} = \mathbf{0} \text{ and } \zeta = 0 & \text{on } \Sigma_T,\\ \mathbf{v}(\,\cdot\,,0) = \mathbf{0} \text{ and } \zeta(\,\cdot\,,0) = 0 & \text{in } \Omega. \end{cases}$$

Multiplying the first equation of Problem (46) by $\mathbf{v}$ we obtain, for all $\gamma > 0$ and for a constant c depending only on k_0, see (38)–(39), and the constant of continuity of $\mathcal{L}$, see (26),

$$(47)\qquad \int_\Omega |\mathbf{v}(T)|^2 + \left(k - \frac{c}{\gamma}\right)\int_{Q_T} |\mathbf{v}|^2 + \int_{Q_T} |\nabla\mathbf{v}|^2 \le \gamma I,$$

with I given by either $I := \int_{Q_T} |\zeta||\nabla\theta_2|$ or $I := \int_{Q_T} |\nabla\zeta||\theta_2|$. Multiplying the first equation of Problem (46) by $\mathbf{v}_\tau$ we obtain

$$(48)\quad \int_{Q_T} |\mathbf{v}_\tau|^2 + \int_\Omega (k|\mathbf{v}(T)|^2 + |\nabla\mathbf{v}(T)|^2) \le \int_{Q_T} |\nabla\mathbf{v}|^2 + \|\zeta\|^2_{L^\infty(Q_T)} \int_{Q_T} |\nabla\theta_2|^2.$$

Finally, multiplying the first equation of Problem (46) by $-\Delta\mathbf{v}$ we have

$$(49)\quad \int_\Omega |\nabla\mathbf{v}(T)|^2 + (k - c)\int_{Q_T} |\nabla\mathbf{v}|^2 + \int_{Q_T} |\Delta\mathbf{v}|^2 \le \|\zeta\|^2_{L^\infty(Q_T)} \int_{Q_T} |\nabla\theta_2|^2.$$

Multiplying now the second equation of Problem (46) by $\Delta\zeta$ we obtain

$$(50)\quad \int_\Omega |\nabla\zeta(T)|^2 + \left(k - \frac{c}{h_0}\right)\int_{Q_T} |\nabla\zeta|^2 + \frac{h_0}{2}\int_{Q_T} |\Delta\zeta|^2 \le \frac{c}{h_0}\int_{Q_T} |\mathbf{v}|^2 + \int_{Q_T} |\nabla\Theta|^2,$$

with $h_0 := \inf_{Q_T} h_\varepsilon^n \ge k(\varepsilon)$, see (36), and with $\Theta := \theta - \theta_2$. Adding (47), (48) and (49) we find, for k large enough

$$(51)\qquad \int_{Q_T} (|\mathbf{v}|^2 + |\nabla\mathbf{v}|^2 + |\Delta\mathbf{v}|^2 + |\mathbf{v}_\tau|^2) \le c\|\zeta\|^2_{L^\infty(Q_T)} \int_{Q_T} |\nabla\theta_2|^2,$$

and therefore, by Sobolev's theorem

$$(52)\qquad \|\mathbf{v}\|_{L^\infty_\sigma(Q_T)} \le c\|\zeta\|_{L^\infty(Q_T)} \|\nabla\theta_2\|_{L^2(Q_T)}.$$

From (47) and (50) we also find

$$(53)\qquad \int_{Q_T} (|\mathbf{v}|^2 + |\nabla\mathbf{v}|^2 + |\nabla\zeta|^2 + h_0|\Delta\zeta|^2) \le c\int_{Q_T} (|\nabla\Theta|^2 + \theta_2^2).$$

Finally, by the Alexandrov's maximum principle, see [20], we have

$$\|\zeta\|_{L^\infty(Q_T)} \le k_0|\Omega|T\|\mathbf{v}\|_{L^\infty_\sigma(Q_T)} + \|\Theta\|_{L^1(0,T;L^\infty(\Omega))}. \tag{54}$$

In estimates (47)–(54) we used the uniform estimates provided by (39). Note that once these estimates are justified we shall obtain (42) from (53).

2. Existence of solutions. Consider the set

$$K := \{h \in L^2(Q_{T_0}) : \|h\|_{L^\infty(Q_{T_0})} < R\},$$

with $R > 0$ and $T_0 < T$ to be fixed. Remind that T has been chosen such that function g defined by (32) satisfies $|g(T)| \le 1/2$. We consider the operator $Q : K \to L^2(Q_{T_0})$ defined by $Q(\widehat{\xi}) := \xi$, with ξ solution of

$$\begin{cases} \mathcal{L}_2(\widehat{\mathbf{v}}, \zeta) = 0 & \text{in } Q_{T_0}, \\ \zeta = 0 & \text{on } \partial\Sigma_{T_0}, \\ \zeta(\,\cdot\,, 0) = 0 & \text{in } \Omega, \end{cases} \tag{55}$$

and $\widehat{\mathbf{v}}$ solution of

$$\begin{cases} \mathcal{L}_1(\widehat{\mathbf{v}}, \widehat{\zeta}) = 0 & \text{in } Q_{T_0}, \\ \widehat{\mathbf{v}} = 0 & \text{on } \partial\Sigma_{T_0}, \\ \widehat{\mathbf{v}}(\,\cdot\,, 0) = \mathbf{0} & \text{in } \Omega, \end{cases} \tag{56}$$

with $\mathcal{L}_1$ and $\mathcal{L}_2$ given in (46). Note that a fixed point of Q is a local in time solution of (46). Since $\widehat{\zeta} \in L^\infty(Q_{T_0})$, Problem (56) has a unique solution with the regularity stated in Lemma 7, see [21]. Therefore, estimates (47)–(49), (51) and (52) are valid for the solution of Problem (56). Due to the regularity of the solution of Problem (56), Problem (55) has a unique solution with the regularity stated in Lemma 7, see [22]. Hence, the a priori estimates (50) and (54) are valid for the solution of Problem (55). To show the existence of a fixed point of Q we use the fixed point theorem of [2], which consists on checking that:

(i) K is convex and weakly compact in $L^2(Q_{T_0})$,
(ii) $Q(K) \subset K$, and
(iii) Q is weakly-weakly sequentially continuous in $L^2(Q_{T_0})$.

Point (i) is straightforward. (ii) Given $\widehat{\zeta} \in K$ we have by (49) that the solution of Problem (56) satisfies

$$\|\widehat{\mathbf{v}}\|_{L^\infty_\sigma(Q_{T_0})} \le \|\nabla\theta_2\|_{L^2(Q_{T_0})}\|\widehat{\zeta}\|_{L^\infty(Q_{T_0})}, \tag{57}$$

and for this $\widehat{\mathbf{v}}$ we have, by (54), that the solution of Problem (55) verifies

$$\|\zeta\|_{L^\infty(Q_{T_0})} \le k_0|\Omega|T_0\|\widehat{\mathbf{v}}\|_{L^\infty_\sigma(Q_{T_0})} + \|\Theta\|_{L^1(0,T_0;L^\infty(\Omega))}. \tag{58}$$

Combining (57) and (58) and using the definition of K we deduce

$$\|\zeta\|_{L^\infty(Q_{T_0})} \le k_0|\Omega|T_0\|\nabla\theta_2\|_{L^2(Q_{T_0})}R + \|\Theta\|_{L^1(0,T_0;L^\infty(\Omega))}. \tag{59}$$

Since the right hand side of (59) tends to zero when $T_0 \to 0$, it is sufficient to take T_0 small enough to obtain

$$\|Q(\widehat{\zeta})\|_{L^\infty(Q_{T_0})} := \|\zeta\|_{L^\infty(Q_{T_0})} < R. \tag{60}$$

Note that these estimates are independent of ε and n. (iii) To show the continuity we consider a sequence $\widehat{\zeta}_j \in K$ with $\widehat{\zeta}_j \to \widehat{\zeta}$ weakly in $L^2(Q_{T_0})$ and prove that $Q(\widehat{\zeta}_j) \to Q(\zeta)$ weakly in $L^2(Q_{T_0})$. Since $\widehat{\zeta}_j$ is bounded in $L^\infty(Q_{T_0})$ it follows from (51) and (52) that the sequence of solutions $\widehat{\mathbf{v}}_j$ of Problem (56) corresponding to $\widehat{\zeta}_j$ is bounded in $L^\infty_\sigma(Q_{T_0}) \cap L^2(0,T_0;W^{2,2}_\sigma(\Omega))$ and therefore there exists a subsequence $\widehat{\mathbf{v}}_j$ such that $\widehat{\mathbf{v}}_j \to \widehat{\mathbf{v}}$ weakly $*$ in $L^\infty_\sigma(Q_{T_0})$, strongly in $L^2(0,T_0;W^{1,2}_\sigma(\Omega))$ and a.e. in Q_{T_0}. Linearity and smoothness of the coefficients of Problem (56) allows us to identify $\widehat{\mathbf{v}}$ as the solution of Problem (56) corresponding to $\widehat{\zeta}$. On the other hand, since $\widehat{\mathbf{v}}_j$ is bounded in $L^\infty_\sigma(Q_{T_0}) \cap L^2(0,T_0;W^{1,2}_\sigma(\Omega))$ it follows from (50) and (54) that the sequence of solutions ζ_j of Problem (55) corresponding to $\widehat{\mathbf{v}}_j$ which, by definition, is $Q(\widehat{\zeta}_j)$, is uniformly bounded in $L^\infty(Q_{T_0}) \cap L^2(0,T_0;H^1_0(\Omega))$ and therefore converges weakly in $L^2(Q_{T_0})$ to an element ζ of K. Again the linearity allows us to identify the limit as the solution of Problem (55) corresponding to $\widehat{\mathbf{v}}$, and therefore to $\widehat{\zeta}$. Hence, the continuity of Q is established and the local existence result follows. Note that adding the estimates (52) and (54) we obtain

$$\|\zeta\|_{L^\infty(Q_{T_0})} + \|\mathbf{v}\|_{L^\infty_\sigma(Q_{T_0})} \leq \frac{1}{1-g(T_0)}\|\Theta\|_{L^1(0,T_0;L^\infty(\Omega))}, \tag{61}$$

which is finite for all $T_0 \in [0,T]$, see (32). Therefore the solution may be extended to the whole interval $[0,T]$ and estimate (43) follows. Finally, uniqueness of solutions and the additional regularity are a consequence of the linearity of Problem (41) and the regularity of coefficients.

Estimate (30) for the solution of Problem (29) is deduced from (53) taking into account $h_0 > 0$, see (28), and $\nabla\theta \in L^2(Q_T)$, see Remark 4. Note that for Problem (41) it only holds $h^n_\varepsilon \geq k(\varepsilon)$, see (38), and even assumption $\nabla\theta \in L^2(Q_T)$ does not ensure a uniform estimate like (30) for the solution of Problem (41). □

Proof of Lemma 8. Since $\partial\Omega$ is regular, Ω has the property of the exterior sphere, i.e., for all $\mathbf{x}_0 \in \partial\Omega$ there exist $R_1 > 0$ and $\mathbf{x}_1 \in \mathbb{R}^N \setminus \overline{\Omega}$ such that

$$B(\mathbf{x}_1, R_1) \cap \overline{\Omega} = \{\mathbf{x}_0\},$$

with $B(\mathbf{x}_1,R_1) := \{\mathbf{x} \in \mathbb{R}^N : |\mathbf{x}-\mathbf{x}_1| < R_1\}$. Consider $\delta > 0$ small enough such that, by setting $R_2 := \delta + R_1$, we have $B(\mathbf{x}_1,R_2) \cap \partial\Omega \neq \emptyset$. We define $\omega := \Omega \cap B(\mathbf{x}_1,R_2)$ and introduce the following notation $a := \|g\|_{L^\infty(Q_T)}$, $b :=$

$(((N-1)/R_1)+1)\|\mathbf{B}\|_{L^\infty(Q_T)}$ and $c := \|\psi\|_{L^\infty(Q_T)}$. We define

$$\mathcal{L}(\psi) := \psi_t + A_\varepsilon \Delta\psi + \mathbf{B}\cdot\nabla\psi \quad \text{and} \quad w(\mathbf{x},t) := \psi(\mathbf{x},t) + \sigma(r),$$

for $(\mathbf{x},t) \in \omega \times (0,T)$ and with $r := |\mathbf{x}-\mathbf{x}_0|$. Function $\sigma \in C^2([R_1,R_2])$ will be chosen such that $\sigma''(r) \geq 0$, $\sigma'(r) \leq 0$ and the maximum of w in $\overline{\omega} \times [0,T]$ is attained in $\{\mathbf{x}_0\} \times [0,T]$. We have

$$\mathcal{L}(w) = -g + A_\varepsilon \Delta\sigma + \mathbf{B}\cdot\nabla\sigma \geq k(\varepsilon)\sigma''(r) + b\sigma'(r) - a,$$

in $\omega\times(0,T)$. Choosing $\sigma(r) := ar/b + C_2 e^{-br/k(\varepsilon)}$, with C_2 an arbitrary constant, we obtain

$$k(\varepsilon)\sigma''(r) + b\sigma'(r) - a = 0$$

and $\sigma''(r) \geq 0$. A straightforwad computation shows that if we choose

$$C_2 \geq k(\varepsilon) a e^{bR_2/k(\varepsilon)}/b^2 \quad \text{then} \quad \sigma'(r) \leq 0. \tag{62}$$

Then we have $\mathcal{L}(w) \geq 0$ in $\omega \times (0,T)$ and by the maximum principle we deduce that w attains its maximum on the parabolic boundary of $\omega \times (0,T)$. In this boundary the values of w may be estimated as follows:

$$\begin{cases} w(\mathbf{x},t) = \sigma(r) \leq \sigma(R_1) & \text{on } (\partial\Omega \cap \partial\omega) \times [0,T], \\ w(\mathbf{x},t) = \psi(\mathbf{x},t) + \sigma(r) \leq c + \sigma(R_2) & \text{on } (\partial B(\mathbf{x}_1,R_2) \cap \partial\omega) \times [0,T], \\ w(\mathbf{x}_0,t) = \sigma(R_1) & \text{on } [0,T], \\ w(\mathbf{x},T) = \sigma(r) \leq \sigma(R_1) & \text{in } \omega. \end{cases}$$

It is not difficult to check that, by taking δ small enough, we can choose C_2 such that (62) and $\sigma(R_1) = c + \sigma(R_2)$ hold. As a consequence we obtain $\nabla w(\mathbf{x}_0,t)\cdot \nu \geq 0$ and by the definition of w and for an appropiate δ we obtain

$$\nabla\psi(\mathbf{x}_0,t)\cdot\nu \geq -C\frac{bc}{k(\varepsilon)} \quad \text{in } [0,T]. \qquad \square$$

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