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TRAVELLING WAVES FOR REACTION-DIFFUSION-CONVECTION SYSTEMS

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1. Introduction

There is a considerable literature (e.g. [6], [14]) on the existence of travelling-wave solutions of reaction-diffusion equations and systems in the form

$$u_t = Au_{xx} + f(u), \quad u \in \mathbb{R}^N,\ x \in \mathbb{R},\ t \in [0,\infty), \tag{1}$$

where A is a real, positive-definite, $N \times N$ matrix and $f : \mathbb{R}^N \to \mathbb{R}^N$ is a continuously differentiable nonlinear function. The vector u may represent, for example, the concentrations of chemicals or the population densities of interacting species, the interactions between components of u being modelled by the *reaction* term $f(u)$ and their *diffusion* by Au_{xx}. Travelling waves are solutions u of (1) in the form

$$u(x,t) = w(x - ct), \tag{2}$$

where $w : \mathbb{R} \to \mathbb{R}^N$ is the profile of the wave which propagates through the one-dimensional spatially homogeneous domain at the (*a priori* unknown) constant velocity c.

This paper is concerned with an extension of the theory to systems with nonlinear dependence on the gradient of u, such as arise in applications in which

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there is an underlying drift (chromatography, convection in chemical reactions or wind effects in biology [9, pp. 292, 322 and 420]). Some work has been done on scalar-valued gradient-dependent problems [14, p. 111]. Here we prove existence theorems which extend that of [14] to gradient-dependent cases. Corresponding stability results are under development and will be presented elsewhere. In Sections 2–4 of this paper, we prove the existence of a monotone travelling-wave solution for the reaction-diffusion-convection system

$$u_t = Au_{xx} + G(u, u_x)u_x + f(u) \tag{3}$$

under the following hypotheses. (Standard notation is recalled at the end of the Introduction.)

(a) A is a positive-definite diagonal matrix.

As in [14], $f : \mathbb{R}^N \to \mathbb{R}^N$ is a continuously differentiable function satisfying

(f1) f is locally monotone;
(f2) S and T are stable equilibria of f and $S < T$;
(f3) there is a (necessarily non-zero) finite number of equilibria E of f with $S < E < T$ and each such E is unstable.
(g1) G is a continuously differentiable, diagonal-matrix-valued function on $\mathbb{R}^N \times \mathbb{R}^N$ and there exist continuous functions $\beta, \gamma : [0, \infty) \to [0, \infty)$ such that for each $u, v \in \mathbb{R}^N$,

$$\|G(u, v)\| \le \gamma(\|u\|)(1 + \beta(\|v\|)),$$

where β is increasing and $\beta(p)/p \to 0$ as $p \to \infty$. Suppose that β_∞ is such that $\beta(p)/p \le \beta_\infty$.
(g2) $G(E, 0)$ is positive-definite at each equilibrium E of f in $[S, T]$.

Thus for some $c \in \mathbb{R}$, we obtain a solution w of the system

$$Aw'' + cw' + G(w, w')w' + f(w) = 0. \tag{4}$$

Note that (g2) can be assumed without loss of generality in a problem where (f3) holds. This is obvious because, by (f3), $dI + G(E, 0)$ is positive-definite at each of the finite number of equilibria E in $[S, T]$ provided d is sufficiently large. Now replace the unknown parameter c with a new parameter $\widehat{c} = c - d$ and replace G with $\widehat{G} = G + dI$. The new system, (4) with $\widehat{c}$ instead of c and $\widehat{G}$ instead of G satisfies (a), (f1)–(f3) and (g1), (g2).

The main result of this paper is in Section 5. There the existence of travelling waves is established without assuming (g1) about the growth of G, provided that G is known to be monotone in a certain weak sense. These results are deduced using the theory of the preceding sections, and contain the previous existence theorems as special cases.

Following [14, Chapters 2 and 3], we use degree theory for $(S)_+$ mappings in a weighted Sobolev space. Although the operators associated with (4) are not $(S)_+$ except when $G \equiv 0$, that difficulty can be overcome by approximation. We consider the modified system

$$Av''(s) + cv'(s) + \sigma_R(s)G(v(s), v'(s))v'(s) + f(v(s)) = 0, \quad s \in \mathbb{R}, \tag{5}$$

where for $R > 1$, $\sigma_R \in C(\mathbb{R}, [0,1])$ is supported in $[-R, R]$, and $\sigma_R(s) \to 1$ as $R \to \infty$ for each $s \in \mathbb{R}$. The corresponding operators are shown to be $(S)_+$ and the existence of a monotone solution to (5) asymptotic to S and T is proved. Solutions to (4) follow by taking the limit as $R \to \infty$. To keep the presentation as short as possible, we refer to [14] when appropriate.

The following is key notation. An *equilibrium* E of $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, a point where $f = 0$, is said to be *stable* if all the eigenvalues of $df[E]$ are in the open left-half complex plane, and *unstable* if there is an eigenvalue in the open right-half plane. When $S, T \in \mathbb{R}^N$ (the source and target) are equilibria of f with $S < T$, system (1) is said to be *bistable* if both S and T are stable. In that case there exist equilibria $E \in [S,T] \setminus \{S,T\}$. (These are called intermediate equilibria; see [4].)

Attention in this paper, as in [13], [14], is restricted to functions f which are *locally monotone*: whenever $u \in \mathbb{R}^N$ is such that $f_i(u) = 0$ for some i, $1 \le i \le N$

$$\partial f_i/\partial u_j > 0 \quad \text{for } j \neq i, \; j = 1, \dots, N, \tag{6}$$

where f_i denotes the ith component function of f. This is a natural hypothesis for mutualistic interactions in biology, and for certain kinds of chemical kinetics. If $w_0 \in \mathbb{R}^N$ is such that $\partial f_i/\partial u_j(w_0) > 0$ for $i \neq j$, $i, j = 1, \dots, N$, then the off-diagonal elements of $df[w_0]$ are positive - that is, $df[w_0]$ is a Perron–Frobenius matrix [10], [12].

2. Degree theory for the analysis of travelling waves

Preliminaries. The notation ":=" means "is defined to equal". For vectors $p, q \in \mathbb{R}^N$, $p < q$ ($p \le q$) means that $p_k < q_k$ ($p_k \le q_k$) for each k, $1 \le k \le N$ and p_k denotes the kth coordinate of p. When $q < p$, $(q,p) = \{x \in \mathbb{R}^N : q < x < p\}$ and $[q,p] = \{x \in \mathbb{R}^N : q \le x \le p\}$. A vector $x \in \mathbb{R}^N$ is *positive* (*non-negative*) if $x > 0$ ($x \ge 0$). The set of non-negative vectors in $\mathbb{R}^N$ will be denoted by $\mathbb{R}^N_+$ and the set of all real $N \times N$ matrices with strictly positive off-diagonal elements by $P^{N\times N}$. The real $N \times N$ matrices will be denoted by $M^{N\times N}$ where the ijth entry of M is M_{ij}. If $M \in M^{N\times N}$ is diagonal, then M_i denotes the ith element on its diagonal. For functions $f : \mathbb{R}^N \to \mathbb{R}^N$, the Fréchet derivative of f at $w \in \mathbb{R}^N$ will be denoted by $df[w]$ and, where necessary, identified it with the Jacobian matrix of partial derivatives of f at w.

For a set Ω in a metric space, the closure will be denoted by $\overline{\Omega}$ and the boundary by $\partial\Omega$. The Euclidian inner product and norm on $\mathbb{R}^N$ will be denoted by $\langle\,\cdot\,,\,\cdot\,\rangle$, and $\|\cdot\|$ and $\|x\|_\infty := \max\{|x_i| \mid 1 \le i \le N\}$ for $x \in \mathbb{R}^N$. For $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^N$, $C^k(\overline{\Omega}, \mathbb{R}^N)$ is the Banach space consisting of functions $f : \Omega \to \mathbb{R}^N$ for which all derivatives of order at most k are bounded and uniformly continuous on Ω with the usual supremum norm, $L_2(\Omega, \mathbb{R}^N)$ is the Hilbert space of square-integrable measurable functions $u : \Omega \to \mathbb{R}^N$ and $W_2^1(\Omega, \mathbb{R}^N)$ is the Sobolev space of functions $u \in L_2(\Omega, \mathbb{R}^N)$ with square-integrable weak derivative. With $\mu(s) := 1 + s^2$, let

$$\omega(s) := \sqrt{\mu(s)} \quad \text{and} \quad \nu(s) := \mu(s)^{-1}\mu'(s), \ s \in \mathbb{R}. \tag{7}$$

For an open subset Ω of $\mathbb{R}$, let $L_{2,\mu}(\Omega, \mathbb{R}^N)$ be the space of measurable functions $u : \mathbb{R} \to \mathbb{R}^N$ with $\|u\|_{L_{2,\mu}(\Omega,\mathbb{R}^N)} := \langle u, u\rangle^{1/2}_{L_{2,\mu}(\Omega,\mathbb{R}^N)} < \infty$, where

$$\langle u, v\rangle_{L_{2,\mu}(\Omega,\mathbb{R}^N)} := \int_\Omega \langle u(s), v(s)\rangle \mu(s)\, ds. \tag{8}$$

Then the weighted Sobolev space $W^1_{2,\mu}(\Omega, \mathbb{R}^N)$ is the set of functions

$$u \in L_{2,\mu}(\Omega, \mathbb{R}^N) \quad \text{for which } u' \in L_{2,\mu}(\Omega, \mathbb{R}^N),$$

with the inner product

$$\langle u, v\rangle_{W^1_{2,\mu}(\Omega,\mathbb{R}^N)} := \langle u, v\rangle_{L_{2,\mu}(\Omega,\mathbb{R}^N)} + \langle u', v'\rangle_{L_{2,\mu}(\Omega,\mathbb{R}^N)}. \tag{9}$$

We state the following without proof.

Lemma 2.1. *Bounded linear operators $M : W_2^1(\Omega, \mathbb{R}^N) \to W^1_{2,\mu}(\Omega, \mathbb{R}^N)$ and $N : W^1_{2,\mu}(\Omega, \mathbb{R}^N) \to W_2^1(\Omega, \mathbb{R}^N)$, are defined by multiplication as follows:*

$$\begin{aligned} Mu(s) &= \omega^{-1}(s)u(s), \quad u \in W_2^1(\Omega, \mathbb{R}^N), \\ \text{and} \quad Nu(s) &= \omega(s)u(s), \qquad u \in W^1_{2,\mu}(\Omega, \mathbb{R}^N). \end{aligned} \tag{10}$$

Also

$$\|u(s)\| \le \|u\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)}/\sqrt{\mu(s)} \quad \text{for } u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \text{ and } s \in \mathbb{R}^N. \tag{11}$$

Degree theory for $(S)_+$ operators is developed in [1], [2], [11] and used in [14, Chapter 2]. Let X denote a real, reflexive, separable Banach space, and let $W : X \to X^*$. Then W is an $(S)_+$ operator if, for $\{u_n\} \subset X$,

$$u_n \rightharpoonup u_0 \text{ in } X, \quad \limsup_{n\to\infty}\, (W(u_n))(u_n - u_0) \le 0 \Rightarrow u_n \to u_0 \quad \text{in } X. \tag{12}$$

Suppose also that W is bounded, demicontinuous (continuous from the strong topology of X to the weak topology of X^*) and $0 \notin W(\partial\Omega)$ where $\Omega \subset X$ is bounded and open. Then $\deg_{(S)_+}(W, \Omega, 0)$ denotes the integer-valued $(S)_+$-degree, which has the usual properties. If $\deg_{(S)_+}(W, \Omega, 0) \ne 0$, then there exists $u \in \Omega$ with $W(u) = 0$. The homotopy property has the following form. If

$W_1, W_2 : X \to X^*$ are bounded demicontinuous $(S)_+$ mappings, and there is a bounded demicontinuous operator $W : \overline{\Omega} \times [0,1] \to X^*$ such that

$$0 \notin W(\partial\Omega \times [0,1]) \tag{13}$$

$$W(u,0) = W_1(u), \quad W(u,1) = W_2(u) \quad \text{for all } u \in \overline{\Omega}, \tag{14}$$

and

$$\begin{aligned} &\text{For } \{u_n, t_n\} \subset \overline{\Omega} \times [0,1], \quad u_n \rightharpoonup u_0 \text{ in } X, \quad t_n \to t_0, \\ &\text{and } \limsup_{n\to\infty} (W(u_n,t_n))(u_n - u_0) \le 0 \Rightarrow u_n \to u_0 \quad \text{in } X, \end{aligned} \tag{15}$$

then W_1 and W_2 are said to be $(S)_+$*-homotopic relative to* Ω, and they have equal degree.

$(S)_+$ **operators for the approximate system.**

Definition 2.2. For $R > 1$, let $\sigma_R \in C(\mathbb{R}, [0,1])$ be a function which equals 1 on $[-R+1, R-1]$, equals 0 outside $[-R, R]$, and is affine elsewhere.

We seek a constant c and a function $w \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that

$$Aw''(s) + cw'(s) + \sigma_R(s)G(w(s), w'(s))w'(s) + f(w(s)) = 0, \quad s \in \mathbb{R}, \tag{16}$$

$$w'(s) > 0, \quad s \in \mathbb{R}, \tag{17}$$

$$w(s) \to S \quad \text{as } s \to -\infty, \qquad w(s) \to T \quad \text{as } s \to \infty. \tag{18}$$

Let

$$\psi(s) = T\alpha(s) + S(1 - \alpha(s)) \quad \text{and} \quad w(s) = u(s) + \psi(s), \tag{19}$$

where $\alpha \in C^\infty(\mathbb{R}, [0,1])$ is a fixed monotone function with $\alpha(s) = 0$ when $s \le -1$ and $\alpha(s) = 1$ when $s \ge 1$. Then w satisfies (16), (17) and (18) if and only if u satisfies

$$(u'' + \psi'') + c(u' + \psi') + \sigma_R G(u + \psi, u' + \psi')(u' + \psi') + f(u + \psi) = 0, \tag{20}$$

$$u'(s) + \psi'(s) > 0, \quad s \in \mathbb{R} \quad \text{and} \quad u(s) \to 0 \quad \text{as } s \to \pm\infty. \tag{21}$$

Following [14, Chapter 2, pp. 123–124], we seek a solution u to (20) and (21) in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Note that estimate (11) implies that $u(s) \to 0$ as $s \to \pm\infty$ every $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. The fact that the constant c is not known *a priori* is overcome by functionalising the parameter, an idea due to Krasnosel'skiĭ. Let $\phi : \mathbb{R} \to \mathbb{R}$ be given by $\phi(s) = e^s$ if $s < 0$ and $\phi(s) = 1$ if $s \ge 0$. For $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$, set

$$\rho(u) := \left(\int_{\mathbb{R}} \|u(s) + \psi(s) - T\|^2 \phi(s)\, ds \right)^{1/2} \quad \text{and} \quad c(u) := \log \rho(u), \tag{22}$$

where ψ is defined in (19). Define $u_h(s) := u(s+h) + \psi(s+h) - \psi(s)$, $s \in \mathbb{R}$ for each $h \in \mathbb{R}$. Then $c(\cdot)$ has the following properties, which are proved in [14, Proposition 1.1, Chapter 2, pp. 134–135].

Lemma 2.3. *The functional c is Lipschitz continuous on bounded subsets of $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ and, for $u\in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$, $c(u_h)$ is monotone in h, and $c(u_h)\to\pm\infty$ as $h\to\mp\infty$.*

We will define $c(w)$ as

$$c(w)=c(u)\quad\text{when } w=u+\psi,\ u\in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N), \tag{23}$$

when there is no danger of ambiguity. Now consider the equation

$$A(u''+\psi'')+c(u)(u'+\psi')+\sigma_R G(u+\psi,u'+\psi')(u'+\psi')+f(u+\psi)=0. \tag{24}$$

It is clear that if $u\in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ satisfies (24), then u satisfies (20) with $c=c(u)$. Conversely, if a function u satisfies (20) with velocity c, then there is some $h\in\mathbb{R}$ with $c(u_h)=c$. Thus u_h satisfies (24) with $\sigma_R(\,\cdot\,)$ replaced by $\sigma_R(\,\cdot+h)$. Whence there is an equivalence between (24) and (20). Henceforth, a solution of (24) and (21) will be sought in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$.

The method is to consider the following continuous deformation of (24):

$$\begin{aligned} A(u''+\psi'')+c(u)(u'+\psi')+\tau\sigma_R G(u+\psi,u'+\psi')(u'+\psi')&\\ +f(u+\psi)=0,\quad \tau\in[0,1].&\end{aligned} \tag{25}$$

When $\tau=1$, (25) becomes (24), whilst when $\tau=0$, (25) corresponds to the system treated in [14].

To invoke degree theory, an $(S)_+$ operator associated with (24) is required, together with a suitable $(S)_+$ homotopy associated with (25). First define $P_R:\ [0,1]\times W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)\to W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)^*$ by

$$\begin{aligned} P_R(\tau,u)(v)=\int_{\mathbb{R}}\langle Au',(v\mu)'\rangle\,ds-\int_{\mathbb{R}}\langle A\psi''+c(u)(u'+\psi'),v\rangle\mu\,ds&\\ -\int_{\mathbb{R}}\langle \tau\sigma_R G(u+\psi,u'+\psi')(u'+\psi')+f(u+\psi),v\rangle\mu\,ds&\end{aligned} \tag{26}$$

for $\tau\in[0,1]$, $u,v\in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$.

Lemma 2.4.

(i) *P_R is well-defined and maps each bounded subset of $[0,1]\times W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ into a bounded subset of $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)^*$.*
(ii) *P_R is jointly continuous in τ and u (from the strong topology of $[0,1]\times W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ to the strong topology of $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)^*$).*
(iii) *Let $u\in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ and $\tau\in[0,1]$. Then u is a solution of $P_R(\tau,u)=0$ if and only if $u\in C^2(\mathbb{R},\mathbb{R}^N)$ and u satisfies (25).*

Proof. The first statement is immediate from the Hölder's inequality, Definition 2.2, the continuity of f, (g1), Lemma 2.3, and Lemma 2.1 together with observations that ν is uniformly bounded and ψ' and ψ'' have compact support. The second statement is a standard consequence of Krasnosel'skiĭ's theorem [5, p. 77]. The third is the usual relation between weak and strong solutions which, in this one-dimensional setting, is straightforward. □

Now we show that a construction in [14, Chapter 2, Section 2, pp. 128–134] remains effective under our hypotheses.

THEOREM 2.5. *There exists a bounded linear positive-definite self-adjoint operator* $S_\mu : W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ *and a function* $\theta : W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \times W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \times [0,1]$, *such that for* $\tau \in [0,1]$ *and* $u, u_0 \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$,

$$(P_R(\tau,u))(S_\mu(u-u_0)) \geq \|u-u_0\|^2_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} + \theta(u,u_0,\tau), \tag{27}$$

and $\theta(u_n,u_0,\tau) \to 0$ *uniformly for* $\tau \in [0,1]$ *as* $u_n \rightharpoonup u_0$ *in* $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$.

This result is significant because (27) can be rewritten as

$$(S^*_\mu P_R(\tau,u))(u-u_0) \geq \|u-u_0\|^2_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} + \theta(u,u_0,\tau). \tag{28}$$

Together with Lemma 2.4, this yields that the operator

$$S^*_\mu P_R : [0,1] \times W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$$

is an admissible $(S)_+$ homotopy.

So if $\Omega \subset W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ is open, bounded and such that

$$0 \notin S^*_\mu P_R([0,1] \times \partial\Omega), \tag{29}$$

then $S^*_\mu P_R(0,\,\cdot\,)$ and $S^*_\mu P_R(1,\,\cdot\,)$ are $(S)_+$ homotopic relative to Ω. Hence

$$\deg_{(S)_+}(S^*_\mu P_R(1,\,\cdot\,),\Omega,0) = \deg_{(S)_+}(S^*_\mu P_R(0,\,\cdot\,),\Omega,0). \tag{30}$$

Further, if $0 \notin S^*_\mu P_R(\tau,\,\cdot\,)(\partial\Omega)$ and $\deg_{(S)_+}(S^*_\mu P_R(\tau,\,\cdot\,),\Omega,0) \neq 0$, then there exists $u \in \Omega$ such that $S^*_\mu P_R(\tau,u) = 0$. So by the last part of Lemma 2.4, $u \in C^2(\mathbb{R},\mathbb{R}^N)$ satisfies (25).

For f satisfying (f1)–(f3), let the matrix $B(s)$ be given by

$$B(s) = \phi_1(s)df[S] + \phi_2(s)df[T], \quad s \in \mathbb{R}, \tag{31}$$

where $\phi_1,\phi_2 \in C^\infty(\mathbb{R},[0,1])$ are such that $\phi_1(s)+\phi_2(s) = 1$ for each $s \in \mathbb{R}$, and $\phi_1(s) = 0$ when $s > 1$, $\phi_2(s) = 0$ when $s < -1$. The next result follows from [14, Chapter 2, Section 2, pp. 128–134] because all the eigenvalues of $df[S]$ and $df[T]$ are in the left-half plane.

Lemma 2.6. *There exists a bounded linear self-adjoint positive-definite operator $S_\mu : W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$, such that for $u \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$,*

$$(L_\mu u)(S_\mu u) \geq \|u\|^2_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} + \theta_\mu(u), \tag{32}$$

where $\theta_\mu : W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \to \mathbb{R}$ is such that $\theta_\mu(u_n) \to 0$ as $u_n \rightharpoonup 0$ in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$, and for $u \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$, $L_\mu u \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)^*$ is defined by

$$(L_\mu u)(v) = \int_{\mathbb{R}} [\langle Au'(s), v'(s)\rangle - \langle B(s)u(s), v(s)\rangle]\mu(s)\, ds \tag{33}$$

for $v \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$.

Proof of Theorem 2.5. Let the sequence $\{u_n\}_{n=1}^\infty \subset W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ be such that $u_n \rightharpoonup u_0$ in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$ as $n \to \infty$, and set $v_n = u_n - u_0$. Then for each $\tau \in [0,1]$, (26) gives that

$$\begin{aligned} P_R(\tau,u_n)(S_\mu v_n) = &\int_{\mathbb{R}} \langle A{u_n}', (S_\mu v_n)'\rangle \mu\, ds + \int_{\mathbb{R}} \langle A{u_n}', \nu(S_\mu v_n)\rangle \mu\, ds \\ &- \int_{\mathbb{R}} \langle A\psi'' + c(u_n)({u_n}' + \psi'), S_\mu v_n\rangle \mu\, ds \\ &- \int_{\mathbb{R}} \langle f(u_n+\psi), S_\mu v_n\rangle \mu\, ds \\ &- \int_{\mathbb{R}} \tau\langle \sigma_R G(u_n+\psi, u_n+\psi')({u_n}'+\psi'), S_\mu v_n\rangle \mu\, ds \end{aligned} \tag{34}$$

where ν is defined in (7). In [14] it is shown that

$$\begin{aligned} &\int_{\mathbb{R}} \langle A{u_n}', (S_\mu v_n)'\rangle \mu + \langle A{u_n}', \nu(S_\mu v_n)\rangle \mu \\ &\quad - \langle A\psi'' + c(u_n)({u_n}'+\psi'), S_\mu v_n\rangle \mu\, ds - \int_{\mathbb{R}} \langle f(u_n+\psi), S_\mu v_n\rangle \mu\, ds \\ &\qquad = \int_{\mathbb{R}} \langle A{v_n}', (S_\mu v_n)'\rangle - \langle Bv_n, S_\mu v_n\rangle \mu\, ds + \widetilde{\theta}(u_n,u) \end{aligned} \tag{35}$$

where $\widetilde{\theta}(u_n,u) \to 0$ as $n \to \infty$.

Now hypothesis (g1) holds, $|\tau\sigma_R| \leq 1$ and σ_R is supported in $[-R,R]$. Therefore there is a constant γ_0, depending only on $\{u_n\}$, such that

$$\begin{aligned} &\bigg|\int_{\mathbb{R}} \tau\sigma_R \langle G(u_n+\psi, {u_n}'+\psi')({u_n}'+\psi'), S_\mu v_n\rangle \mu\, ds\bigg| \\ &\quad \leq \gamma_0 \int_{|s|\leq R} \{\|{u_n}'+\psi'\| + \beta_\infty \|{u_n}'+\psi'\|^2\} \|S_\mu v_n\| \mu\, ds \to 0 \quad \text{as } n \to \infty \end{aligned} \tag{36}$$

uniformly for $\tau \in [0,1]$, because $S_\mu v_n \to 0$ uniformly on $[-R,R]$ as $n \to \infty$ and $\{u_n\}$ is bounded in $W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)$. Thus there exists a functional θ such that

$\theta(u_n, u_0, \tau) \to 0$ as $u_n \rightharpoonup u_0$ in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ uniformly for $\tau \in [0,1]$, and

$$(P_R(\tau, u_n))(S_\mu v_n) = \int_{\mathbb{R}} [\langle A v_n{}'(s), (S_\mu v_n)'(s)\rangle - \langle B(s) v_n(s), (S_\mu v_n(s)\rangle]\mu(s)\, ds + \theta(u_n, u_0, \tau).$$

Therefore

$$\begin{aligned}(P_R(\tau, u_n))(S_\mu v_n) &= (L_\mu v_n)(S_\mu v_n) + \theta(u_n, u_0, \tau) \\ &\geq \|u_n - u_0\|^2_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} + \theta_\mu(v_n) + \theta(u_n, u_0, \tau).\end{aligned}$$

Here L_μ, S_μ and θ_μ are as in Lemma 2.6. The result follows. □

3. A priori estimates for monotone approximations

The goal is to prove that the set of all solutions u of equation (65) is bounded, independently of $\tau \in [0,1]$ and $R > 0$, in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$.

A priori estimates for a general system. We first consider solutions w of

$$\begin{cases} Aw'' + cw' + \sigma_R \tau G(w, w')w' + f(w) = 0, \\ w = u + \psi, \ u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N), \end{cases} \tag{37}$$

where f satisfies (f1)–(f3), G and σ_R are given by Definition 2.2, and $\tau \in [0,1]$. In this section the velocity $c \in \mathbb{R}$ is regarded as a parameter, and not the functional (22). The representation of c as a functional will be re-introduced in the next section to get a uniform *a priori* bound for monotone solutions. The initial step is to obtain estimates for the absolute value of derivatives of solutions of (37) that are independent of $c \in \mathbb{R}$, $\tau \in [0,1]$ and R. Note that it is *not* assumed yet that the solutions are monotone.

Lemma 3.1. *Let $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfy* (37) *for some $c \in \mathbb{R}$, $\tau \in [0,1]$ and $R > 0$. Then u, u', $u'' \in L_\infty(\mathbb{R}, \mathbb{R}^N)$, and there exists $K > 0$ such that $\|u'(s)\| \leq K/\sqrt{\mu(s)}$ for each $s \in \mathbb{R}$ and hence $\|u'(s)\| \to 0$ as $n \to \infty$.*

Proof. That $u \in L_\infty(\mathbb{R}, \mathbb{R}^N)$ follows from estimate (11). Since u satisfies (37), Clearly $-\psi'' - cA^{-1}[u' + \psi'] \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Since u' is bounded on $[-R, R]$, $\tau\sigma_R G(u + \psi, u' + \psi')(u' + \psi') \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Also $f(u + \psi) \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. Consequently $u'' \in L_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ and $u' \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$. The decay estimate for $u' \in L_\infty(\mathbb{R}, \mathbb{R}^N)$ now follows from (11). That $u'' \in L_\infty(\mathbb{R}, \mathbb{R}^N)$ follows from (37), the boundedness of ψ, ψ', u and u' and the continuity of f and G. □

Theorem 3.2. *Let $M > 0$. Suppose that w satisfies* (37) *for some $c \in \mathbb{R}$, $\tau \in [0,1]$ and $R > 0$, with $\|w\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} < M$. Then there exist $N_1(M)$, $N_2(M) > 0$, independent of w, c and R, such that*

$$\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} < N_1(M) \quad \textit{and} \quad \|w''\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} < N_2(M).$$

Proof. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(s) := \langle Aw'(s), w'(s)\rangle, s \in \mathbb{R}$. Then h is continuously differentiable and $h \geq \min_{i=1,\dots,N}\{A_i\}\|w'\|^2$, since A is positive-definite. Also, Lemma 3.1 implies that $h(s) \to 0$ as $|s| \to \infty$, since $\psi'(s) = 0$ when $|s| > 1$. So h attains a maximum at a point $s_0 \in \mathbb{R}$ where $h'(s_0) = 0$. Since A is symmetric, $\langle Aw''(s_0), w'(s_0)\rangle = 0$. So taking the inner product of the left-hand side of (37) with $w'(s_0)$ yields, by hypothesis (g1) and Definition 2.2, that

$$|c|\|w'(s_0)\|^2 \leq \gamma(w(s_0))(1+\beta(\|w'(s_0)\|))\|w'(s_0)\|^2 + \|f(w(s_0))\|\|w'(s_0)\|. \tag{38}$$

To find a uniform bound for w' there is no loss of generality in supposing that that $\|w'(s_0)\| \geq 1$. Since $\|w\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} < M$ and γ and f are continuous, there exist $\gamma_0, \alpha_0 > 0$, depending only on M, such that $\gamma(w(s)) < \gamma_0$, and $\|f(w(s))\| < \alpha_0$ for all $s \in \mathbb{R}$. Hence

$$|c| \leq \gamma_0(1+\beta(\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)})) + \alpha_0. \tag{39}$$

Lemma 3.1 implies that w' and w'' are uniformly bounded on $\mathbb{R}$. So Landau's inequality [8, Theorem 5.3.1, p. 167] and the fact that w satisfies (37) together yield that

$$\begin{aligned}\|w'\|^2_{L_\infty(\mathbb{R},\mathbb{R}^N)} &\leq 4\|w\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}\|A^{-1}\|\|(cw' + \sigma_R\tau G(w,w')w' + f(w))\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \\ &\leq 4\|w\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}\|A^{-1}\|\{(2\gamma_0(1+\beta(\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}))\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \\ &\quad + \alpha_0(1+\|w'\|_{\infty(\mathbb{R},\mathbb{R}^N)})\}.\end{aligned}$$

Therefore

$$\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \leq 4M\|A^{-1}\|\{(2\gamma_0(1+\beta(\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}) + 2\alpha_0\}, \tag{40}$$

since we have assumed that $\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \geq 1$. This proves the existence of $N_1(M) > 0$ as in the statement of the theorem.

The corresponding bound for w'' comes from the fact that, by inequality (38), $|c|\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}$ is bounded by a constant depending only on M. Therefore the ith equation of (37) yields that for each $s \in \mathbb{R}$,

$$|w_i''(s)| \leq A_i^{-1}(\sup_{s\in\mathbb{R}}|cw_i'(s)| + \sup_{s\in\mathbb{R}}|\tau G_i(w(s),w'(s))w_i'(s)| + \sup_{s\in\mathbb{R}}|f(w(s))|). \tag{41}$$

The existence of $N_2(M) > 0$ as in the statement of the theorem follows from (39) and (41). □

To obtain an *a priori* estimate for the speed of monotone solutions to (37), we require some preliminary results on the signs of the component functions of f. The first is proved in [13].

LEMMA 3.3. *Let $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ satisfy* (f1) *and* (f2). *Let $p, q > 0$ denote Perron–Frobenius eigenvectors of $df[S]$ and $df[T]$ respectively. Then there exist $t, \zeta, \varepsilon > 0$ such that for each $i \in \{1, \ldots, N\}$, $\varepsilon < tp_i, \varepsilon < tq_i$,*

$$\begin{aligned} x \in \Gamma_i := \{x \in \mathbb{R}^N : S \leq x \leq S + tp,\ x_i = S_i + tp_i\}, \\ y \in \mathbb{R}^N \textit{with} \max_{1\leq k\leq N} |y_k - x_k| \leq \varepsilon \Rightarrow f_i(y) < 0, \\ x \in \Lambda_i := \{x \in \mathbb{R}^N : T - tq \leq x \leq T,\ x_i = T_i - tq_i\}, \\ y \in \mathbb{R}^N, \textit{with} \max_{1\leq k\leq N} |y_k - x_k| \leq \varepsilon \Rightarrow f_i(y) > 0. \end{aligned}$$

This leads to bounds on the velocity c of monotone solutions to (37), independently of the choice of R.

THEOREM 3.4. *There exists $\gamma > 0$, depending only on f, G and A, such that if w satisfies* (37) *for some $c \in \mathbb{R}$, $\tau \in [0, 1]$ and $R > 0$ and $w'(s) > 0$ for all $s \in \mathbb{R}$, then $|c| < \gamma$.*

PROOF. Let w be as in the statement of the theorem. The fact that $w(s)$ converges monotonically to S as $s \to -\infty$ will yield a lower bound for c. The existence of an upper bound can be proved similarly, using the monotonic convergence of $w(s)$ to T as $s \to \infty$.

Since $w(s) \to S$ as $s \to -\infty$ and $w'(s) > 0$ for each s, there exists $s_0 \in \mathbb{R}$ and $i \in \{1, \ldots, N\}$ such that $w(s_0) > S$ and $w(s_0) \in \Gamma_i$, where Γ_i is defined in (3.3). Now $w(s_0) \in \Gamma_i \Rightarrow w_i(s_0) = tp_i$, and $tp_i > \varepsilon > 0$, where ε is as in Lemma 3.3. Choose $s_1 \in \mathbb{R}$ such that $w_i(s_0) - w_i(s_1) = \varepsilon$. Since w is increasing, $s_1 < s_0$. Integrating the ith equation of (37) from s_1 to s_0 gives

$$A_i(w_i'(s_0) - w_i'(s_1)) + c\varepsilon + \int_{s_1}^{s_0} \sigma_R(s)\tau G_i(w(s), w'(s))w_i'(s)\,ds + \int_{s_1}^{s_0} f_i(w(s))\,ds = 0. \tag{42}$$

Since w is a monotone solution between S and T, $\|w\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \leq \|S\| + \|T - S\|$. Thus by Theorem 3.2, there exists $N_1 > 0$, independent of w, c and R, such that $\|w'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} \leq N_1$, $|A_i(w_i'(s_0) - w_i'(s_1))| \leq 2\|A\|N_1$, and there exists $\gamma_1 > 0$ such that $\|\tau G_i\|_{L_\infty(\mathbb{R}^N\times\mathbb{R}^N,\mathbb{R})} \leq \gamma_1$ for each $i \in \{1, \ldots, N\}$. Hence

$$\begin{aligned} \left| \int_{s_1}^{s_0} \sigma_R(s)\tau G_i(w(s), w'(s))w_i'(s)\,ds \right| &\leq \gamma_1 \int_{s_1}^{s_0} |w_i'(s)|\,ds \\ &= \gamma_1 \int_{s_1}^{s_0} w_i'(s)\,ds = \gamma_1\varepsilon, \end{aligned} \tag{43}$$

since w is monotone. We now observe from the monotonicity of w and Lemma 3.3 that

$$f_i(w(s)) < 0 \quad \text{for } s_1 \leq s \leq s_0. \tag{44}$$

Therefore, $c \geq -\gamma_1 - 2\|A\|N_1/\varepsilon$, which gives a lower bound for the velocity c as required. □

Our next result shows that the assumption of the local monotonicity of f (condition (f1)) and the fact that G is diagonal, forces monotone solutions to be strictly monotone.

Lemma 3.5. *For any $c \in \mathbb{R}$, let w satisfy* (37), *when $\tau\sigma_R$ is replaced by any function $\sigma \in C(\mathbb{R}, [0,1])$ and G is replaced with any continuously differentiable, diagonal-matrix-valued function G. Suppose that for each $s \in \mathbb{R}$, $w'(s) \geq 0$. Then either there exists $E \in \mathbb{R}^N$ such that*

$$w(s) \equiv E \quad \text{for each } s \in \mathbb{R}, \quad \text{or} \quad w'(s) > 0 \quad \text{for each } s \in \mathbb{R}. \tag{45}$$

Proof. Suppose that there exists $s_0 \in \mathbb{R}$ for which $w'(s_0)$ has a zero component; without loss of generality, say $w_1'(s_0) = 0$. Consider the first equation of (37),

$$A_1 w_1''(s) + c w_1'(s) + \sigma(s) G_1(w(s), w'(s)) w_1'(s) + f_1(w(s)) = 0, \quad s \in \mathbb{R}. \tag{46}$$

If $f_1(w(s_0)) \neq 0$, (46) implies that $w_1''(s_0) \neq 0$, which says that w_1 has either a maximum or a minimum at s_0. Since this contradicts the monotonicity of w_1 we conclude that $f_1(w(s_0)) = 0$, and that $w_1''(s_0) = 0$. Now for $s \in \mathbb{R}$, there exists $\theta(s) \in (0,1)$ such that

$$\begin{aligned} f_1(w(s)) &= f_1(w(s_0)) + df[w(s_0) + \theta(s)\{w(s) - w(s_0)\}](w(s) - w(s_0)) \\ &= \sum_{i=1}^{N} \frac{\partial f_1}{\partial w_i}(w(s_0) + \theta(s)\{w(s) - w(s_0)\})(w_i(s) - w_i(s_0)) \end{aligned} \tag{47}$$

since $f_1(w(s_0)) = 0$. Define $v(s) = w_1(s) - w_1(s_0), s \in \mathbb{R}$. Then for $s \leq s_0$, $v(s) \leq v(s_0) = 0$. Moreover, (46) and (47) yield that

$$\begin{aligned} A_1 v''(s) &+ \{c + \sigma(s) G_1(w(s), w'(s))\} v'(s) \\ &+ \frac{\partial f_1}{\partial w_1}[w(s_0) + \theta(s)\{w(s) - w(s_0)\}] v(s) \\ &= -\sum_{i=2}^{N} \frac{\partial f_1}{\partial w_i}[w(s_0) + \theta(s)\{w(s) - w(s_0)\}](w_i(s) - w_i(s_0)). \end{aligned} \tag{48}$$

It follows from the local monotonicity of f (condition (f1)) and the fact that $f_1(w(s_0)) = 0$ that $\dfrac{\partial f_1}{\partial w_i}(w(s_0)) > 0$ for $i \neq 1$. Thus since $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, there exists a neighbourhood U of $w(s_0)$ such that

$$\frac{\partial f_1}{\partial w_i}(w(s_0)) > 0 \quad \text{for each } i \neq 1 \quad \text{and each } E \in U. \tag{49}$$

Let $s_1 < s_0$ be such that $w(s) \in U$ for each $s \in \mathbb{R}$, $s_1 \leq s \leq s_0$. It follows from (48), the monotonicity of w and the fact that $v(s_0) = 0$ that

$$A_1 v''(s) + \{c + \sigma(s) G_1(w(s), w'(s))\} v'(s) + \frac{\partial f_1}{\partial w_1}[w(s_0) + \theta(s)\{w(s) - w(s_0)\}] v(s) \geq 0 \tag{50}$$

for $s_1 \leq s \leq s_0$. Since G is continuous and $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, the coefficients of v and of v' are bounded independently of s. Hence the Hopf Boundary Point Lemma [7] yields that either $v'(s_0) > 0$, or $v(s) = v(s_0) = 0$ for each $s \in (s_1, s_0)$. By the choice of s_0, $v'(s_0) = 0$. So $v(s) = v(s_0) = 0$ for each $s \in (s_1, s_0)$. In fact, $v(s) = 0$ for each s, $-\infty < s \leq s_0$. For if $\beta := \inf_{s \leq s_0}\{v(t) = 0 \text{ for } s \leq t \leq s_0\}$ were finite, then $w_1'(\beta) = 0$. The above argument then yields the existence of a left-neighbourhood of β on which $v = 0$, which contradicts the definition of β. A similar argument shows that $v(s) = 0$ for each s, $s_0 \leq s \leq \infty$. Hence

$$w_1(s) = w_1(s_0) \quad \text{for each } s \in \mathbb{R}. \tag{51}$$

Suppose now that there exists $j \in \{2, \ldots, N\}$ such that $w_j \not\equiv w_j(s_0)$. It follows from the above argument that $w_j'(s_0) > 0$. Hence for s in a neighbourhood of s_0, $w_j(s) \neq w_j(s_0)$. Now by (51), $v(s) = 0$ for each $s \in \mathbb{R}$, where $v(s) := w_1(s) - w_1(s_0)$. Thus (48) gives that

$$\sum_{i=2}^{N} \frac{\partial f_1}{\partial w_i}[w(s_0) + \theta(s)\{w(s) - w(s_0)\}](w_i(s) - w_i(s_0)) = 0 \tag{52}$$

for each $s \in \mathbb{R}$. Let $s \leq s_0$ be sufficiently close to s_0 that

$$\frac{\partial f_1}{\partial w_i}[w(s_0) + \theta(s)\{w(s) - w(s_0)\}] > 0$$

for each $i \in \{2, \ldots, N\}$ and $w_j(s) \neq w_j(s_0)$. Then the monotonicity of w contradicts (52). So $w(s) \equiv w(s_0)$, and hence (45) holds with $E := w(s_0)$. The result follows. □

The next result shows that monotone solutions of (37) satisfy uniform exponential estimates in neighbourhoods of S and T. Note that these exponential estimates, and the neighbourhoods in which they are valid, are independent of the velocity c (because c is *a priori* bounded), $\tau \in [0, 1]$ and the choice of R.

Theorem 3.6. *There exists κ, α, β and $\delta > 0$ such that if w satisfies* (37) *for some $c \in \mathbb{R}$, $\tau \in [0, 1]$ and $R > 0$, and $w'(s) > 0$ for all $s \in \mathbb{R}$, then*

$$\|w(s) - S\| \leq \kappa e^{\alpha(s - s_0)}, \quad \|w'(s)\| \leq \kappa e^{\alpha(s - s_0)}, \tag{53}$$

where $s_0 \in \mathbb{R}$ is the unique point such that $\|w(s_0) - S\| = \delta$ and $s \leq s_0$. Moreover,

$$\|w(s) - T\| \leq \kappa e^{-\beta(s - t_0)}, \quad \|w'(s)\| \leq \kappa e^{-\beta(s - t_0)}, \tag{54}$$

where $t_0 \in \mathbb{R}$ *is the unique point such that* $\|w(t_0) - T\| = \delta$ *and* $s \geq t_0$. *In addition, for* $E \in \mathbb{R}^N$,

$$0 < \|E - S\| \leq \delta \Rightarrow f(E) \neq 0, \quad \textit{and} \quad 0 < \|E - T\| \leq \delta \Rightarrow f(E) \neq 0. \tag{55}$$

Proof. It will suffice to prove (53) (the proof of (54) is similar). We begin by considering the equation (37) linearised at $w = S$, when σ_R is replaced by 1. We will show that solutions λ of the corresponding characteristic equation $\mathrm{Det}N(\lambda, c, \tau) = 0$ are uniformly bounded away from the imaginary axis, independently of $|c| \leq \gamma$, where γ is as in Theorem 3.4, and $\tau \in [0, 1]$. Here

$$N(\lambda, c, \tau) = \lambda^2 A + c\lambda I + \lambda\tau G(S, 0) + df[S].$$

Suppose that $\lambda = i\mu$ satisfies $\mathrm{Det}N(\lambda, c, \tau) = 0$. Then $-ic\mu$ is an imaginary eigenvalue of $-\mu^2 A + df[S] + i\mu\tau G(S, 0)$. However by f(2) the Perron–Frobenius eigenvalue of $df[S]$ is negative and hence all the eigenvalues of the matrix $-\mu^2 A + df[S]$ are in the left half plane (decreasing the elements of a matrix $M \in P^{N\times N}$ decreases the Perron–Frobenius eigenvalue $\mu_{PF}(M)$). The same is therefore true of the eigenvalues of $-\mu^2 A + df[S] + i\mu\tau G(S, 0)$. (See [14, Lemma 4.1, p. 234].) This is a contradiction. Therefore, since solutions λ of $\mathrm{Det}N(\lambda, c, \tau) = 0$ depend continuously on c and τ, the set of solutions for (c, τ) in the compact set $[-\gamma, \gamma] \times [0, 1]$, is bounded away from the imaginary axis.

With this observation in hand, the proof of [3, Chapter 13, Theorem 4.1] shows that there exist positive numbers δ, κ, α and β (independent of $(c, \tau) \in [-\gamma, \gamma] \times [0, 1]$) such that if w satisfies equation (37) when σ_R is replaced by 1, then the conclusions of this theorem hold. (Note that taking $\tau = 0$ is the same as replacing σ_R by a function which is identically zero, so that case also is covered.)

Finally, for any $(c, \tau) \in [-\gamma, \gamma] \times [0, 1]$, suppose that w is a monotone solution of equation (37) joining S to T. Let $\|w(s_0) - S\| = \delta$. Note that s_0 may be anywhere on the real line so the precise value of $\sigma_R(s_0) \in [0, 1]$ is unknown. However, $\|w(s) - S\| \leq \delta$ for all $s \leq s_0$, by the monotonicity of w, and consequently we have established uniform exponential decay of $w(s)$ to S as $s \to -\infty$ is on the set $(-\infty, s_0) \setminus Y$, where $Y := (-R, -R+1) \cup (R-1, R))$ is the set upon which $\sigma_R \notin \{0, 1\}$. Since the decay estimates already established are uniform and since Y has length at most 2 (independently of R), the required result is immediate.□

The following theorem gives the relation between the asymptotic behaviour of certain solutions of an ordinary differential equation and the eigenvectors of a nonlinear-in-λ linear eigenvalue problem.

Theorem 3.7. *Let* $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ *and* $E \in \mathbb{R}^N$ *be such that* $f(E) = 0$, $df[w_0] \in P^{N\times N}$ *and* $\mu_{PF}(df[w_0]) \neq 0$. *Let* $A \in M^{N\times N}$ *be a positive-definite* *diagonal matrix, G be continuously differentiable, and the constant ϱ be either* 0 *or* 1*. Suppose that $w \in C^2(\mathbb{R}, \mathbb{R}^N)$ satisfies*

$$Aw'' + cw' + \varrho\tau G(w, w')w' + f(w) = 0, \quad s \in \mathbb{R}, \tag{56}$$

for some $c \in \mathbb{R}$ and $\tau \in [0,1]$. Further, suppose that

$$w(s) \not\equiv E \quad \text{and} \quad w(s) \to E \text{ as } s \to \infty \ (s \to -\infty) \tag{57}$$

and there exists $s_0 \in \mathbb{R}$ and $M > 0$ such that

$$s \geq s_0 \ (s \leq s_0) \Rightarrow w(s) \leq E \ (w(s) \geq E) \quad \text{and} \quad \|w'(s)\| \leq M. \tag{58}$$

Then there exist $\lambda < 0$ ($\lambda > 0$) and a vector $q \in \mathbb{R}^N$ such that $q > 0$, and

$$(\lambda^2 A + \lambda c I + \lambda \varrho \tau G(w_0, 0) + df[w_0])q = 0; \tag{59}$$

that is, there is a stable (unstable) monotone eigenvalue of the travelling-wave problem linearized at E.

Proof. Consider the case when $s \to \infty$, and without loss of generality assume $E = 0$. For the proof in the case of the linear equation

$$Aw'' + (c + D)w' + Bw = 0, \tag{60}$$

where D is a diagonal matrix, we refer the reader to the argument in the proof of [14, Lemma 2.4, p. 161]. To treat the nonlinear problem, consider translates w_n of the solution w such that $\|w_n(0)\| = \sup_{s \geq 0} \|w_n(s)\| = n^{-1}$ and let $v_n(s) := n^{-1} w_n(s)$ for $s \geq 0$. Then

$$A{v_n}'' + c{v_n}' + \varrho G(w_n, {w_n}'){v_n}' + Bv_n + \frac{R(w_n)}{\|w_n(0)\|}, \quad s > 0,$$

where $B := df[0]$ and $\|R(y)\|/\|y\| \to 0$ as $y \to \infty$. The *a priori* bounds of Theorem 3.2 enable the use of the Arzela–Ascoli theorem to prove the existence of a non-trivial monotone solution v of the linear problem (60) with $D := G(0,0)$. The result follows. □

The importance of the above theorem for the gradient-dependent travelling-wave problem lies in the following application.

Lemma 3.8. *Suppose that* f(1)–f(3) *and* (g2) *hold and that $E \in (S,T)$ is such that $f(E) = 0$. Then, for fixed $c \in \mathbb{R}$ and $\tau \in [0,1]$, there cannot exist two functions $w_1, w_2 \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that w_1 and w_2 satisfy* (56) *with $\varrho = 1$ and* 0*, respectively and*

$$w_1(s) \to E \quad \text{as } s \to -\infty, \quad w_1'(s) \geq 0 \quad \text{for sufficiently large } -s, \tag{61}$$

$$w_2(s) \to E \quad \text{as } s \to \infty, \quad w_2'(s) \geq 0 \quad \text{for sufficiently large } s. \tag{62}$$

Further, for fixed $c \in \mathbb{R}$ and $\tau \in [0,1]$, there cannot exist two such functions $w_1, w_2 \in C^2(\mathbb{R},\mathbb{R}^N)$ both of which satisfy (56) *with $\varrho = 1$, or both of which satisfy* (56) *with $\varrho = 0$.*

Proof. Suppose that there is $c \in \mathbb{R}$ such that functions w_1, w_2 as in the first statement of the lemma exist. Then it follows from Theorem 3.7 that there are real numbers $\lambda_1 > 0$, $\lambda_2 < 0$ and vectors $q_1, q_2 > 0$, such that

$$(\lambda_1^2 A + \lambda_1 cI + \lambda_1 \tau G(E,0) + df[E])q_1 = 0, \tag{63}$$

and

$$(\lambda_2^2 A + \lambda_2 cI + df[E])q_2 = 0. \tag{64}$$

Now (63) says that the Perron–Frobenius eigenvalue of $\lambda_1^2 A+\lambda_1 cI+\lambda_1\tau G(E,0)+df[E]$ is zero. Also, when $\lambda = 0$ the Perron–Frobenius eigenvalue of $\lambda^2 A + \lambda cI + \lambda\tau G(E,0) + df[E]$ is positive, by hypothesis f(3). But it is shown in [4] that for c fixed, the Perron–Frobenius eigenvalue is a strictly convex function of λ. Therefore, since $\lambda_1 > 0$ and A is positive-definite, the Perron–Frobenius eigenvalue of $\lambda^2 A + \lambda cI + \lambda\tau G(E,0) + df[E]$ is positive for all negative λ. In particular, this is the case when $\lambda = \lambda_2$. However, since $G(E,0)$ is positive definite by (g2), it follows that the Perron–Frobenius eigenvalue of $(\lambda_2^2 A+\lambda_2 cI+df[E])$ is positive. This contradicts (64). The first part is proven.

For the proof of the second part, recall that the Perron–Frobenius eigenvalue of $(\lambda^2 A + \lambda cI + \lambda\tau\varrho G(E,0) + df[E])$ is a convex function of λ which, by the hypothesis of the theorem, is positive at $\lambda = 0$. But, by Theorem 3.7 and the present hypotheses, there exists $\lambda_1 > 0 > \lambda_2$ such that the Perron–Frobenius eigenvalue of $\lambda_1^2 A + \lambda_i cI + \lambda_i\tau\varrho G(E,0) + df[E]$ for $i = 1, 2$ is zero. This is a contradiction. The proof is now complete. □

Estimates for the approximate system that are independent of R. Consider now the system

$$\begin{cases} A(w'') + c(w)w' + \tau\sigma_R G(w,w')(w') + f(w) = 0, \\ w = u + \psi,\ u \in W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N) \cap C^2(\mathbb{R},\mathbb{R}^N), \end{cases} \tag{65}$$

where $R > 0$ and σ_R is as in Section 2, (a), (f1)–(f3), (g1) and (g2) hold, and $c(w)$ is defined in (23). We will prove *a priori* lower bounds on $w_i'(s)$, $i \in \{1,\dots,N\}$, when $w(s)$ is outside the δ-neighbourhoods of S and T constructed in Theorem 3.6. The following lemma is the key.

Lemma 3.9. *Suppose w is a monotone solution of* (65) *for some $R > 0$ and $\tau \in [0,1]$. Let δ be defined as in Theorem* 3.6. *Then the set $\{t \in \mathbb{R} \mid \|w(t) - T\| = \delta\}$ is bounded independently of R, τ and w.*

PROOF. From the definition of $c(w)$, it is immediate that the set $\{t \in \mathbb{R} \mid \|w(t) - T\| = \delta\}$ is bounded above independently of τ and R, for otherwise the set of all possible $c(w)$ for monotone w would be unbounded, contrary to the assertion in Theorem 3.4. Also, because of (54) and the definition of $c(w)$, the set $\{\tau \in \mathbb{R} \mid \|w(t) - T\| = \delta\}$ is bounded below independently of τ and R, for otherwise $c(w)$, w monotone, would be unbounded. This completes the proof.□

THEOREM 3.10. *Let w satisfy* (65) *for some $\tau \in [0,1]$ and $R > 0$, and suppose that $w'(s) > 0$ for all $s \in \mathbb{R}$. Then there exists $\chi > 0$, independent of u, τ and R, such that*

$$w_i'(s) > \chi, \quad i = 1, \dots, N, \tag{66}$$

when $\|w(s) - S\| \geq \delta$ and $\|w(s) - T\| \geq \delta$, where δ is as in Theorem 3.6.

PROOF. Suppose that the result is false. Then there exist sequences $\{w^k\}$, $\{\tau_k\}$, $\{R_k\}$ and $\{s_k\}$ and $i_0 \in \{1, \dots, N\}$ such that w^k satisfies (65) with $\tau = \tau_k$ and $R = R_k$,

$$\|w^k(s_k) - S\| \geq \delta, \quad \|w^k(s_k) - T\| \geq \delta, \quad \text{and} \quad 0 < (w_{i_0}^k)'(s_k) < k^{-1}. \tag{67}$$

Since Theorem 3.4 gives that $\{c(w_k)\}$ is bounded, there no loss of generality in assuming that $i_0 = 1$, $\tau_k \to \tau_0$ and $c(w_k) \to c_0$ as $k \to \infty$.

We consider first the possibility that $R_k \to \infty$ as $k \to \infty$. Let $\{t_k\} \subset \mathbb{R}$ be defined by $\|w^k(t_k) - T\| = \delta$. Lemma 3.9 ensures that $\{t_k\}$ is bounded. Let

$$c_k = c(w_k), \quad v^k(s) = w^k(s + t_k), \quad \text{and} \quad \sigma^k(s) = \sigma_{R_k}(s + t_k), \tag{68}$$

so that $\|v^k(0) - T\| = \delta$ and

$$A(v^k)'' + c_k(v^k)' + \tau_k \sigma^k G(v^k, (v^k)')(v^k)' + f(v^k) = 0, \quad s \in \mathbb{R}. \tag{69}$$

Also, $(w_1^k)'(s_k) \to 0$ as $k \to \infty$, and

$$\|v^k(s_k - t_k) - S\| \geq \delta, \quad \|v^k(s_k - t_k) - T\| \geq \delta. \tag{70}$$

Since w^k is a monotone solution connecting S and T, $\{\|w^k\|_{L_\infty(\mathbb{R},\mathbb{R}^N)}\}$ is bounded and, by Theorem 3.2, there exist $N_1, N_2 > 0$,

$$\|(v^k)'(s)\| \leq N_1, \quad \|(v^k)''(s)\| \leq N_2 \quad \text{for all } s \in \mathbb{R}. \tag{71}$$

Note that since $\{t_k\}$ is bounded, and $R_k \to \infty$, $\sigma^k \to 1$ as $k \to \infty$ uniformly on compact intervals. Since G and f are continuous, an Arzela–Ascoli argument means that there is no loss of generality in further assuming that there exists $v \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that $v^k \to v$ in $C^2([-r, r], \mathbb{R}^N)$ for each $r > 0$. Passing to the limit in (69) yields that

$$Av''(s) + c_0 v'(s) + \tau_0 G(v(s), v'(s))v'(s) + f(v(s)) = 0, \quad s \in \mathbb{R}, \tag{72}$$

where

$$\|v(0)-T\|=\delta \quad S\le v(s)\le T,\quad s\in\mathbb{R}\quad\text{and}\quad v'(s)\ge 0,\quad s\in\mathbb{R}. \tag{73}$$

Clearly there exist $p,q\in\mathbb{R}^N$, with $S\le p\le q\le T$ such that

$$v(s)\to p\quad\text{as } s\to-\infty\quad\text{and}\quad v(s)\to q\quad\text{as } s\to\infty. \tag{74}$$

Moreover, since $\|v''(s)\|\le N_2$ by (71), Landau's inequality gives that $v'(s)\to 0$ as $|s|\to\infty$. Hence by (74) and (72), $v''(s)\to 0$ as $|s|\to\infty$. So using (74), we find that $f(p)=f(q)=0$. It thus follows from (73) that $q=T$, because δ satisfies (55).

Next suppose, for contradiction, that $p=S$. By (70), (74) and the fact that $v^k\to v$ pointwise, $\{s_k-t_k\}$ is a bounded sequence, so has a convergent subsequence, say $s_k-t_k\to x_0\in\mathbb{R}$. Now $v_1'(x_0)=0$, since $(v_1^k)'(s_k-t_k)\to 0$. But this contradicts Lemma 3.5 since v satisfies (72) and (73) holds. So $p\ne S$.

Recall (f3) that there are only a finite number of zeros E of f with $S\le E\le T$ and therefore, by local monotonicity, ε can be chosen so that

$$f(E)=0,\ |w_{0_i}-p_i|\le\varepsilon\quad\text{for some } i\in\{1,\dots,N\}\Rightarrow E=p. \tag{75}$$

Let $\Sigma:=(\varepsilon,\dots,\varepsilon)$ and for $n\in\mathbb{N}$, choose $x_n\in\mathbb{R}$ such that $p\le v(x_n)\le p+\Sigma/4n\sqrt{N}$. Since $v^k(x_n)\to v(x_n)$ as $k\to\infty$, we can choose k_n $(>k_{n-1})$ such that $p\le v^{k_n}(x_n)\le p+\Sigma/2n\sqrt{N}$. Since $v^{k_n}(s)\to S$ monotonically as $s\to-\infty$, there exists unique $\alpha_{k_n}\in\mathbb{R}$ such that $\|v^{k_n}(\alpha_{k_n})-p\|=\varepsilon$ and $\|v^{k_n}(s)-p\|\ge\varepsilon$ for $s\le\alpha_{k_n}$. For each $n\in\mathbb{N}$, define $\widetilde{v}^{k_n}(s)=v^{k_n}(s+\alpha_{k_n})$, $s\in\mathbb{R}$. Then $\|\widetilde{v}^{k_n}(0)-p\|=\varepsilon$ for each n, and $S\le\widetilde{v}^{k_n}(0)\le p+\Sigma/n$. Arguing as in the construction of the function v above, we obtain the existence of $\widetilde{v}\in C^2(\mathbb{R},\mathbb{R}^N)$ and $\widetilde{\sigma}\in C^1(\mathbb{R},[0,1])$ such that (for a subsequence) $\widetilde{v}^{k_n}\to\widetilde{v}$ uniformly on compact subsets of $\mathbb{R}$,

$$\|\widetilde{v}(0)-p\|=\varepsilon,\quad S\le\widetilde{v}(0)\le p,\quad \widetilde{v}'(s)\ge 0,\quad s\in\mathbb{R}, \tag{76}$$

and

$$A\widetilde{v}''+c_0\widetilde{v}'+\tau_0\widetilde{\sigma}G(\widetilde{v},\widetilde{v}')\widetilde{v}'+f(\widetilde{v})=0, \tag{77}$$

where here $\widetilde{\sigma}\equiv 0$ or 1 for s sufficiently large. (We do not know *a priori* that the sequence $\{\alpha_{n_k}\}$ is bounded, hence the ambiguity about the limit of $\widetilde{\sigma}_k$ in this case.) As before, there exist $p',q'\in\mathbb{R}^N$ such that $S\le p'\le q'\le T$, and $\widetilde{v}(s)\to p'$ as $s\to-\infty$ and $\widetilde{v}(s)\to q'$ as $s\to\infty$.

Now by condition (f3), $\mu_{PF}(df[E])>0$ for every E, $S<E<T$ with $f(E)=0$. Theorem 2.4 of [4] implies the existence of a zero E of f such that $\mu_{PF}(df[E])<0$ in the order interval between two zeros whose Fréchet derivatives have positive Perron–Frobenius eigenvalues. Hence there is no zero E of f such that $S<E<p$ or $p<E<T$. Thus $p'=S$ since $\|\widetilde{v}(0)-p\|=\varepsilon$. Also,

(75) yields that $q' \in \{p, T\}$. If $q' = T$, then as earlier in this proof, the points $s_{k_n} - t_{k_n} - \alpha_{k_n}$ at which $(v_1^{k_n})'$ tends to zero are contained in a finite interval in $\mathbb{R}$. So there exists $z_0 \in \mathbb{R}$ such that $(\widetilde{v_1})'(z_0) = 0$, which contradicts Lemma 3.5 since (76) and (77) hold. Hence $q' = p$.

Thus we have $\widetilde{v}$ satisfying (77), where $\widetilde{\sigma}$ is either 0 or 1, with $\widetilde{v}(s) \leq p, s \in \mathbb{R}$, $\widetilde{v}(s) \to p$ as $s \to \infty$, and v satisfying (72) with $v(s) \geq p, s \in \mathbb{R}$, $v(s) \to p$ as $s \to -\infty$. But since $\mu_{PF}(df[p]) > 0$, this contradicts Lemma 3.8. The result follows for the case $R_k \to \infty$.

If $\{R_k\}$ is bounded, $\sigma_{R_k} \to \sigma_R$ (or a subsequence) uniformly on compact intervals, as $k \to \infty$, for some finite R. If $\{s_k\}$ is bounded, a standard Ascoli–Arzela argument yields a solution of equation (37) which violates Lemma 3.5. If $\{s_k\}$ is unbounded then a subsequence converges to $-\infty$, by Lemma 3.9. The argument for the case $R_k \to \infty$ may be repeated to find two solutions of equation (56) with $\varrho = 0$, contradicting the last part of Lemma 3.8. This completes the proof. □

Let $\mathcal{W}$ denote the set of all monotone functions $w \in C^2(\mathbb{R}, \mathbb{R}^N)$ satisfying (65) for some and $R > 1$ and $\tau \in [0, 1]$.

Theorem 3.11. *There exists constant $C > 0$ such that if $w \in \mathcal{W}$,*

$$\|w - \psi\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} = \|u\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \leq C. \tag{78}$$

Proof. By Lemma 3.9 and Theorem 3.10, it is immediate that if $w \in \mathcal{W}$ and δ is defined as in Theorem 3.6, then the set $\{s \in \mathbb{R} : \|w(s) - S\| = \delta\}$ is bounded independently of R, τ and w. The result is now immediate from Theorems 3.2 and 3.6. □

4. Existence of monotone solutions

In [14, Chapters 2 and 3, Theorem 1.1, p. 153] it is shown that for $G \equiv 0$, there exists $c \in \mathbb{R}$ for which there is a heteroclinic orbit from S to T of (4), the components of which are monotone. We first show that for each $R > 0$, there exists $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ such that $u'(s) + \psi'(s) > 0$ for each $s \in \mathbb{R}$, u satisfies (20), and $u(s) + \psi(s) \to S, T$ as $s \to -\infty, +\infty$.

Existence of monotone solutions of the approximate system. Fix $R > 0$. With the aim of constructing a set $\Omega \subset W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ satisfying (29), the first theorem gives that non-monotone solutions $w_N = u_N + \psi$ (N for non-monotone) of (24) are bounded away from monotone solutions $w_M = u_M + \psi$ (M for monotone in $W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$). In the following, write $u_M = w_M - \psi$ when $w'_M(s) > 0$ for each $s \in \mathbb{R}$ and $u_N = w_N - \psi$ when there exists $s_0 \in \mathbb{R}$ and i such that $(w_{N_i})'(s_0) \leq 0$. Since the proof of this result is a minor modification of the proof of the corresponding result in [14, Proposition 1.2, Chapter 3, pp. 167–169], we omit it.

Theorem 4.1. *Let $R > 0$. Then there exists $r_R > 0$ such that if $u_M, u_N \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ satisfy* (25) *with corresponding $\tau = \tau_M, \tau_N \in [0,1]$ for this R, and $w_M = u_M + \psi$ and $w_N = u_N + \psi$ are monotone and non-monotone respectively, then*

$$\|u_M - u_N\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} \geq r_R. \tag{79}$$

Now define $\Omega \subset W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$ by

$$\Omega := \bigcup_{R>0} \{v \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) : \|u_R - v\|_{W^1_{2,\mu}(\mathbb{R},\mathbb{R}^N)} < r_R/2, \quad \text{for some monotone } u_R \text{ satisfying (25)}\}.$$

Clearly Ω is open, and bounded by Theorem 3.11. Moreover,

$$S_\mu : W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \to W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N)$$

constructed in Theorem 2.5, and P_R, defined for $R > 0$ by (26), satisfy (29):

$$0 \notin S^*_\mu P_R([0,1] \times \partial\Omega).$$

To see this note that if $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ is non-monotone and satisfies (25) then Theorem 4.1 yields that $u \notin \overline{\Omega}$. On the other hand, if w is monotone, then $u \in \Omega^o$ (the interior of Ω).

Hence (30) holds:

$$\deg_{(S)_+}(S^*_\mu P_R(1, \cdot\,), \Omega, 0) = \deg_{(S)_+}(S^*_\mu P_R(0, \cdot\,), \Omega, 0).$$

Now when $\tau = 0$, (25) reduces to the system studied by the Vol'perts [14, Chapter 3], and it is implicit in their proof that $\deg_{(S)_+}(S^*_\mu P_R(1, \cdot\,), \Omega, 0) = 1$. (This follows by first simplifying slightly the argument for existence in [14, pp. 169–170] by constructing $\widetilde{\Omega}$ analogous to Ω above, and pursuing their homotopy argument.)

The following is then immediate from the existence property of degree, (29), and the fact that there are no non-monotone solutions of (25) in $\overline{\Omega}$.

Theorem 4.2. *Let $R > 0$ be given. Then there exists $u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ such that $u'(s) + \psi'(s) > 0$ for each $s \in \mathbb{R}$, and*

$$A(u'' + \psi'') + c(u)(u' + \psi') + \sigma_R G(u + \psi, u' + \psi')(u' + \psi') + f(u + \psi) = 0.$$

Moreover, estimate (78) *in Theorem* 3.11 *is valid for the same constant C.*

Monotone solutions of the autonomous system. Here we combine the uniform *a priori* estimates for monotone solutions of Section 3 with the existence of monotone solutions for the approximate system proved above. (Since (g2) can be assumed without loss of generality, we do not cite it in the hypothesis of this theorem.)

THEOREM 4.3. *Let* (a), (f1)–(f3) *and* (g1) *hold. Then there exists*

$$w \in C^2(\mathbb{R}, \mathbb{R}^N) \text{ and } c \in \mathbb{R} \quad \text{such that } w'(s) > 0 \quad \text{for each } s \in \mathbb{R},$$

$w(s) \to S, T$ *as* $s \to \pm\infty$, *and* $u(x,t) = w(x - ct)$ *is a solution of*

$$u_t = Au_{xx} + G(u, u_x)u_x + f(u).$$

Moreover, estimate (78) *in Theorem* 3.11 *is valid for the same constant* C.

PROOF. For each $n \in \mathbb{N}$, let $u_n \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N)$ be such that

$$u_n'(s) + \psi'(s) > 0 \quad \text{for each } s \in \mathbb{R}, \tag{81}$$

and

$$A(u_n''+\psi'')+c(u_n)[u_n'+\psi']+\sigma_n G(u_n+\psi, u_n'+\psi')(u_n'+\psi')+f(u_n+\psi) = 0. \tag{82}$$

By Lemma 3.9 and Theorem 3.10, there is a bounded interval J (independent of n) of the real line with the property that if $s \notin J$ then $w_m(s)$ lies in the δ-neighbourhood of S or T described in Theorem 3.6. Since w_m is monotone for each n, $\{w_n\}$ has a subsequence which converges everywhere on $\mathbb{R}$, to a function w say. By the Arzela–Ascoli theorem (see Theorem 3.2) the convergence and that of derivatives is uniform on compact intervals — in particular, on J. A simple limiting argument now gives that the limiting function is a solution of the travelling-wave equation with $c = c(w)$. Since w is monotone and contained in $[S, T]$, it converges to S as $s \to -\infty$ and to T as $s \to \infty$ because the convergence of the $\{w_j\}$ is uniform on J and δ satisfies (55). The final estimate follows from Theorem 4.2 and Fatou's lemma. This completes the proof. □

5. The general case

Here we indicate how the results of the preceding theory lead to an existence theory under the following hypothesis, which may be regarded as an extension of (g1). This generalisation takes account of possible monotonicity of the component functions of G.

(G) G is a continuously differentiable, diagonal-matrix-valued function on $\mathbb{R}^N \times \mathbb{R}^N$ and there are continuous non-decreasing functions β, γ : $[0, \infty) \to [0, \infty)$ such that for all $i \in \{1, \ldots, N\}$, a, $b \in \mathbb{R}^N$ with

$\|a\|$, $\|b\| \le M$ and p, q in $\mathbb{R}^N$ with $0 \le q \le \|p\|_\infty \mathbf{1}$ where $\mathbf{1} = (1, 1, \ldots 1) \in \mathbb{R}^N$,

$$G_i(a, q) - G_i(b, p) \le \gamma(M)(1 + \beta(\|p\|_\infty))$$

where $\beta(p)/p \to 0$ as $p \to \infty$, and β_∞ is such that $\beta(p)/p \le \beta_\infty$.

Clearly if G satisfies (g1), it satisfies (G), but not vice-versa.

Remark. An existence theory for bistable systems involving G satisfying (G) immediately yields an equivalent existence result for bistable systems for which $-G$ satisfies (G). This is a consequence of the following observation. Suppose that for some $c \in \mathbb{R}$, $u \in C^2(\mathbb{R}, \mathbb{R}^N)$ is a monotone solution of system (4) with f satisfying (f1)–(f3), such that $u(s) \to S$ as $s \to -\infty$ and $u(s) \to T$ as $s \to \infty$. Let $w(s) = S + T - u(-s)$ for each $s \in \mathbb{R}$. Then

$$Aw'' + \widetilde{c}w' - G(S + T - w, w')w' - f(S + T - w) = 0, \tag{83}$$

where $\widetilde{c} = -c$, w is monotone, and $w(s) \to S, T$ as $s \to -\infty, \infty$ respectively. Furthermore, $-f(S + T - \cdot)$ satisfies (f1)–(f3). We are grateful to H. Matano for pointing out the significance of this change of variables for our problem. In particular, it leads to a theory of existence of travelling waves whenever G_i is a monotonic function of w_i' alone, since we can take $\beta = \gamma \equiv 0$ in (G) when G_i is an *increasing* function of w_i'.

Our approach to the problem under the hypothesis (G) is to introduce a truncation G_K of the mapping G, where G_K satisfies (g1) for each $K \in \mathbb{N}$. We will prove the existence of a monotone travelling-wave solution of equation (4) via the known existence of such solutions when G is replaced by G_K together with a uniform-in-K *a priori* bound on the derivative of these solutions.

For each $K > 0$, let $\psi_K \in C^1(\mathbb{R}, \mathbb{R})$ be a monotone function with bounded range, for which $\psi_K(\alpha) = \alpha$ for all $\alpha \in \mathbb{R}$ with $|\alpha| \le K$ and $|\psi_K(\alpha)| \le |\alpha|$ for all $\alpha \in \mathbb{R}$. Define $\chi_K : \mathbb{R}^N \to \mathbb{R}^N$ by $(\chi_K)_i(p) = \psi_K(p_i)$, $p \in \mathbb{R}^N$, $i \in \{1, \ldots, N\}$. Let $G_K(u, v) = G(u, \chi_K(v))$. Note that for all $i \in \{1, \ldots, N\}$, $a, b \in \mathbb{R}^N$ with $\|a\|$, $\|b\| \le M$ and p, q in $\mathbb{R}^N$ with $0 \le q \le \|p\|_\infty \mathbf{1}$ in $\mathbb{R}^N$,

$$(G_K)_i(a, q) - (G_K)_i(b, p) = G_i(a, \chi_K(q)) - G_i(b, \chi_K(p)) \tag{84}$$

$$\le \gamma(M)(1 + \beta(\|\chi_K(p)\|_\infty)) \le \gamma(M)(1 + \beta(\|p\|_\infty)). \tag{85}$$

So G_K satisfies hypothesis (G), uniformly in K.

Then, since G is continuous, G_K satisfies (g1) for a function β that is independent of K. The existence, for some $c \in \mathbb{R}$, of a monotone solution $w = w_K$ of

$$\begin{cases} Aw'' + cw' + G_K(w, w')w' + f(w) = 0, \\ w = u + \psi, \ u \in W^1_{2,\mu}(\mathbb{R}, \mathbb{R}^N) \cap C^2(\mathbb{R}, \mathbb{R}^N), \\ w' > 0, \end{cases} \tag{86}$$

for each $K \in \mathbb{N}$ is therefore guaranteed by Theorem 4.3.

THEOREM 5.1. *Suppose that w_K satisfies* (86) *for some $c \in \mathbb{R}$ and $K \in \mathbb{N}$. Then there exists $N_1 > 0$, independent of K and c, such that*

$$\|{w_K}'\|_{L_\infty(\mathbb{R},\mathbb{R}^N)} < N_1.$$

PROOF. To prove the existence of N_1, suppose without loss of generality that for a given K the first component of w_K has the property that

$$(87)\quad {(w_K)_1}'(s_0) = \max_{s\in\mathbb{R}} {(w_K)_1}'(s) = \max_{i\in\{1,\dots,N\}} \max_{s\in\mathbb{R}} (w_K)_i'(s) \geq {(w_K)_1}'(s_0) > 0,$$

where $s_0 \in \mathbb{R}$ is chosen such that

$$(88)\qquad \|(w_K)_1'\|_\infty = {(w_K)_1}'(s_0).$$

For convenience in what follows, let v denote $(w_K)_1$ and let $(G_K)_1(w_K, {w_K}')$ be denoted by $g(w_K, {w_K}')$. Then $v' > 0$ and v satisfies the equation

$$A_1 v'' + cv' + g(w_K, {w_K}')v' + f_1(w_K) = 0$$

on $\mathbb{R}$. Since v' has its maximum at s_0,

$$c = -g(w_K(s_0), {w_K}'(s_0)) - \frac{f_1(w_K(s_0))}{v'(s_0)},$$

and therefore for $s \in \mathbb{R}$

$$A_1 v''(s) + \{g(w_K(s), {w_K}'(s)) - g(w_K(s_0), {w_K}'(s_0))\}v'(s) + \left\{f_1(w_K(s)) - \frac{f_1(w_K(s_0))}{v'(s_0)}v'(s)\right\} = 0.$$

Now

$${A_1}^{-1}\left\|f_1(w_K(s)) - \frac{f_1(w_K(s_0))}{v'(s_0)}v'(s)\right\| \leq M,$$

for some constant M, since $w_K \in [S, T]$ and since $0 < v'(s)/v'(s_0) \leq 1$, by (88). Hence, since v is an increasing function, for $s \in \mathbb{R}$

$$(89)\qquad \begin{aligned} -v''(s) &\leq {A_1}^{-1}\{g(w_K(s), {w_K}'(s)) - g(w_K(s_0), {w_K}'(s_0))\}v'(s) + M \\ &\leq {A_1}^{-1}\gamma_0(1 + \beta(v'(s_0))v'(s) + M, \end{aligned}$$

for some constant γ_0, by (87), (88) and (G). Let $m = \|S\| + \|S - T\|$. Then $|v(s)| \leq m$ and since v is twice continuously differentiable, integration by parts gives that for all $s \in \mathbb{R}$

$$v(s+1) = v(s) + v'(s) + \int_0^1 (1-t)v''(s+t)\,dt,$$

from which it follows that

$$
\begin{aligned}
(90)\qquad 0 < v'(s_0) &= v(s_0+1) - v(s_0) - \int_0^1 (1-t)v''(s_0+t)\,dt \\
&\leq 2m + {A_1}^{-1}\gamma_0(1+\beta(v'(s_0))\int_0^1 (1-t)v'(s_0+t)\,dt + M/2 \\
&\leq 2m + 2m{A_1}^{-1}\gamma_0(1+\beta(v'(s_0)) + M/2.
\end{aligned}
$$

From the fact that $\beta(p)/p \to 0$ as $p \to \infty$, it follows that $v'(s_0)$ is bounded independently of c and K. The existence of N_1 as required then follows from (87) and (88). This completes the proof. □

THEOREM 5.2. *Let* (a), (f1)–(f3) *and* (G) *hold. Then there exists* $w \in C^2(\mathbb{R},\mathbb{R}^N)$ *and* $c \in \mathbb{R}$ *such that* $w'(s) > 0$ *for each* $s \in \mathbb{R}$, $w(s) \to S, T$ *as* $s \to \pm\infty$, *and* $u(x,t) = w(x-ct)$ *is a solution of*

$$u_t = Au_{xx} + G(u,u_x)u_x + f(u).$$

PROOF. Theorem 5.1 and the definition of G_K together show that for K sufficiently large and w_K a solution of (86), $G_K(w_K, w'_K) = G(w_K, w'_K)$. Whence w_K satisfies (4). The result follows. □

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