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SOLUTIONS OF SUPERLINEAR AT ZERO ELLIPTIC EQUATIONS VIA MORSE THEORY

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Dedicated to the memory of Professor Mark Aleksandrovich Krasnosel'skiĭ

In this note we study the existence of nontrivial solutions of the Dirichlet problem

$$
\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}
$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary. We assume that $f \in C(\mathbb{R},\mathbb{R})$ satisfies $f(0) = 0$, so the constant function $u \equiv 0$ is a trivial solution of (1). We are interested in the existence of nontrivial solutions when f is superlinear at zero, that is near zero it looks like $O(u|u|^{\nu-2})$ for some $\nu \in (1,2)$. More precisely, we assume that f and its primitive

$$
F(u) = \int_0^u f(\xi)\,d\xi,
$$

satisfy the following conditions:

(f_1) for some $\nu \in (1,2)$ there are constants $r, a_r > 0$ such that

$$
F(u) \geq a_r|u|^\nu \quad \text{for} \quad |u| \leq r,
$$

(f_2) $F(u) - uf(u)/2 > 0$ for all $u \neq 0$.

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We seek solutions of (1) in the Sobolev space $\mathbf{H} = \mathbf{H}_0^1(\Omega)$ under the usual norm

$$\|u\| = \left(\int_\Omega |\nabla u|^2 \, dx \right)^{1/2}.$$

It is known, that under an additional assumption

(f) for some $p \in (1, 2^*)$ (where $2^* = 2N/(N-2)$ if $N \geq 3$ and $2^* = \infty$ if $N = 1, 2$) there exists a constant $a_1 > 0$ such that

$$|f(u)| \leq a_1(1 + |u|^{p-1}) \quad \text{for all} \quad u \in \mathbb{R},$$

the solutions of equation (1) correspond to the critical points of the "energy" functional

$$\varphi(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(u) \, dx,$$

which is of class $\mathcal{C}^1$ on $\mathbf{H}$.

There are many papers in which problem (1) with superlinear at zero nonlinearities is treated by means of the method of positive operators; see for example [9] for the earlier work of Krasnosel'skiĭ et al. and the recent paper [2]. There are only a few papers that we know which consider the problem (1) with superlinear at zero function f using variational methods. Recently, in [5] (see also [1], [8], [2] for previous results of this sort), Bartsch and Willem using a suitable variant of critical point theory for functionals with symmetries have considered the case when f is superlinear at zero and odd. Under some assumptions on f at infinity, they established the existence of infinitely many (pairs of) solutions of (1) with negative "energy" φ.

The main goal of this paper is to establish the existence of nontrivial solutions of (1) without any symmetry assumptions on f. Our approach is based on the Morse theory for isolated critical points developed by Chang [7] (see also [10]). In Section 1 we show that under assumptions (f), (f_1), (f_2), the Morse critical groups at zero for φ are trivial. In other words, zero is a homologically "invisible" critical point of φ. Combining this result with different conditions on f at infinity, we obtain the existence of multiple solutions of (1). More precisely, we consider the cases when f is

(a) coercive (Section 2),

(b) asymptotically linear (Section 3).

It is a pity but we can not consider the case when f is

(c) asymptotically superlinear

since it is incompatible with our assumption (f_2). We discuss this problem in Section 4 together with some partial results for this case for two point boundary value problem, when condition (f_2) can be relaxed. A description of another technique for this case is given in the recent article [3].

1. Morse critical groups at zero

Hereafter, as usual, for $c \in \mathbb{R} \cup \{\infty\}$ we write

$$\varphi^c = \{u \in \mathbf{H} : \varphi(u) \le c\}, \quad \dot{\varphi}^c = \{u \in \mathbf{H} : \varphi(u) < c\},$$

for the closed and open sublevel sets of φ. The closed ball in $\mathbf{H}$ of radius $\rho > 0$ with the center at the origin will be denoted by B_ρ.

We recall [7], [10] that the *Morse critical groups* of an isolated critical point u with value $c = \varphi(u)$ are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \{\varphi^c \cap U\} \setminus \{u\}),$$

where U is a closed neighbourhood of u and H_k is the k-th (singular) homology group with coefficients in a field $\mathcal{F}$ (for the topological notions mentioned in the paper, see [12]). Due to the excision property of homology, it is known that $C_k(\varphi, u)$ is independent of U. The main result in this section is the following.

Theorem 1. *Let us assume that f satisfies* (f), (f_1), (f_2) *and let zero be an isolated critical point of φ. Then the Morse critical groups for φ at zero are trivial, i.e.*

$$C_k(\varphi, 0) = 0 \quad \textit{for all} \;\; k.$$

To prove the theorem we need some preliminaries lemmas.

Lemma 1.1. *Let us assume that f satisfies* (f), (f_1). *Then for each $u \neq 0$ zero is a strict local maximum for the function $\varphi(\tau u)$, $\tau \in \mathbb{R}$.*

Proof. By (f), (f_1) there is a constant $A_r > 0$ such that

$$|F(u)| \le A_r |u|^p \quad \text{for all} \;\; |u| \ge r.$$

We set for $u \in \mathbf{H}$ and $h > 0$

$$\Omega^h(u) = \{x \in \Omega : |u| \le h\}, \quad \Omega_h(u) = \Omega \setminus \Omega^h(u).$$

Taking into account the absolute continuity of the Lebesgue integral for each $u \neq 0$, we obtain

$$\begin{aligned}
\varphi(\tau u) &= \frac{\tau^2}{2}\|u\|^2 - \int_\Omega F(\tau u)\,dx \\
&\le \frac{\tau^2}{2}\|u\|^2 - a_r \tau^\nu \int_{\Omega^{r/\tau}(u)} |u|^\nu dx + A_r \tau^p \int_{\Omega_{r/\tau}(u)} |u|^p\,dx \\
&= \frac{\tau^2}{2}\|u\|^2 - a_r \tau^\nu \|u\|_{L_\nu}^\nu + a_r \tau^\nu \int_{\Omega_{r/\tau}(u)} |u|^\nu dx + A_r \tau^p \int_{\Omega_{r/\tau}(u)} |u|^p\,dx \\
&= -a_r \|u\|_{L_\nu}^\nu \tau^\nu + o(\tau^\nu) < 0,
\end{aligned}$$

for $\tau > 0$ sufficiently small. The proof is complete. □

Remark 1.1. Similarly we can proove that for every finite dimensional subspace $\widetilde{\mathbf{E}} \subset \mathbf{H}$ zero there is a point of strict local maximum for the restricted functional $\varphi_{|\widetilde{\mathbf{E}}}$. On the other hand, zero can not be a point of local maximum for φ, since by (f) it has the form $\|\cdot\|^2+$"*weakly continuous*". Roughly speaking, the origin looks like a "pseudo"-maximum point (see [11], where the functionals with a similar geometry "at infinity" are studied).

Lemma 1.2. *Let us assume that f satisfies* (f), (f_2). *Then for each $u \neq 0$*

$$\frac{d}{d\tau}\varphi(\tau u) > \frac{2}{\tau}\varphi(\tau u) \quad \textit{for all} \ \ \tau > 0. \tag{2}$$

Proof. By (f_2) we obtain

$$\begin{aligned}\frac{d}{d\tau}\varphi(\tau u) &= \langle \nabla\varphi(\tau u), u\rangle \\ &= \tau\|u\|^2 - \int_\Omega uf(\tau u)\,dx = \frac{2}{\tau}\left\{\frac{\tau^2}{2}\|u\|^2 - \int_\Omega \frac{1}{2}(\tau u)f(\tau u)\,dx\right\} \\ &> \frac{2}{\tau}\left\{\frac{1}{2}\|\tau u\|^2 - \int_\Omega F(\tau u)\,dx\right\} = \frac{2}{\tau}\varphi(\tau u).\end{aligned}$$

The proof is complete. □

Remark 1.2. In fact, (f), (f_2) implies

$$\langle \nabla\varphi(u), u\rangle/2 > \varphi(u) \quad \text{for all} \ \ u \neq 0.$$

This means, that functional φ has no positive critical values.

Now we are in the position for proving Theorem 1.

Proof of Theorem 1. Let us fix $\rho > 0$ such that zero is a unique critical point of φ in B_ρ, to prove the theorem it is enough to verify if

(a) $\varphi^0 \cap B_\rho$ is contractible in itself,
(b) $\{\varphi^0 \cap B_\rho\} \setminus \{0\}$ is contractible in itself.

Indeed, let us suppose that (a) and (b) hold. Then taking a pair

$$(A, B) = (\varphi^0 \cap B_\rho, \{\varphi^0 \cap B_\rho\} \setminus \{0\}),$$

we obtain an exact sequence (see [12])

$$\begin{aligned}\cdots \to H_1(B) \to H_1(A,B) \to H_0(B) \to H_0(A) \to H_0(A,B) \to 0 \\ = \cdots \to 0 \to H_1(A,B) \to \mathcal{F} \to \mathcal{F} \to H_0(A,B) \to 0.\end{aligned}$$

Hence $C_k(\varphi, 0) = H_k(A, B) = 0$ for all k.

Claim (a). We observe that the set $\varphi^0 \cap B_\rho$ is star-shaped that is, for $u \in \varphi^0 \cap B_\rho$,

$$\tau u \in \varphi^0 \cap B_\rho \quad \text{for all} \ \ \tau \in [0, 1].$$

Really, if we suppose that there exists $\tau_0 \in [0,1]$ such that $\varphi(\tau_0 u) > 0$, then from (2) it follows that

$$\frac{d}{d\tau}\varphi(\tau_0 u) > 0,$$

and by monotonicity arguments we obtain $\varphi(u) > 0$ – a contradiction.

So $\varphi^0 \cap B_\rho$ is contractible in itself. As a required deformation we can take a mapping $h : [0,1] \times \varphi^0 \cap B_\rho \to \varphi^0 \cap B_\rho$ defined by the formula

$$h_t(u) = (1-t)u.$$

Claim (b). From Lemmas 1.1 and 1.2 by monotonicity arguments it follows that for each $u \in \{B_\rho \setminus \dot{\varphi}^0\} \setminus \{0\}$ there exists a unique positive solution $\tau(u) \in (0,1]$ of the equation

$$\varphi(\tau u) = 0.$$

We observe the continuity of function $\tau(u)$. Fix $u \in \{B_\rho \setminus \dot{\varphi}^0\} \setminus \{0\}$. By (2) we have

$$\frac{d}{d\tau}\varphi(\tau(u)u) > 0.$$

Hence the Implicit Function Theorem implies the continuity of function $\tau(u)$ in some neighbourhood of u and therefore $\tau : \{B_\rho \setminus \dot{\varphi}^0\} \setminus \{0\} \to (0,1]$ being continuous.

Define a mapping $r : B_\rho \setminus \{0\} \to \{\varphi^0 \cap B_\rho\} \setminus \{0\}$ by the formula

$$r(u) = \begin{cases} \tau(u)u & \text{if } \ u \in \{B_\rho \setminus \dot{\varphi}^0\} \setminus \{0\}, \\ u & \text{if } \ u \in \{\dot{\varphi}^0 \cap B_\rho\} \setminus \{0\}. \end{cases}$$

The continuity of r follows from the continuity of τ and the fact that by definition $\tau(u) = 1$ as $\varphi(u) = 0$. Moreover, by definition

$$r(u) = u \quad \text{for } \ u \in \{\varphi^0 \cap B_\rho\} \setminus \{0\}.$$

Thus r is a retraction of $B_\rho \setminus \{0\}$ to $\{\varphi^0 \cap B\rho\} \setminus \{0\}$. So since $\mathbf{H}$ is infinite dimensional, then $B_\rho \setminus \{0\}$ is contractible in itself. But the retracts of a contractible space are also contractible [12] and hence $\{\varphi^0 \cap B_\rho\} \setminus \{0\}$ is contractible in itself. The proof is complete. □

Remark 1.3. The global condition (f_2) seems to be too restrictive. Indeed, to prove Theorem 1 we need only the existence of $\rho > 0$ such that zero is a unique critical point of φ in B_ρ and

$$\frac{d}{d\tau}\varphi(\tau u) > \frac{2}{\tau}\varphi(\tau u) \quad \text{for } \ u \in B_\rho \setminus \{0\}.$$

This can be guaranteed by the assumption

(A) zero is a point of strict local minimum for the functional

$$\delta(u) = \int_\Omega F(u) - \frac{1}{2}uf(u)\,dx.$$

Unfortunately, in the general situation for the local positiveness of δ in a neighbourhood of zero we need the global positiveness of the function $F(u)-uf(u)/2$, that is condition (f_2). This condition can be relaxed only in the case $N=1$ when $\mathbf{H}_0^1(\Omega)\subset\mathbf{L}_\infty$ (see Section 4 for details).

Now let us recall (see [7], [10]) that in the case when φ satisfies *the Palais–Smale condition*

(PS) each sequence $(u_n)\subset\mathbf{H}$ such that $\varphi(u_n)\to c$ and $\|\nabla\varphi(u_n)\|\to 0$ contains a convergent subsequence,

and for $[a,b]\subseteq\mathbb{R}\cup\{\infty\}$ the critical set

$$K_a^b=\{u\in\mathbf{H}:a\le\varphi(u)\le b,\ \nabla\varphi(u)=0\},$$

is finite, we have the following *Morse relations* between the Morse critical groups and homological characterization of sublevel sets:

$$H_k(\varphi^b,\varphi^a)=\bigoplus_{K_a^b}C_k(\varphi,u). \tag{3}$$

In applications one often has $H_k(\varphi^b,\varphi^a)$ to be nontrivial for some k. Thus the Morse relations (3) provide the existence of a critical point $u\in K_a^b$ with $C_k(\varphi,u)\ne 0$. But since $C_k(\varphi,0)=0$ for all k, we can conclude that $u\ne 0$, and thus the problem (1) has a nontrivial solution. The nontriviality of $H_k(\varphi^b,\varphi^a)$ can be provided by different assumptions on f at infinity. Let us consider some examples.

2. Coercive problem

In this section we shall suppose that f is Hölder continuous and satisfies the usuall "coercive" condition

(f_3) $\lim_{|u|\to\infty}f(u)/u<\lambda_1$, where $\lambda_1>0$ is the first eigenvalue of $-\Delta$.

Theorem 2. *Let us assume that f is Hölder continuous and satisfies* (f_1)–(f_3). *Then problem* (1) *has at least three nontrivial solutions.*

Proof. By condition (f_3), there exists an $\alpha\in(0,\lambda_1)$ and a constant $a>0$ such that $f(u)\le\alpha u+a$ for $u>0$ and $f(u)\ge-\alpha u-a$ for $u<0$. Let u_α be a solution of the equation

$$\begin{cases}-\Delta u=\alpha u+a & \text{in }\Omega,\\ u=0 & \text{on }\partial\Omega.\end{cases}$$

Then, by the Strong Maximum Principle u_α is bounded on Ω and $u_\alpha>0$ on Ω, and hence u_α is a super-solution of equation (1).

According to the standart cut-off technique (see f.e. [7], [13]) we may replace f by a Hölder continuous function satisfying (f), (f_1)–(f_3). Then φ is a functional of class $\mathcal{C}^1$ on $\mathbf{H}$ and its critical points are the solutions of the original equation (1). Moreover, by (f) and (f_3) functional φ is bounded from below and satisfies (PS)-condition. Hence φ reaches a minimum on each closed set $\mathbf{M} \subseteq \mathbf{H}$. We observe that φ has at least two local minimum points on $\mathbf{H}$ (see [7], [13] for analogous argument).

Let u^+ be the minimum of φ on

$$\mathbf{M}^+ = \{u \in \mathbf{H} : 0 \leq u \leq u_\alpha\}.$$

Since 0 and u_α are the pair of bounded sub- and super-solution of (1), u^+ is a solution of (1), moreover, since f is Hölder continuous, according to the general regularity result u^+ is a classical solution of (1) that is $u^+ \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\Omega)$. Then by the Hopf Maximum Principle we have

$$\begin{cases} 0 < u^+ < u_\alpha & \text{in } \ \Omega, \\ \dfrac{\partial}{\partial \nu} u^+ < 0 & \text{on } \ \partial\Omega, \\ \dfrac{\partial}{\partial \nu}(u^+ - u_\alpha) > 0 & \text{on } \ \partial\Omega. \end{cases} \tag{4}$$

From (4) it follows that if

$$\|v - u^+\|_{\mathcal{C}^1} < \varepsilon,$$

with ε small, then $0 < v < u_\alpha$. Therefore, u^+ is a local minimum for φ in the $\mathcal{C}^1$ topology. Hence the Brezis–Nirenberg result (see [6]) implies that u^+ is also a local minimum for φ in the topology of $\mathbf{H}$. Moreover, we can see that $\varphi(u^+) < 0$.

By the same arguments we obtain another local minimum $u^- < 0$ with $\varphi(u^-) < 0$.

Now we establish the existence of the third nontrivial critical point of φ. We take a, b satisfying

$$a < \inf_{\mathbf{H}} \varphi \leq m \leq M \leq b < 0,$$

where $m = \min\{\varphi(u^-), \varphi(u^+)\}$ and $M = \max\{\varphi(u^-), \varphi(u^+)\}$ and assume that $K_a^b = \{u^+, u^-\}$. Since u^+, u^- are the local minima for φ, we have (see [7])

$$C_k(\varphi, u^+) = C_k(\varphi, u^-) = \begin{cases} \mathcal{F} & k = 0, \\ 0 & k \neq 0. \end{cases}$$

Then by the Morse relations (3) we have

$$H_k(\varphi^b, \varphi^a) = \begin{cases} \mathcal{F} \oplus \mathcal{F} & k = 0, \\ 0 & k \neq 0. \end{cases}$$

On the other hand,

$$H_k(\varphi^b, \varphi^a) = H_k(\varphi^b, \emptyset) = H_k(\varphi^b) \quad \text{for all } \ k,$$

and hence

$$H_k(\varphi^b) = \begin{cases} \mathcal{F} \oplus \mathcal{F} & k = 0, \\ 0 & k \neq 0. \end{cases}$$

Taking the pair $(\mathbf{H}, \varphi^b)$ we obtain an exact sequence

$$\begin{aligned}\cdots \to H_1(\mathbf{H}) \to H_1(\mathbf{H}, \varphi^b) \to H_0(\varphi^b) \to H_0(\mathbf{H}) \to H_0(\mathbf{H}, \varphi^b) \to 0\\ = \cdots \to 0 \to H_1(\mathbf{H}, \varphi^b) \to \mathcal{F} \oplus \mathcal{F} \to \mathcal{F} \to H_0(\mathbf{H}, \varphi^b) \to 0.\end{aligned}$$

Hence $H_1(\mathbf{H}, \varphi^b) \neq 0$ and according to the Morse relations (3), functional φ has a critical point $\overline{u}$ such that $\varphi(\overline{u}) \geq b$ and $C_1(\varphi, \overline{u}) \neq 0$. But since $C_1(\varphi, 0) = 0$ it follows that $\overline{u} \neq 0$. The proof is complete. □

Remark 2.1. The existence of two local minima for the functional φ for f satisfiyng coercive condition (f$_3$) and certain assumptions at zero are well-known for smooth [7] or even Lipshitz [13] function f. It is incompatible with assumption (f$_1$). On the other hand (f$_1$) does not contradict the Hölder continuity of f.

3. Asymptotically linear problem

In this section we consider an asymptotically linear function f. Let

$$0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots,$$

be the eigenvalues of $-\Delta$. We assume that

(f$_4$) $f(u) = \lambda u + g(u)$ with bounded on $\mathbb{R}$ function g,

and denote by G the primitive of g .

Theorem 3. *Let us assume that f satisfies* (f$_1$), (f$_2$), (f$_4$) *and one of the following conditions takes place:*

(i) $\lambda \in (\lambda_n, \lambda_{n+1})$,
(ii) $\lambda = \lambda_n$ *and* $G(u) \to \infty$ *as* $|u| \to \infty$

for some $n \in \mathbb{N}$. Then problem (1) *has at least one nontrivial solution.*

Proof. Since (f$_4$) implies (f), functional φ is of class $\mathcal{C}^1$ on $\mathbf{H}$. We consider a decomposition of $\mathbf{H}$ into direct sum $\mathbf{H}^+ \oplus \mathbf{H}^-$, where $\mathbf{H}^-$ is the n-dimensional subspace of $\mathbf{H}$ spanned by the eigenfunctions of $-\Delta$ corresponding to the $\{\lambda_1, \ldots, \lambda_n\}$. It is well-known (see, e.g. [7], [4]) that in both cases (i), (ii) the functional φ satisfies (PS). Moreover,

$$\inf_{\mathbf{H}^+} \varphi = m > -\infty,$$

and

$$\varphi \to -\infty \quad \text{for } u \in \mathbf{H}^- \text{ as } \|u\| \to \infty.$$

Suppose that φ has only a finite number of critical points. Then (see [7], [4]) for $a \ll m$ sufficiently small we have

$$H_n(\mathbf{H}, \varphi^a) \neq 0,$$

and according to the Morse relations (3) the functional φ has a critical point $\overline{u}$ such that $\varphi(\overline{u}) \geq a$ and $C_n(\varphi, \overline{u}) \neq 0$. But, since $C_n(\varphi, 0) = 0$, it follows that $\overline{u} \neq 0$. The proof is complete. □

REMARK 3.1. In the nonresonance case (i) Theorem 3 remains true if a bounded function g in (f_4) is replaced by a function which behaves like $o(u)$ at infinity (see [7] for details).

REMARK 3.2. Case (ii) of the theorem is the simplest variant of the Landesman–Laser resonance condition. There are many much more "delicate" resonance conditions at infinity, including the "strong resonance" case, which should lead to the existence of nontrivial critical points. Unfortunately, all results of such sort that we know employ functionals of class $\mathcal{C}^2$ (see f.e. [7], [4]), the latter is incompatible with the assumption (f_1).

4. Two point boundary value problem and concluding remarks

Let us consider the case $N = 1$ and $\Omega = (0, \pi)$. So the problem (1) turns into a two point boudary value problem for the ordinary differential equation

$$\begin{cases} u'' = f(u), \\ u(0) = u(\pi) = 0. \end{cases} \tag{5}$$

In this case we have the inclusion $\mathbf{H}_0^1(0, \pi) \subset \mathbf{L}_\infty$, so the "global" condition (f_2) can be replaced by the weaker "local" condition

$(\mathrm{f}_2)_{\mathrm{loc}}$ there exists $r > 0$ such that

$$F(u) - \frac{1}{2} u f(u) > 0 \quad \text{for } \ 0 < |u| < r.$$

Using the above introduced notation, we obtain the following analogue of Theorem 1.

THEOREM 4. *Let us assume that $N = 1$, f satisfies (f_1), $(\mathrm{f}_2)_{\mathrm{loc}}$ and let zero be an isolated critical point of φ. Then the Morse critical groups for φ at zero are trivial, i.e.*

$$C_k(\varphi, 0) = 0 \quad \textit{for all } \ k.$$

PROOF. According to Remark 1.3 to prove the theorem it suffices to verify the assumption (A). Since $N = 1$ there exists $A > 0$ such that

$$\|u\|_{\mathbf{L}_\infty} \leq A\|u\|.$$

We set $\rho = 1/A$. Then for $u \in B_\rho$ we have $\|u\|_{\mathbf{L}_\infty} \leq r$ and hence

$$\delta(u) = \int_\Omega F(u) - \frac{1}{2} u f(u)\, dx > 0 \qquad \text{for all } \ u \in B_\rho \setminus \{0\}.$$

The proof is complete. □

Using Theorem 4 we can obtain analougues of Theorems 2 and 3 for problem (5) with condition (f_2) replaced by $(\mathrm{f}_2)_{\mathrm{loc}}$. Similar results can be obtained for systems of ordinary differential equations, that is for the second order Hamiltonian system.

Now let us consider the asymtotically superlinear function f (it is possible since $(\mathrm{f}_2)_{\mathrm{loc}}$ is a local condition). We suppose that f satisfies the following assumption "at infinity":

(f_5) for some $\mu \in (2, \infty)$ there exists $R > 0$ such that

$$0 < \mu F(u) \le u f(u) \quad \text{for} \quad |u| > R.$$

Unfortunately, we can not state any general existence result in this case even when $N = 1$. Indeed, let's assume that f satisfies (f), (f_1), $(\mathrm{f}_2)_{\mathrm{loc}}$, (f_5). Then (see [7], [11]) from (f_5) it follows that for all $a \ll 0$ sufficiently small we have

$$H_k(\mathbf{H}, \varphi^a) = 0 \quad \text{for all} \quad k.$$

According to Theorem 4 Morse critical groups for φ at zero are also trivial. So we cannot deduce direcly from the Morse relations (3) the existence of even one different from zero critical point.

Finally, let us consider the problem

$$\begin{cases} u'' = \lambda f(u), \\ u(0) = u(\pi) = 0, \end{cases} \tag{6}$$

where λ is a real parameter. We suppose that f satisfies (f), (f_1), (f_5). Then the results of Ambrosetti et al. ([2], see Remark 2.5) can be easily adopted to establish the existence of at least two positive and two negative nontrivial solutions

$$u_0^+, \; u_1^+, \; u_0^-, \; u_1^-,$$

of (6) for all $\lambda > 0$ suficciently small. Here u_0^+, u_0^- are local minima and u_1^+, u_1^- are of the mountain–pass type. We can conjecture that in this case, under the additional assumption $(\mathrm{f}_2)_{\mathrm{loc}}$, the Morse relations (3) imply the existence of the fifth nontrivial solution $\overline{u}$ of (6) with $C_2(\varphi, \overline{u}) \neq 0$, but, at the moment, we have no precise proof of this. Recently, the existence of at least sixth nontrivial solutions of (6) in the case of an arbitrary N was established under some strong additional assumptions on f [3].

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