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MULTIPLE POSITIVE SOLUTIONS FOR SOME NONLINEAR ELLIPTIC SYSTEMS

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0. Introduction

In this paper we study, via variational methods, the existence and multiplicity of positive solutions of the following systems of nonlinear elliptic equations:

$$k_1\Delta u + V_u(u,v) = 0 \qquad \text{in } \Omega, \tag{0.1}$$

$$k_2\Delta v + V_v(u,v) = 0 \qquad \text{in } \Omega, \tag{0.2}$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \qquad \text{on } \partial\Omega, \tag{0.3}$$

$$u(x) > 0, \quad v(x) > 0 \qquad \text{in } \Omega, \tag{0.4}$$

where $k_1,\ k_2 > 0$ are positive constants, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$ and $V(u,v) \in C^2(\mathbb{R}^2,\mathbb{R})$. We refer to [CdFM], [CM], [dFF], [dFM] and [HvV] for variational study of such elliptic systems. However, it seems that the multiplicity of positive solutions for such elliptic systems is not well studied.

Here, we study a case related to some models (with diffusion) in mathematical biology, ecology, etc., and we consider the case where (0.1)–(0.3) have 4 constant non-negative solutions $(0,0)$, $(a,0)$, $(0,b)$, $(u_0,v_0) \in \mathbb{R}^2$ (a, b, u_0, $v_0 > 0$), that is, solutions of $V_u(u,v) = V_v(u,v) = 0$, and 2 constant solutions $(a,0)$, $(0,b)$ are

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stable and other 2 solutions $(0,0)$, (u_0,v_0) are unstable as steady solutions of the evolution problem:

$$u_t = V_u(u,v), \quad v_t = V_v(u,v).$$

Moreover, we assume

$$0 = V(0,0) < V(u_0,v_0) < \min\{V(a,0),\, V(0,b)\}, \tag{0.5}$$

and we look for non-constant positive solutions of (0.1)–(0.4). More precisely, we assume the following conditions on $V(u,v)$:

(V0) $V \in C^2(\mathbb{R}^2,\mathbb{R})$ and $V(u,v)$ is even in both variables u, v, that is, $V(u,v) = V(-u,v) = V(u,-v)$ for all u, v.

(V1) There exist a, b, u_0, $v_0 > 0$ such that $V_u(u,v) = V_v(u,v) = 0$ $(u,\, v \ge 0)$ implies

$$(u,v) \in \{(0,0),(a,0),(0,b),(u_0,v_0)\}.$$

Moreover,

1° $V(0,0)$, $V(a,0)$, $V(0,b)$, $V(u_0,v_0)$ satisfy relation (0.5).
2° $V(u,v)$ attains a non-degenerate local minimum at $(0,0)$.
3° $V(u,v)$ attains non-degenerate local maxima at $(a,0)$, $(0,b)$.
4° (u_0,v_0) is a non-degenerate saddle point of $V(u,v)$.

(V2) There exists $R_0 > 0$ such that
1° $V_u(u,v) < 0$ for $(u,v) \in [R_0,\infty) \times [0,\infty)$.
2° $V_v(u,v) < 0$ for $(u,v) \in [0,\infty) \times [R_0,\infty)$.

(V3)
1° $\frac{\partial}{\partial u}(V_u(u,0)/u) < 0$ for $u \in (0,R_0]$.
2° $\frac{\partial}{\partial v}(V_v(0,v)/v) < 0$ for $v \in (0,R_0]$.

(V4) $V_{uv}(u,v) < 0$ for all $(u,v) \in [0,R_0] \times [0,R_0]$.

We denote the eigenvalues of $-\Delta$ under Neumann boundary conditions by

$$0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots .$$

Now we can state our main result.

Theorem 0.1.

(i) *Assume* (V0)–(V3) *and*

$$\det\left(\lambda_2\begin{bmatrix} k_1 & 0 \\ 0 & k_2\end{bmatrix} - \begin{bmatrix} V_{uu}(u_0,v_0) & V_{uv}(u_0,v_0) \\ V_{uv}(u_0,v_0) & V_{vv}(u_0,v_0)\end{bmatrix}\right) < 0. \tag{0.6}$$

Then (0.1)–(0.4) *have at least one non-constant positive solution.*

(ii) *In addition to the assumptions of* (i), *assume* (V4) *and*

$$\det\left(\lambda_j\begin{bmatrix} k_1 & 0 \\ 0 & k_2\end{bmatrix} - \begin{bmatrix} V_{uu}(u_0,v_0) & V_{uv}(u_0,v_0) \\ V_{uv}(u_0,v_0) & V_{vv}(u_0,v_0)\end{bmatrix}\right) \ne 0 \quad \textit{for all } j \in \mathbb{N}. \tag{0.7}$$

Then (0.1)–(0.4) *have at least two non-constant positive solutions.*

We have a stronger result when $N = 1$.

Theorem 0.2. *Assume* $N = 1$ *and* (V0)–(V3). *Suppose that*

$$m \equiv \max\left\{\ell \in \mathbb{N} : \det\left(\lambda_\ell \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} - \begin{bmatrix} V_{uu}(u_0,v_0) & V_{uv}(u_0,v_0) \\ V_{uv}(u_0,v_0) & V_{vv}(u_0,v_0) \end{bmatrix}\right) < 0\right\} \geq 2. \tag{0.8}$$

Then (0.1)–(0.4) *have at least* $2(m-1)$ *non-constant positive solutions.*

Remark 0.3. Conditions (0.6) and (0.8) are conditions on the unstability of (u_0, v_0) or on the Morse index of (u_0, v_0) of the corresponding functional, and they are satisfied if k_1 and k_2 are sufficiently small. In Section 1.3, we will see that the integer m defined in (0.8) is equal to the Morse index of (u_0, v_0) and (0.7) holds if and only if (u_0, v_0) is non-degenerate. See Remark 1.9.

In mathematical biology, the existence and multiplicity of positive solutions of the following nonlinear elliptic systems is important (c.f. [CC], [CL], [D1], [D3], [D4], [LL]):

$$k_1\Delta u + f(u,v)u = 0 \quad \text{in } \Omega, \tag{0.9}$$

$$k_2\Delta v + g(u,v)v = 0 \quad \text{in } \Omega, \tag{0.10}$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{0.11}$$

$$u(x) > 0, \quad v(x) > 0 \quad \text{in } \Omega. \tag{0.12}$$

where $f(u,v)$, $g(u,v) \in C^1(\mathbb{R}^2, \mathbb{R})$ are functions satisfying $f_v(u,v) < 0$ and $g_u(u,v) < 0$ for all u, v. Unknown functions $u(x)$, $v(x)$ correspond to the population densities of 2 species and conditions $f_v < 0$, $g_u < 0$ describe a certain *competition* environment. Solutions of (0.9)–(0.12) can be regarded as positive steady-states for evolution problem:

$$u_t = k_1\Delta u + f(u,v)u, \quad v_t = k_2\Delta v + g(u,v)v, \tag{0.13}$$

and they are considered as co-existence states for 2 species.

We consider the so-called *bistable* situation — we assume that (0.13) has 2 stable constant steady-states $(a, 0)$, $(0, b)$ and 2 unstable steady-states $(0, 0)$, (u_0, v_0) — and we try to find non-constant positive steady-states. A typical example of such situation is

$$f(u,v) = a_1 - b_1u - c_1v, \quad g(u,v) = a_2 - b_2u - c_2v, \tag{0.14}$$

with $a_1/b_1 > a_2/b_2$ and $a_1/c_1 < a_2/c_2$. This model is called the Lotka–Volterra competition model.

In this paper we assume also that the system has a variational structure, and we study (0.1)–(0.4) under conditions (V0)–(V4). We remark that (V0) should be regarded as a condition on the behavior of $V(u,v)$ near $\{u=0\}\cup\{v=0\}$. In particular, it follows that $V_u(0,v)=V_v(u,0)=0$ for all u, v. Under this condition, $W(x,y)=V(\sqrt{x},\sqrt{v})$ is a function of class C^1 on $[0,\infty)\times[0,\infty)$, and our system (0.1)–(0.4) can be written in the form (0.9)–(0.12) with $f(u,v)=2W_x(u^2,v^2)$, $g(u,v)=2W_y(u^2,v^2)$ (see Section 1.1). Condition (V1) says that the system is bistable and (V4) means that the system gives a competition model.

An example of $V(u,v)$ satisfying (V0)–(V4) is

$$V(u,v)=\frac{a}{2}u^2+\frac{c}{2}v^2-\frac{b}{4}u^4-\frac{d}{4}v^4-\frac{1}{2}u^2v^2,$$

with $bc<a<c/d$. The corresponding system is

$$k_1\Delta u+(a-bu^2-v^2)u=0 \qquad \text{in } \Omega, \tag{0.15}$$

$$k_2\Delta v+(c-u^2-dv^2)v=0 \qquad \text{in } \Omega, \tag{0.16}$$

$$\frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0 \qquad \text{on } \partial\Omega, \tag{0.17}$$

$$u(x)>0, \quad v(x)>0 \qquad \text{in } \Omega. \tag{0.18}$$

As a special case of our Theorems 0.1 and 0.2, we have:

THEOREM 0.4. *Assume $bc<a<c/d$.*

(i) *If*

$$\det\left(\lambda_2\begin{bmatrix}k_1 & 0\\ 0 & k_2\end{bmatrix} + \frac{2}{1-bd}\begin{bmatrix}\frac{b(c-ad)}{\sqrt{(a-bc)(c-ad)}} & \sqrt{(a-bc)(c-ad)}\\ & d(a-bc)\end{bmatrix}\right)<0, \tag{0.19}$$

then (0.15)–(0.18) *have at least one non-constant positive solution.*

(ii) *In addition to the assumptions of* (i), *assume*

$$\det\left(\lambda_j\begin{bmatrix}k_1 & 0\\ 0 & k_2\end{bmatrix} + \frac{2}{1-bd}\begin{bmatrix}\frac{b(c-ad)}{\sqrt{(a-bc)(c-ad)}} & \sqrt{(a-bc)(c-ad)}\\ & d(a-bc)\end{bmatrix}\right)\neq 0$$

for all $j\in\mathbb{N}$. Then (0.15)–(0.18) *have at least two non-constant positive solutions.*

(iii) *Assume $N=1$. Let*

$$m\equiv\max\left\{\ell\in\mathbb{N}:\det\left(\lambda_\ell\begin{bmatrix}k_1 & 0\\ 0 & k_2\end{bmatrix} + \frac{2}{1-bd}\begin{bmatrix}\frac{b(c-ad)}{\sqrt{(a-bc)(c-ad)}} & \sqrt{(a-bc)(c-ad)}\\ & d(a-bc)\end{bmatrix}\right)<0\right\},$$

and assume $m\geq 2$. Then (0.15)–(0.18) *have at least $2(m-1)$ non-constant positive solutions.*

Remark 0.5. (i) Condition (0.19) is satisfied if k_1, k_2 are sufficiently small.

(ii) We remark that a more general case:

$$\begin{aligned} k_1\Delta u + (a_1 - b_1u^2 - c_1v^2)u &= 0 \qquad \text{in } \Omega,\\ k_2\Delta v + (a_2 - b_2u^2 - c_2v^2)v &= 0 \qquad \text{in } \Omega,\\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} &= 0 \qquad \text{on } \partial\Omega,\\ u(x) > 0, \quad v(x) &> 0 \qquad \text{in } \Omega, \end{aligned}$$

can be reduced to (0.15)–(0.18) after a suitable scaling procedure. (0.15)–(0.18) may be regarded as a variational version of the Lotka–Volterra competition model.

(iii) The original Lotka–Volterra competition model (0.9)–(0.12) with (0.13) does not have a variational structure and our theorems are not applicable. It seems that the existence of one non-constant positive solution is not known in general bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$, even if k_1 and k_2 are sufficiently small. For the case $N = 1$, we refer to Nakashima [N]. See Remark 5.5.

The existence and the multiplicity of a scalar elliptic problem:

$$-\Delta u = g(u) \qquad \text{in } \Omega, \tag{0.20}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{0.21}$$

is rather well-studied via variational arguments. We refer to Struwe [St] and references therein.

Especially, Hofer [H1], [H2], [H3] proved the existence of at least 4 non-zero solutions (including positive, negative and sign-changing ones) of (0.20)–(0.21) under conditions:

1° $g \in C^1(\mathbb{R}, \mathbb{R})$.
2° $\limsup_{|u|\to\infty} g(u)/u < \lambda_1$.
3° $g(0) = 0$.
4° $g'(0) \in (\mu_i, \mu_{i+1})$ for some $i \geq 2$.

Here we denote by $0 < \mu_1 < \mu_2 \leq \ldots$ the eigenvalues of $-\Delta$ under Dirichlet boundary conditions. To show the existence of 4 non-zero solutions, Hofer first found positive and negative solutions via minimizing argument. Next, he used the mountain pass theorem and a Morse theoretic argument to obtain two more solutions.

We apply essentially the same idea to our problem (0.1)–(0.4). We remark that we look only for positive solutions of (0.1)–(0.4) and that, if we admit sign-changing solutions, the number of solutions (including sign-changing ones) goes to infinity as $(k_1, k_2) \to (0, 0)$. See Remark 1.10 below.

In the following sections, we give proofs of our Theorems 0.1 and 0.2. In Section 1, we give some preliminaries and some a priori estimates for (0.1)–(0.4). In Sections 2–4, we prove Theorem 0.1. We use the mountain pass theorem and an idea from Hofer [H2], [H3] (c.f. [H1]). Here, Morse indices and Leray–Schauder degree theory play an important role.

In Section 5, we give a proof of Theorem 0.2. A relation between the numbers of zeros of $(u'(x), v'(x))$ and the Morse index of (u,v) is a key of the proof.

1. Preliminaries

In this section, first we give a modification of the given potential $V(u,v)$, so that the corresponding functional is of class C^2 and coercive in a suitable function space. Second, we obtain a priori L^∞-estimates for critical points, and we state some fundamental properties of the corresponding functional.

1.1. Modification of $V(u,v)$. Let $R_0 > 0$ be a positive constant defined in (V2). We choose a smooth function $\nu(s) : [0,\infty) \to \mathbb{R}$ such that

(i) $\nu(s) \in [0,1]$ for all s,
(ii) $\nu'(s) \geq 0$ for all s,
(iii) $\nu(s) = \begin{cases} 1 & \text{for } s \geq 9R_0^2, \\ 0 & \text{for } s \leq 4R_0^2, \end{cases}$

and set

$$\widetilde{V}(u,v) = (1-\nu(u^2+v^2))V(u,v) - \nu(u^2+v^2)((u^2+v^2)/2 + C_0),$$

where $C_0 > 0$ is a positive constant which will be determined in the following lemma.

Lemma 1.1. *Suppose that $V(u,v)$ satisfies* (V0)–(V3) (*resp.* (V0)–(V4)). *Then, for sufficiently large $C_0 > 0$, the modified potential $\widetilde{V}(u,v)$ also satisfies* (V0)–(V3) (*resp.* (V0)–(V4)).

Proof. Suppose that $V(u,v)$ satisfies (V0)–(V3), then clearly (V0), (V3) hold for $\widetilde{V}(u,v)$. Moreover, if $V(u,v)$ satisfies (V4), $\widetilde{V}(u,v)$ also satisfies (V4). We remark

$$\begin{aligned}\widetilde{V}_u(u,v) = {}&(1-\nu(u^2+v^2))V_u(u,v) - \nu(u^2+v^2)u \\ &- 2u\nu'(u^2+v^2)(V(u,v) + (u^2+v^2)/2 + C_0), \\ \widetilde{V}_v(u,v) = {}&(1-\nu(u^2+v^2))V_v(u,v) - \nu(u^2+v^2)v \\ &- 2v\nu'(u^2+v^2)(V(u,v) + (u^2+v^2)/2 + C_0),\end{aligned}$$

and

$$\operatorname{supp} \nu'(u^2+v^2) \subset \{(u,v) : 4R_0^2 \leq u^2+v^2 \leq 9R_0^2\}.$$

We set

$$C_0 = \max_{4R_0^2 \le u^2+v^2 \le 9R_0^2} |V(u,v)| + 1.$$

Thus we can see that (V2) holds for $\widetilde{V}(u,v)$.

To verify (V1) for $\widetilde{V}(u,v)$, suppose u, $v \ge 0$ satisfy $\widetilde{V}_u(u,v) = \widetilde{V}_v(u,v) = 0$, then by (V2) for $\widetilde{V}(u,v)$ we have $(u,v) \in [0,R_0] \times [0,R_0]$. Thus by (V1) for $V(u,v)$, we see that (V1) also holds for $\widetilde{V}(u,v)$. □

Next we give an a priori estimate for the following problem:

$$k_1 \Delta u + \widetilde{V}_u(u,v) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$k_2 \Delta v + \widetilde{V}_v(u,v) = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

We use the notation

$$\|u(x)\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x\in\Omega} |u(x)|.$$

PROPOSITION 1.2. *Suppose that* $(u,v) \in H^1(\Omega) \times H^1(\Omega)$ *is a solution of* (1.1)–(1.3). *Then* (u,v) *satisfies*

$$\|u(x)\|_{L^\infty(\Omega)},\ \|u(x)\|_{L^\infty(\Omega)} \le R_0,$$

and satisfies the original problem (0.1)–(0.3).

PROOF. Suppose that $(u,v) \in H^1(\Omega) \times H^1(\Omega)$ satisfies (1.1)–(1.3). By a standard regularity argument, we see $(u,v) \in L^\infty(\Omega) \times L^\infty(\Omega)$. Let $D = \{x \in \overline{\Omega} : u(x) \ge R_0\}$. By (V2) we have

$$k_1 \Delta u = -\widetilde{V}_u(u(x), v(x)) > 0 \quad \text{in } D.$$

Thus, by the maximal principle, $u(x)$ cannot take maximum in D. Therefore, $D = \emptyset$ and $u(x) \le R_0$ in $\overline{\Omega}$. Similarly, we have $-R_0 \le u(x)$ in $\overline{\Omega}$. Thus $\|u(x)\|_{L^\infty(\Omega)} \le R_0$. In a similar way we can also show $\|v(x)\|_{L^\infty(\Omega)} \le R_0$. □

By the above proposition, solutions of (1.1)–(1.3) are also solutions of (0.1)–(0.3) and there is no need to distinguish $\widetilde{V}(u,v)$ and $V(u,v)$. Therefore in what follows, we assume that $V(u,v)$ satisfies

(V5) For some constant $C_0 > 0$,

$$V(u,v) = -(u^2+v^2)/2 - C_0 \quad \text{for } u^2+v^2 \ge 9R_0^2.$$

We don't distinguish (0.1)–(0.3) and (1.1)–(1.3).

Here we give a remark on condition (V0). As in the Introduction, we define a function $W(u,v) : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$W(x,y) = V(\sqrt{x}, \sqrt{y}) \quad \text{for } x, y \ge 0. \tag{1.4}$$

From (V0)–(V3), it follows that

(W0) $W \in C^1([0,\infty)\times[0,\infty),\mathbb{R}) \cap C^2((0,\infty)\times(0,\infty),\mathbb{R})$.

(W1) $W_x(0,0) > 0$, $W_y(0,0) > 0$.

(W2) $W_x(x,y) < 0$ for $(x,y) \in [R_0^2,\infty)\times[0,\infty)$, $W_y(x,y) < 0$ for $(x,y) \in [0,\infty)\times[R_0^2,\infty)$.

(W3) $W_{xx}(x,0) < 0$ for $x \in (0,R_0^2]$, $W_{yy}(0,y) < 0$ for $y \in (0,R_0^2]$.

(W4) $W_{xy}(x,y) < 0$ for $(x,y) \in [0,R_0^2]\times[0,R_0^2]$.

(W5) $W(x,y) = -(x+y)/2 - C_0$ for $x+y \geq 9R_0^2$.

By Definition (1.4), we have $V(u,v) = W(u^2,v^2)$ and our equation can be written as

$$k_1\Delta u + 2W_x(u^2,v^2)u = 0 \qquad \text{in } \Omega, \tag{1.5}$$

$$k_2\Delta v + 2W_y(u^2,v^2)v = 0 \qquad \text{in } \Omega, \tag{1.6}$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \qquad \text{on } \partial\Omega. \tag{1.7}$$

This form of equations has a similarity with equations (0.9)–(0.10), and it is convenient to apply the maximal principle.

1.2. Variational formulation. In what follows, we assume (V0)–(V3) and (V5). We set

$$I(u,v) = \int_\Omega \left[\frac{k_1}{2}|\nabla u|^2 + \frac{k_2}{2}|\nabla v|^2 - V(u,v)\right] dx : E \to \mathbb{R},$$

where $E = H^1(\Omega)\times H^1(\Omega)$. We use the notation

$$\|(u,v)\|_E^2 = \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2,$$
$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u|^2\,dx\right)^{1/2}.$$

We denote the duality product between E^* and E by $\langle\,\cdot\,,\,\cdot\,\rangle$. By virtue of (V5), we have

Proposition 1.3. *Assume that $V(u,v)$ satisfies* (V0)–(V3) *and* (V5). *Then*

(i) *$I(u,v) \in C^2(E,\mathbb{R})$.*

(ii) *$I(u,v)$ is coercive in the following sense:*

$$I(u,v) \to \infty \quad \textit{as } \|(u,v)\|_E \to \infty.$$

(iii) *$I(u,v)$ satisfies the Palais–Smale condition.*

Proof. By (V5), it is easy to see that $I(u,v)$ is of class C^2 on E. Since $V(u,v)$ satisfies

$$V(u,v) \leq -(u^2+v^2)/2 + C_1 \quad \text{in } \mathbb{R}^2,$$

for some constant $C_1 > 0$, we have

$$\begin{aligned} I(u,v) \geq &\frac{k_1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{k_2}{2}\|\nabla v\|^2_{L^2(\Omega)} \\ &+ \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{1}{2}\|v\|^2_{L^2(\Omega)} - C_1|\Omega| \quad \text{for all } (u,v) \in E. \end{aligned}$$

Thus we have (ii). We can also deduce (iii) from (ii) by a standard argument.□

We also have.

Proposition 1.4. *Suppose that* $(u,v) \in E$ *is a critical point of* $I(u,v)$. *Then it satisfies* (0.1)–(0.3) *and*

$$\|u\|_{L^\infty(\Omega)},\ \|v\|_{L^\infty(\Omega)} \leq R_0,$$

where R_0 *is given in* (V2).

Proof. Suppose that $(u,v) \in E$ is a critical point of $I(u,v)$. Then it satisfies

$$\int_\Omega (k_1 \nabla u \nabla \phi + k_2 \nabla v \nabla \psi - V_u(u,v)\phi - V_v(u,v)\psi)\, dx = 0 \quad \text{for all } (\phi,\psi) \in E.$$

Thus, by a standard argument, we can see that $(u,v) \in E$ is a solution of (0.1)–(0.3). The second assertion follows from Proposition 1.2. □

In what follows, we will try to find critical points $(u,v) \in E$ of $I(u,v)$ with positivity condition (0.4). The following lemma is important to distinguish positive solutions from non-negative solutions.

Lemma 1.5. *Suppose that* $(u,v) \in E$ *is a non-negative critical point of* $I(u,v)$, *that is,* $I'(u,v) = 0$ *and* $u(x) \geq 0$, $v(x) \geq 0$ *in* $\overline{\Omega}$. *Then*

$$(u(x), v(x)) \equiv (0,0),\ (a,0),\ (0,b),$$

or $u(x) > 0$, $v(x) > 0$ *in* $\overline{\Omega}$.

To prove the above lemma, we use the form (1.5)–(1.7) of our problem. We set

$$A = 2 \max_{\substack{x\geq 0,\, y \geq 0 \\ x+y \leq 9R_0^2}} \max\{|W_x(x,y)|,\ |W_y(x,y)|\} + 1, \tag{1.8}$$

and we rewrite (1.5)–(1.7) as

$$-k_1 \Delta u + (A - 2W_x(u^2,v^2))u = Au \qquad \text{in } \Omega, \tag{1.9}$$

$$-k_2 \Delta v + (A - 2W_y(u^2,v^2))v = Av \qquad \text{in } \Omega, \tag{1.10}$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \qquad \text{on } \partial\Omega. \tag{1.11}$$

We remark that by (1.8) and (W5)

$$A - 2W_x(x,y) \geq 1, \quad A - 2W_y(x,y) \geq 1 \quad \text{for all } (x,y) \in [0,\infty) \times [0,\infty). \tag{1.12}$$

Proof of Lemma 1.5. Suppose that $I'(u,v) = 0$ and $u(x)$, $v(x) \geq 0$. By Proposition 1.4, (u,v) satisfies (1.9)–(1.11). By (1.12),

$$A - 2W_x(u(x)^2, v(x)^2) \geq 1, \quad A - 2W_y(u(x)^2, v(x)^2) \geq 1, \quad \text{in } \overline{\Omega}.$$

Applying the maximal principle to (1.9), since the right hand sides are non-negative, we see that $u(x) \geq 0$ and $u(x) \not\equiv 0$ imply $u(x) > 0$ in $\overline{\Omega}$. Similarly, $v(x) \geq 0$ and $v(x) \not\equiv 0$ imply $v(x) > 0$ in $\overline{\Omega}$.

Therefore, a non-negative critical point $(u,v) \in E$ of $I(u,v)$ is a positive solution of (0.1)–(0.3) if $u(x) \not\equiv 0$ and $v(x) \not\equiv 0$. The conclusion of Lemma 1.5 follows from the next lemma which deals with the case $u(x) \equiv 0$ or $v(x) \equiv 0$. □

Lemma 1.6.

(i) *Suppose that $(u(x), 0) \in E$ is a non-negative critical point of $I(u,v)$, then*

$$(u(x), 0) \equiv (0,0) \text{ or } (a, 0).$$

(ii) *Suppose that $(0, v(x)) \in E$ is a non-negative critical point of $I(u,v)$, then*

$$(0, v(x)) \equiv (0,0) \text{ or } (0, b).$$

Proof. We prove (i). (ii) can be proved in a similar way. Since $(u,0)$ is a critical point of $I(u,v)$, we have

$$\begin{aligned} -k_1\Delta u + (A - 2W_x(u^2, 0))u &= Au \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

By the maximal principle, we see that $u(x) \geq 0$ and $u(x) \not\equiv 0$ imply $u(x) > 0$ in $\overline{\Omega}$. Thus we may assume that $u(x) > 0$ in $\overline{\Omega}$, which will take place if $(u(x), 0) \not\equiv (0,0)$.

Since $(a, 0)$ is a critical point of $V(u,v)$, we have $W_x(a^2, 0) = 0$. Thus $\widehat{u}(x) = u(x) - a$ satisfies

$$-k_1\Delta\widehat{u} - 2(W_x(u^2, 0) - W_x(a^2, 0))u = 0.$$

For a suitable $\theta(x) \in (0,1)$, we have

$$-k_1\Delta\widehat{u} - 2W_{xx}((1-\theta(x))u^2 + \theta(x)a^2, 0)(u^2 - a^2)u = 0.$$

That is,

$$\begin{aligned} -k_1\Delta\widehat{u} - 2u(u+a)W_{xx}((1-\theta)u^2 + \theta a^2, 0)\widehat{u} &= 0 \qquad \text{in } \Omega, \\ \frac{\partial \widehat{u}}{\partial n} &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

We multiply $\widehat{u}(x)$ and integrate over Ω, then we get

$$k_1\|\nabla\widehat{u}\|^2_{L^2(\Omega)} - 2\int_\Omega u(u+a)W_{xx}((1-\theta)u^2+\theta a^2,0)|\widehat{u}|^2\,dx = 0.$$

By (V3), in other words, (W3), we can see that $\widehat{u}(x)\equiv 0$. That is, $(u(x),0)\equiv(a,0)$. □

1.3. Morse indices and some properties of $I(u,v)$. For a critical point $(u,v)\in E$ of $I(u,v)$, we have for $(\phi,\psi)\in E$

$$\langle\, I''(u,v)(\phi,\psi)\,,\,(\phi,\psi)\,\rangle = \int_\Omega [k_1|\nabla\phi|^2 + k_2|\nabla\psi|^2 - (D^2V(u,v)(\phi,\psi),(\phi,\psi))]\,dx. \tag{1.13}$$

Here we denote by $D^2V(u,v)$ the Hessian matrix of $V(u,v)$. We define the Morse index $\operatorname{index} I''(u,v)$ and the augmented Morse index $\operatorname{index}_0 I''(u,v)$ at $(u,v)\in E$ by

$$\begin{aligned}\operatorname{index} I''(u,v) = \max\{\,&\dim H : H\subset E \text{ is a subspace such that}\\ &\langle\, I''(u,v)(\phi,\psi)\,,\,(\phi,\psi)\,\rangle < 0 \text{ for all } (\phi,\psi)\in H\setminus\{(0,0)\}\},\\ \operatorname{index}_0 I''(u,v) = \max\{\,&\dim H : H\subset E \text{ is a subspace such that}\\ &\langle\, I''(u,v)(\phi,\psi)\,,\,(\phi,\psi)\,\rangle \le 0 \text{ for all } (\phi,\psi)\in H\}.\end{aligned}$$

We say that (u,v) is *non-degenerate* if and only if $\operatorname{index} I''(u,v) = \operatorname{index}_0 I''(u,v)$. Roughly speaking, $\operatorname{index} I''(u,v)$ (resp. $\operatorname{index}_0 I''(u,v)$) is a number of negative (resp. non-positive) eigenvalues of $I''(u,v)$, and (u,v) is non-degenerate if and only if 0 is not an eigenvalue of $I''(u,v)$.

For a constant solution $(\overline{u},\overline{v})\in\mathbb{R}^2\subset E$, the Morse index $I''(\overline{u},\overline{v})$ can be represented in the following way.

Lemma 1.7. *For $(\overline{u},\overline{v})\in\mathbb{R}^2\subset E$,*

$$\operatorname{index} I''(\overline{u},\overline{v}) = \sum_{j=1}^{\infty} \#\ \textit{of negative eigenvalues of}\ \left(\lambda_j\begin{bmatrix}k_1 & 0\\ 0 & k_2\end{bmatrix} - D^2V(\overline{u},\overline{v})\right), \tag{1.14}$$

$$\operatorname{index}_0 I''(\overline{u},\overline{v}) = \sum_{j=1}^{\infty} \#\ \textit{of non-positive eigenvalues of}\ \left(\lambda_j\begin{bmatrix}k_1 & 0\\ 0 & k_2\end{bmatrix} - D^2V(\overline{u},\overline{v})\right), \tag{1.15}$$

and $(\overline{u},\overline{v})$ is non-degenerate if and only if

$$\det\left(\lambda_j\begin{bmatrix}k_1 & 0\\ 0 & k_2\end{bmatrix} - D^2V(\overline{u},\overline{v})\right)\neq 0 \quad \textit{for all } j\in\mathbb{N}. \tag{1.16}$$

Here $0 = \lambda_1 < \lambda_2 \leq \ldots$ *are eigenvalues* (*counting multiplicities*) *of* $-\Delta$ *under Neumann boundary conditions. We remark that* $\left(\lambda_j \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} - D^2V(\overline{u},\overline{v})\right)$ *is positive definite for a sufficiently large* j *and the sums in* (1.14) *and* (1.15) *are finite.*

Proof. Using eigenfunction expansion, we write

$$\phi(x) = \sum_{j=1}^{\infty} \phi_j e_j(x), \quad \psi(x) = \sum_{j=1}^{\infty} \psi_j e_j(x),$$

where $e_j(x)$ are eigenfunctions of $-\Delta$ corresponding to the eigenvalues λ_j, and we assume $\int_\Omega e_i(x)e_j(x)\,dx = \delta_{ij}$. From (1.13) it follows that

$$\begin{aligned}
&\langle\, I''(\overline{u},\overline{v})(\phi,\psi)\,,\,(\phi,\psi)\,\rangle \\
&\quad = \sum_{j=1}^{\infty}\left(k_1\lambda_j|\phi_j|^2 + k_2\lambda_j|\psi_j|^2 - \left(D^2V(\overline{u},\overline{v})\begin{bmatrix}\phi_j\\ \psi_j\end{bmatrix}, \begin{bmatrix}\phi_j\\ \psi_j\end{bmatrix}\right)\right) \\
&\quad = \sum_{j=1}^{\infty}\left(M(\lambda_j;\overline{u},\overline{v})\begin{bmatrix}\phi_j\\ \psi_j\end{bmatrix}, \begin{bmatrix}\phi_j\\ \psi_j\end{bmatrix}\right).
\end{aligned}$$

Here we use the notation: $M(\lambda;\overline{u},\overline{v}) = \lambda\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} - D^2V(\overline{u},\overline{v})$. We also denote by $\operatorname{index} M(\lambda;\overline{u},\overline{v})$ (resp. $\operatorname{index}_0 M(\lambda;\overline{u},\overline{v})$) the number of negative (resp. non-positive) eigenvalues of $M(\lambda;\overline{u},\overline{v})$. We see

$$\begin{aligned}
\operatorname{index} I''(\overline{u},\overline{v}) &= \sum_{j=1}^{\infty} \operatorname{index} M(\lambda_j;\overline{u},\overline{v}), \\
\operatorname{index}_0 I''(\overline{u},\overline{v}) &= \sum_{j=1}^{\infty} \operatorname{index}_0 M(\lambda_j;\overline{u},\overline{v}).
\end{aligned}$$

Thus we obtain (1.14) and (1.15). Since $\operatorname{index} M(\lambda_j;\overline{u},\overline{v}) \leq \operatorname{index}_0 M(\lambda_j;\overline{u},\overline{v})$, $(\overline{u},\overline{v})$ is non-degenerate if and only if

$$\operatorname{index} M(\lambda_j;\overline{u},\overline{v}) = \operatorname{index}_0 M(\lambda_j;\overline{u},\overline{v}) \quad \text{for all } j \in \mathbb{N}.$$

That is, 0 is not an eigenvalue of $M(\lambda_j;\overline{u},\overline{v})$ for all $j \in \mathbb{N}$. This is nothing but (1.16). Finally, we remark that

$$\begin{aligned}
&\operatorname{index} M(\lambda;\overline{u},\overline{v}) \text{ and } \operatorname{index}_0 M(\lambda;\overline{u},\overline{v}) \\
&\qquad\qquad \text{are non-increasing functions of } \lambda.
\end{aligned} \tag{1.17}$$

$$\operatorname{index} M(\lambda;\overline{u},\overline{v}) = \operatorname{index}_0 M(\lambda;\overline{u},\overline{v}) = 0 \text{ for sufficiently large } \lambda > 1. \tag{1.18}$$

Thus the sums in (1.14) and (1.15) are finite. □

As a corollary, we have.

COROLLARY 1.8.

(i) $\operatorname{index} I''(a,0) = \operatorname{index}_0 I''(a,0) = 0$, $\operatorname{index} I''(0,b) = \operatorname{index}_0 I''(0,b) = 0$ *and* $(a,0)$, $(0,b)$ *are strict local minimums of* $I(u,v)$ *in* E.

(ii)

$$\begin{aligned}\operatorname{index} I''(u_0,v_0) \\ = \max\left\{\ell \in \mathbb{N} : \det\left(\lambda_\ell \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} - D^2V(u_0,v_0)\right) < 0\right\},\end{aligned} \tag{1.19}$$

and if we regard (k_1,k_2) *as a parameter*

$$\operatorname{index} I''(u_0,v_0) \ge 1 \quad \textit{for all } (k_1,k_2) \in (0,\infty)\times(0,\infty), \tag{1.20}$$

$$\operatorname{index} I''(u_0,v_0) \to \infty \quad \textit{as } (k_1,k_2) \to (0,0). \tag{1.21}$$

Moreover, (u_0,v_0) *is non-degenerate if and only if*

$$\det\left(\lambda_j \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} - D^2V(u_0,v_0)\right) \ne 0 \quad \textit{for all } j \in \mathbb{N}. \tag{1.22}$$

PROOF. By assumption (V1), $-D^2V(a,0)$ and $-D^2V(0,b)$ are positive definite. Thus we can obtain (i) from Lemma 1.7. For (ii), by assumption (V1), we have

$$\operatorname{index} M(0;u_0,v_0) = \operatorname{index} - D^2V(u_0,v_0) = 1.$$

Thus, by (1.17) we have for $\lambda \ge 0$

$$\operatorname{index} M(\lambda;u_0,v_0) = \begin{cases} 1 & \text{if } \det M(\lambda;u_0,v_0) < 0, \\ 0 & \text{if } \det M(\lambda;u_0,v_0) \ge 0. \end{cases}$$

Therefore we get (1.19). (1.20)–(1.21) can be deduced from (1.19) easily. (1.22) follows from (1.16). □

REMARK 1.9. Using Corollary 1.8, we can represent conditions (0.6)–(0.8) in Theorems 0.1 and 0.2 in terms of $I''(u_0,v_0)$; condition (0.6) is equivalent to $\operatorname{index} I''(u_0,v_0) \ge 2$, (0.7) holds if and only if (u_0,v_0) is non-degenerate, and (0.8) is nothing but $m = \operatorname{index} I''(u_0,v_0) \ge 2$.

REMARK 1.10. We can see as in Corollary 1.8 that

$$\begin{aligned} &\operatorname{index} I''(0,0) \ge 2 \quad && \text{for all } (k_1,k_2) \in (0,\infty)\times(0,\infty), \\ &\operatorname{index} I''(0,0) \to \infty && \text{as } (k_1,k_2) \to (0,0). \end{aligned}$$

Since $I(u,v)$ is even and coercive (Proposition 1.3), we can apply Clark's theorem ([C], see also Theorem 9.1 of Rabinowitz [R2]) to see that the number of solutions (including positive, negative and sign-changing ones) of (0.1)–(0.3) goes to infinity as $(k_1,k_2) \to (0,0)$.

2. Proof of of Theorem 0.1(i)

In this section, we give a proof to (i) of Theorem 0.1. In what follows, we identify the set of constant functions in E and $\mathbb{R}^2$. We use the notation:

$$P = \{(u,v) \in E : u(x) \geq 0 \text{ and } v(x) \geq 0 \text{ in } x \in \overline{\Omega}\}.$$

Since $I(\overline{u},\overline{v}) = -|\Omega| V(\overline{u},\overline{v})$ for $(\overline{u},\overline{v}) \in \mathbb{R}^2$, critical values corresponding to constant solutions satisfy the following relation:

$$\max\{I(a,0),\, I(0,b)\} < I(u_0,v_0) < I(0,0) = 0.$$

This follows from (V1). By Corollary 1.8(i), $(a,0)$, $(0,b)$ are strict local minima of $I(u,v)$ in E. Therefore, we will find a non-constant positive solution through a version of the mountain pass theorem. We consider the following set of paths:

$$\Gamma = \{\gamma(s) \in C([0,1],E) : \gamma(0)(x) = (a,0), \quad \gamma(1)(x) = (0,b) \text{ for all } x \in \overline{\Omega}\}, \tag{2.1}$$

$$\Gamma_+ = \{\gamma(s) \in \Gamma : \gamma(s) \in P \text{ for all } s \in [0,1]\}, \tag{2.2}$$

$$\Gamma_{c+} = \{\gamma(s) \in \Gamma_+ : \gamma(s) \in \mathbb{R}^2 \text{ for all } s \in [0,1]\}. \tag{2.3}$$

We define

$$\beta = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I(\gamma(s)), \tag{2.4}$$

$$\beta_+ = \inf_{\gamma \in \Gamma_+} \max_{s \in [0,1]} I(\gamma(s)), \tag{2.5}$$

$$\beta_{c+} = \inf_{\gamma \in \Gamma_{c+}} \max_{s \in [0,1]} I(\gamma(s)). \tag{2.6}$$

We have

Proposition 2.1.

(i) $\max\{I(a,0), I(0,b)\} < \beta = \beta_+ \leq \beta_{c+} = I(u_0,v_0) < 0$.

(ii) *There exists a* $(u_*,v_*) \in E$ *such that*

$$(u_*,v_*) \in P, \tag{2.7}$$

$$I(u_*,v_*) = \beta, \tag{2.8}$$

$$I'(u_*,v_*) = 0, \tag{2.9}$$

$$\operatorname{index} I''(u_*,v_*) \leq 1. \tag{2.10}$$

Proof. First we show $\beta = \beta_+$. Since $\Gamma_+ \subset \Gamma$, $\beta \leq \beta_+$ is clear. For $\gamma(s) = (u(s)(x), v(s)(x)) \in \Gamma$, setting

$$\widetilde{\gamma}(s) = (|u(s)(x)|, |v(s)(x)|) \in \Gamma_+,$$

we have $I(\widetilde{\gamma}(s)) \leq I(\gamma(s))$ for all $s \in [0,1]$. Therefore

$$\beta_+ \leq \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} I(\widetilde{\gamma}(s)) \leq \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} I(\gamma(s)) = \beta.$$

That is, $\beta = \beta_+$. Since $(a,0)$, $(0,b)$ are strict local minima of $I(u,v)$, we can also see

$$\max\{I(a,0),\, I(0,b)\} < \beta = \beta_+.$$

Since $I(u,v)$ satisfies the Palais–Smale condition (Proposition 1.3), we can see that $\beta = \beta_+$ is a critical value of $I(u,v)$, that is, there exists $(u_*,v_*) \in E$ satisfying (2.8) and (2.9).

To see (2.7), we use the Ekeland's principle. Since $\beta = \beta_+$, we can find a path $\gamma_n \in \Gamma_+$ such that

$$\max_{s\in[0,1]} I(\gamma_n(s)) \to \beta.$$

By the Ekeland's principle, we can find $(u_n,v_n) \in E$ such that

$$\operatorname{dist}\left((u_n,v_n), \gamma_n([0,1])\right) \to 0, \tag{2.11}$$

$$I(u_n,v_n) \to \beta, \tag{2.12}$$

$$I'(u_n,v_n) \to 0. \tag{2.13}$$

Since $I(u,v)$ satisfies the Palais–Smale condition, we can choose a convergent subsequence, denoted by (u_n,v_n), such that $(u_n,v_n) \to (u_*,v_*)$ in E. By (2.12), (2.13), we have (2.8), (2.9). Since $\gamma_n([0,1]) \subset P$, (2.7) follows from (2.11).

Property (2.10) can be deduced just as in Tanaka [T]. See also Fang and Ghoussoub [FG]. Therefore we get (ii). We set

$$\Gamma_c = \{\gamma(s) \in \Gamma : \gamma(s) \in \mathbb{R}^2 \text{ for all } s \in [0,1]\}, \qquad \beta_c = \inf_{\gamma\in\Gamma_c} \max_{s\in[0,1]} I(\gamma(s)).$$

We remark

$$I(u,v) = -|\Omega| V(u,v) \quad \text{on } \mathbb{R}^2. \tag{2.14}$$

Arguing as above in $\mathbb{R}^2$ instead of E, we can see $\beta_c = \beta_{c+} > \max\{I(a,0),\, I(0,b)\}$ and there exists a $(\overline{u},\overline{v}) \in [0,\infty) \times [0,\infty)$ such that

$$I(\overline{u},\overline{v}) = \beta_c, \quad I'(\overline{u},\overline{v}) = 0, \quad \text{index } -D^2V(\overline{u},\overline{v}) \leq 1.$$

By (2.14), $(\overline{u},\overline{v})$ is a critical point of $-|\Omega|V(u,v)$ in $\mathbb{R}^2$. Thus by (V1) we see $(\overline{u},\overline{v}) = (u_0,v_0)$ and $\beta_+ = I(u_0,v_0)$. Thus we obtain (i). □

Using Lemma 1.5 and assumption (0.6), we give a proof of Theorem 0.1(i).

Proof of Theorem 0.1(i). Let (u_*, v_*) be a critical point of $I(u,v)$ obtained in Proposition 2.1. It satisfies

$$\begin{gathered} u_*(x) \geq 0,\ v_*(x) \geq 0 \quad \text{in } \overline{\Omega}, \\ (u_*, v_*) \notin \{(0,0),\, (a,0),\, (0,b)\}, \\ \operatorname{index} I''(u_*, v_*) \leq 1. \end{gathered}$$

By assumption (0.6), that is, $\operatorname{index} I''(u_0, v_0) \geq 2$,

$$(u_*, v_*) \not\equiv (u_0, v_0). \tag{2.15}$$

Thus by Lemma 1.5, (u_*, v_*) is a positive solution of (0.1)–(0.4). Moreover, by (V1) and (2.15), (u_*, v_*) is non-constant. □

Remark 2.2. We can find a path $\overline{\gamma}(s) \in \Gamma_{c+}$ such that

$$\max_{s\in[0,1]} I(\overline{\gamma}(s)) = \beta_{c+} = I(u_0, v_0).$$

If (0.6) holds, i.e., $\operatorname{index} I''(u_0, v_0) \geq 2$, we modify $\overline{\gamma}(s)$ near (u_0, v_0) "in E" to obtain a path $\gamma \in \Gamma$ such that

$$\max_{s\in[0,1]} I(\gamma(s)) < \beta_{c+} = I(u_0, v_0).$$

Thus, under the assumption of Theorem 0.1(i), we have

$$\max\{I(a,0),\, I(0,b)\} < \beta = I(u_*, v_*) < \beta_{c+} = I(u_0, v_0) < 0. \tag{2.16}$$

3. Proof of Theorem 0.1(ii)

In this section, we will prove the existence of at least 2 non-constant positive solutions under additional assumptions (V4) and (0.7). We remark that (0.7) ensures the non-degeneracy of (u_0, v_0). We use (V4) together with Krein–Rutman theory to compute the local degree at the mountain pass critical point $(u_*, v_*) \in E$ obtained in the previous section (see Section 4.5).

For technical reasons, we modify $I(u,v)$ and consider the following functional $J(u,v) : E \to \mathbb{R}$ defined as

$$J(u,v) = \int_\Omega \left[\frac{k_1}{2}|\nabla u|^2 + \frac{k_2}{2}|\nabla v|^2 - V(u,v) + \frac{A}{2}(u^2 - u_+^2 + v^2 - v_+^2)\right] dx, \tag{3.1}$$

where $A > 0$ is a constant defined in (1.8) and $u_+ = \max\{u, 0\}$. First of all, we have

Lemma 3.1.

(i) $J(u,v) \geq I(u,v)$ *for all* $(u,v) \in E$.
(ii) $J(u,v) = I(u,v)$ *for all* $(u,v) \in P$.
(iii) $J(u,v) \in C^1(E,\mathbb{R})$.
(iv) $J(u,v)$ *is coercive and satisfies the Palais–Smale condition.*
(v) $(u,v) \in E$ *is a critical point of* $J(u,v)$ *if and only if* (u,v) *is a non-negative solution of* (0.1)–(0.3).
(vi) *Moreover, if* $(u,v) \neq (0,0)$, $(a,0)$, $(0,b)$, *then* (u,v) *is a positive solution of* (0.1)–(0.4).

Proof. (i)–(iii) are clear, and we can prove (iv) as in the proof of Proposition 1.3. Suppose that $(u,v) \in E$ is a critical point of $J(u,v)$. That is, (u,v) satisfies

$$\begin{aligned} -k_1\Delta u + (A - 2W_x(u^2,v^2))u &= Au_+ && \text{in } \Omega, \\ -k_2\Delta v + (A - 2W_y(u^2,v^2))v &= Av_+ && \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Here we use the notation $W(u^2,v^2) = V(u,v)$ as in Section 1.2. Since the right hand sides are non-negative and $A - 2W_x(u^2,v^2) \geq 1$ and $A - 2W_y(u^2,v^2) \geq 1$ in $\overline{\Omega}$, we can see that $u(x)$, $v(x) \geq 0$ in $\overline{\Omega}$. Thus we get (v).

By (v), critical points of $J(u,v)$ are also critical points of $I(u,v)$. Thus (vi) follows from Lemma 1.5. $\square$

The advantage of the functional $J(u,v)$ is that there are no negative or sign-changing critical points, and that all critical points of $J(u,v)$ are non-negative solutions of (0.1)–(0.3). The short point of $J(u,v)$ is its regularity; $J(u,v)$ belongs only to $C^1(E,\mathbb{R})$ not in $C^2(E,\mathbb{R})$, and this property is not convenient to apply Morse theoretic arguments.

To give a proof to (ii) of Theorem 0.1, we argue indirectly and assume

(A) The set of non-negative solutions of (0.1)–(0.3) is

$$\{(0,0),\ (a,0),\ (0,b),\ (u_0,v_0),\ (u_*(x),v_*(x))\},$$

where $(u_*(x),v_*(x)) \in E$ is a positive non-constant solution obtained in (i) of Theorem 0.1.

Being inspired by the work of Dancer [D2], we apply Leray–Schauder degree theory to our problem. We regard $E^* \simeq E$ by the Hilbert structure and denote by $\deg(J',0,B_R(u,v))$ the Leray–Schauder degree of

$$J' : E \to E^* \simeq E,$$

with respect to 0 and the ball

$$B_R(u,v)=\{(\phi,\psi)\in E:\|(\phi,\psi)-(u,v)\|_E<R\}.$$

For an isolated critical point $(u,v)\in E$ of $J(u,v)$, we define *local degree* by

$$\deg_{\mathrm{loc}}(J',(u,v))=\lim_{\varepsilon\to 0}\deg(J',0,B_\varepsilon(u,v)).$$

We remark that $\deg(J',0,B_\varepsilon(u,v))$ does not depend on ε if (u,v) is the unique critical point of $J(u,v)$ in $B_\varepsilon(u,v)$. In Section 4 we prove

Proposition 3.2. *Assume* (A). *Then*

(D.1) $\deg_{\mathrm{loc}}(J',(0,0))=0$.
(D.2) $\deg_{\mathrm{loc}}(J',(a,0))=\deg_{\mathrm{loc}}(J',(0,b))=1$.
(D.3) $\deg_{\mathrm{loc}}(J',(u_0,v_0))=(-1)^{\mathrm{index}\,I''(u_0,v_0)}$.
(D.4) $\deg_{\mathrm{loc}}(J',(u_*,v_*))=-1$.

We also prove the following proposition in Section 4.

Proposition 3.3. *For a sufficiently large $R\geq 1$,*

(D.5) $\deg(J',0,B_R(0,0))=1$.

Using Propositions 3.2 and 3.3, we can give a proof of Theorem 0.1(ii).

Proof of Theorem 0.1(ii). Assume (A). Then we have

$$\begin{aligned}\deg(J',0,B_R(0,0))=&\deg_{\mathrm{loc}}(J',(0,0))+\deg_{\mathrm{loc}}(J',(a,0))\\&+\deg_{\mathrm{loc}}(J',(0,b))+\deg_{\mathrm{loc}}(J',(u_0,v_0))\\&+\deg_{\mathrm{loc}}(J',(u_*,v_*)).\end{aligned}\tag{3.2}$$

However, (D.1)–(D.4) and (D.5) are incompatible with (3.2). Thus $J(u,v)$ possesses another critical point, that is, (0.1)–(0.4) have at least 2 non-constant positive solutions. □

4. Proofs of Propositions 3.2 and 3.3

In this section, we give proofs of Propositions 3.2 and 3.3 under assumption (A).

4.1. Some regularity property on homotopy. We will use homotopy invariance property of the Leray–Schauder degree repeatedly. First, we state some regularity property.

Let $f_\theta(u,v):\mathbb{R}^2\to\mathbb{R}$, $\theta\in[0,1]$ be a homotopy such that

(i) $f_\theta(u,v)\in C^1([0,1]\times\mathbb{R}^2,\mathbb{R})$.
(ii) $f_0(u,v)=V(u,v)$ for all $(u,v)\in\mathbb{R}^2$.

(iii) There exists a constant $C > 0$ independent of u, v, θ such that

$$|f_\theta(u,v)| \le C(|u|^2 + |v|^2 + 1), \tag{4.1}$$

$$|\partial_u f_\theta(u,v)|,\ |\partial_v f_\theta(u,v)| \le C(|u| + |v| + 1). \tag{4.2}$$

We set

$$K_\theta(u,v) = \int_\Omega \left[\frac{k_1}{2}|\nabla u|^2 + \frac{k_2}{2}|\nabla v|^2 - f_\theta(u,v)\right] dx : E \to \mathbb{R}.$$

LEMMA 4.1. *Suppose that* $(\underline{u},\underline{v}) \in E$ *is a critical point of* $K_\theta(u,v)$ *for all* $\theta \in [0,1]$. *Then for any* $\varepsilon > 0$ *there exists a* $\delta = \delta(\varepsilon) > 0$ *independent of* $\theta \in [0,1]$ *such that if* $(u,v) \in E$ *satisfies*

$$(u,v) \in \overline{B_\delta(\underline{u},\underline{v})} \text{ and } K'_\theta(u,v) = 0 \text{ for some } \theta \in [0,1],$$

then

$$\|u(x) - \underline{u}(x)\|_{L^\infty(\Omega)} < \varepsilon \quad \text{and} \quad \|v(x) - \underline{v}(x)\|_{L^\infty(\Omega)} < \varepsilon.$$

PROOF. We consider an operator $T_\theta : E \to E$ defined by

$$T_\theta \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (-k_1\Delta + 1)^{-1}(\partial_u f_\theta(u,v) + u) \\ (-k_2\Delta + 1)^{-1}(\partial_v f_\theta(u,v) + v) \end{bmatrix}.$$

Under the assumptions (4.1)–(4.2), we can see that T_θ is continuous uniformly in θ as an operator:

$$T_\theta : L^p(\Omega) \times L^p(\Omega) \to W^{2,p}(\Omega) \times W^{2,p}(\Omega) \quad \text{for all } p \in (1,\infty).$$

We set $S_\theta = T_\theta \circ T_\theta \circ \ldots \circ T_\theta$ ($[N/2] + 1$ times). Then

$$S_\theta : L^2(\Omega) \times L^2(\Omega) \to L^\infty(\Omega) \times L^\infty(\Omega),$$

is continuous uniformly in θ. Thus for any $(\underline{u},\underline{v}) \in E$ and $\varepsilon > 0$, we can choose a $\delta = \delta(\varepsilon) > 0$ such that

$$\|(u,v) - (\underline{u},\underline{v})\|_{L^2(\Omega)\times L^2(\Omega)} \le \delta \ \Rightarrow\ \|S_\theta(u,v) - S_\theta(\underline{u},\underline{v})\|_{L^\infty(\Omega)\times L^\infty(\Omega)} < \varepsilon.$$

Now suppose that $(\underline{u},\underline{v})$ is a critical point of $K_\theta(u,v)$. Then $(\underline{u},\underline{v})$ is a fixed point of T_θ and S_θ. Suppose that $(u,v) \in B_\delta(\underline{u},\underline{v})$ satisfies $K'_\theta(\underline{u},\underline{v}) = 0$. Then (u,v) is also a fixed point of S_θ, and we get

$$\|(u,v) - (\underline{u},\underline{v})\|_{L^\infty(\Omega)\times L^\infty(\Omega)} = \|S_\theta(u,v) - S_\theta(\underline{u},\underline{v})\|_{L^\infty(\Omega)\times L^\infty(\Omega)} < \varepsilon.$$

Thus the proof is completed. □

4.2. Proof of (D.1). In this subsection, we prove (D.1), i.e., we consider a homotopy defined by

$$\begin{aligned} f_\theta(u,v) = {} & (1-\theta)W(u^2,v^2) + \theta(W_x(0,0)u^2 \\ & + W_y(0,0)v^2) - \frac{A}{2}(u^2 - u_+^2 + v^2 - v_+^2), \\ K_\theta(u,v) = {} & \int_\Omega \left[\frac{k_1}{2}|\nabla u|^2 + \frac{k_2}{2}|\nabla v|^2 - f_\theta(u,v)\right] dx. \end{aligned}$$

We remark $W_x(0,0) > 0$, $W_y(0,0) > 0$ by (W1), and we choose $\varepsilon > 0$ such that

$$|W_x(0,0) - W_x(u^2,v^2)| \le W_x(0,0)/2, \tag{4.3}$$

$$|W_y(0,0) - W_y(u^2,v^2)| \le W_y(0,0)/2, \tag{4.4}$$

for all $|u|$, $|v| \le \varepsilon$.

We apply Lemma 4.1 and set $\delta = \delta(\varepsilon) > 0$. Then we have

Lemma 4.2. *For $\delta = \delta(\varepsilon) > 0$ defined as above,*

$$K'_\theta(u,v) \ne 0 \quad \textit{for all } \theta \in [0,1] \textit{ and } (u,v) \in \partial B_\delta(0,0).$$

Proof. Suppose that $(u,v) \in E$ satisfies $\|(u,v)\|_E \le \delta$ and $K'_\theta(u,v) = 0$. By Lemma 4.1,

$$\|u\|_{L^\infty(\Omega)},\ \|v\|_{L^\infty(\Omega)} < \varepsilon, \tag{4.5}$$

and

$$\begin{aligned} -k_1\Delta u + (A - 2W_x(0,0) + 2(1-\theta)(W_x(0,0) & \\ - W_x(u^2,v^2)))u & = Au_+ \quad \text{in } \Omega, \end{aligned} \tag{4.6}$$

$$\begin{aligned} -k_2\Delta v + (A - 2W_y(0,0) + 2(1-\theta)(W_y(0,0) & \\ - W_y(u^2,v^2)))u & = Av_+ \quad \text{in } \Omega, \end{aligned} \tag{4.7}$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{4.8}$$

Applying the maximal principle as in the proofs of Lemmas 1.5 and 3.1, we find $(u,v) \in P$. Next we integrate (4.6) over Ω.

$$\int_\Omega [-2W_x(0,0) + 2(1-\theta)(W_x(0,0) - W_x(u^2,v^2))]u\,dx = 0.$$

By (4.3) and (4.5), we find $\int_\Omega u\,dx = 0$. That is, $u = 0$. Similarly, we can see $v = 0$. Therefore $K'_\theta(u,v) \ne 0$ for $(u,v) \in \partial B_\delta(0,0)$. □

As a corollary, we have

Corollary 4.3.

$$\deg_{\mathrm{loc}}(J', (0,0)) = \deg(J', 0, B_\delta(0,0)) = \deg(K_1', 0, B_\delta(0,0)).$$

Next we see

Lemma 4.4. $\deg(K_1', 0, B_\delta(0,0)) = 0$.

Proof. We introduce another homotopy:

$$\begin{aligned} g_\theta(u,v) &= W_x(0,0)u^2 + W_y(0,0)v^2 - \frac{A}{2}(u^2 - u_+^2 + v^2 - v_+^2) + \theta(u+v), \\ L_\theta(u,v) &= \int_\Omega \left[\frac{k_1}{2}|\nabla u|^2 + \frac{k_2}{2}|\nabla v|^2 - g_\theta(u,v)\right] dx. \end{aligned}$$

We remark that $L_0(u,v) = K_1(u,v)$. We claim that

$$L_\theta'(u,v) = 0 \text{ if and only if } (u,v) = (0,0) \text{ and } \theta = 0. \tag{4.9}$$

In fact, suppose that (u,v) and θ satisfy $L_\theta'(u,v) = 0$, that is,

$$-k_1\Delta u + (A - 2W_x(0,0))u = Au_+ + \theta \qquad \text{in } \Omega, \tag{4.10}$$

$$-k_2\Delta v + (A - 2W_y(0,0))v = Av_+ + \theta \qquad \text{in } \Omega, \tag{4.11}$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \qquad \text{on } \partial\Omega. \tag{4.12}$$

Integrating (4.10) over Ω, we get

$$(A - 2W_x(0,0)) \int_\Omega u\, dx = A \int_\Omega u_+\, dx + \theta|\Omega|.$$

Thus,

$$(A - 2W_x(0,0)) \int_\Omega u_+\, dx \geq A \int_\Omega u_+\, dx + \theta|\Omega|.$$

Therefore, we have $\theta = 0$ and $u_+ \equiv 0$. Using (4.10) again, we get $u \equiv 0$ by the maximal principle. Similarly we have $v \equiv 0$. Thus we get (4.9).

By (4.9), we see that

$$\begin{aligned} &L_\theta'(u,v) \neq 0 \quad \text{for all } \theta \in [0,1] \text{ and } (u,v) \in \partial B_\delta(0,0), \\ &L_1'(u,v) \neq 0 \quad \text{for all } (u,v) \in B_\delta(0,0). \end{aligned}$$

Thus we get

$$\deg(K_1', 0, B_\delta(0,0)) = \deg(L_0', 0, B_\delta(0,0)) = \deg(L_1', 0, B_\delta(0,0)) = 0. \qquad \square$$

Proof of (D.1). Combining Corollary 4.3 and Lemma 4.4, we obtain (D.1) from Proposition 3.2. $\square$

4.3. Proof of (D.2). By Corollary 1.8, $(a,0)$ and $(0,b)$ are strict local minimums of $I(u,v)$ in E. By definition (3.1) of $J(u,v)$, we see that $(a,0)$ and $(0,b)$ are also strict local minimums of $J(u,v)$ in E. Thus by the result of Amann [A] and Rabinowitz [R1], we get (D.2).

4.4. Proof of (D.3). Since $I(u,v)\in C^2(E,\mathbb{R})$ and $I''(u_0,v_0)$ is non-degenerate by assumption (0.7), we have

$$\deg_{\mathrm{loc}}(I'',(u_0,v_0))=(-1)^{\operatorname{index} I''(u_0,v_0)}. \tag{4.13}$$

To prove (D.3), we need

Lemma 4.5. *Suppose that $(\overline{u},\overline{v})$ is an isolated critical point of $J(u,v)$ corresponding to a positive solution of* (0.1)–(0.4). *Then*

$$\deg_{\mathrm{loc}}(J',(\overline{u},\overline{v}))=\deg_{\mathrm{loc}}(I',(\overline{u},\overline{v})).$$

Proof. We set

$$\begin{aligned} h_\theta(u,v) &= V(u,v)-\theta\frac{A}{2}(u^2-u_+^2+v^2-v_+^2),\\ M_\theta(u,v) &= \int_\Omega\left[\frac{k_1}{2}|\nabla u|^2+\frac{k_2}{2}|\nabla v|^2-h_\theta(u,v)\right]dx. \end{aligned}$$

Let $\varepsilon=\min\{\inf_\Omega\overline{u}(x),\,\inf_\Omega\overline{v}(x)\}/2>0$, and, by Lemma 4.1, we find a $\delta>0$ such that

$$(u,v)\in\overline{B_\delta(\overline{u},\overline{v})} \text{ and } M_\theta'(u,v)=0 \ \Rightarrow \|u-\overline{u}\|_{L^\infty(\Omega)},\|v-\overline{v}\|_{L^\infty(\Omega)}<\varepsilon.$$

In particular, $u(x)>0$, $v(x)>0$ in $\overline{\Omega}$. Thus, all critical points in $\overline{B_\delta(\overline{u},\overline{v})}$ of $M_\theta'(u,v)$ are positive solutions of (0.1)–(0.4), and they are critical points of $J(u,v)$. Since $(\overline{u},\overline{v})$ is an isolated critical point of $J(u,v)$, we may assume that the unique critical point of $J(u,v)$ in $\overline{B_\delta(\overline{u},\overline{v})}$ is $(\overline{u},\overline{v})$. Thus for $(u,v)\in\overline{B_\delta(\overline{u},\overline{v})}$

$$M_\theta'(u,v)=0 \quad\text{if and only if}\quad (u,v)=(\overline{u},\overline{v}).$$

Thus we get

$$M_\theta'(u,v)\neq 0 \quad\text{for all } \theta\in[0,1] \text{ and } (u,v)\in\partial B_\delta(\overline{u},\overline{v}).$$

Therefore,

$$\deg_{\mathrm{loc}}(J',(\overline{u},\overline{v}))=\deg_{\mathrm{loc}}(M_1',(\overline{u},\overline{v}))=\deg_{\mathrm{loc}}(M_0',(\overline{u},\overline{v}))=\deg_{\mathrm{loc}}(I',(\overline{u},\overline{v})). \qquad\square$$

Proof of (D.3). Applying Lemma 4.5 with $(\overline{u},\overline{v})=(u_0,v_0)$, we see

$$\deg_{\mathrm{loc}}(J',(u_0,v_0))=\deg_{\mathrm{loc}}(I',(u_0,v_0)).$$

Thus (D.3) follows from (4.13). $\square$

4.5. Proof of (D.4). To prove (D.4), we use an idea from Hofer [H2], [H3]. Condition (V4) and Krein–Rutman theory will also play an important role. We need the following

Definition 4.6. Let X be a Banach space and $\Phi \in C^1(X, \mathbb{R})$. Suppose that $x_0 \in X$ is a critical point and set $d = \Phi(x_0)$. We say that x_0 is of *mountain pass type with respect to* Φ if for all neighbourhoods $O \subset X$ of x_0 the topological space $O \cap \mathring{\Phi}^d$ is non-empty and not path-connected.

Here we use the notation

$$\mathring{\Phi}^d = \{x \in X;\, \Phi(x) < d\}, \quad \Phi^d = \{x \in X;\, \Phi(x) \le d\}.$$

We have the following characterization of (u_*, v_*).

Proposition 4.7. *Assume* (A). *Then* (u_*, v_*) *is of mountain pass type with respect to* $I(u, v)$.

Proof. We prove this assertion essentially as in Hofer [H2], [H3]. First, we remark that

$$\inf_{\gamma \in \Gamma} \max_{s \in [0,1]} J(\gamma(s)) = \beta, \tag{4.14}$$

where Γ and β are given in (2.1) and (2.4). This comes from Lemma 3.1(ii) and $\beta = \beta_+$.

We argue indirectly and assume that for some open neighbourhood O of (u_*, v_*) the set $O \cap \mathring{I}^\beta$ is path-connected. We can choose a neighbourhood $U \subset O$ of (u_*, v_*) such that $\operatorname{dist}(\partial O, U) > 0$. We also choose $\overline{\varepsilon} > 0$ so small that the only critical value of "J" in $(\beta - \overline{\varepsilon}, \beta + \overline{\varepsilon})$ is β (see (2.16)).

By a standard construction of a deformation flow (c.f. Lemma 1 of [H2]), we can find an $\varepsilon \in (0, \overline{\varepsilon})$ and a deformation $\sigma \in C([0,1] \times E, E)$ such that

$$\sigma(\{1\} \times (J^{\beta+\varepsilon} \setminus U)) \subset J^{\beta-\varepsilon}, \tag{4.15}$$

$$\sigma([0,1] \times \overline{U}) \subset O, \tag{4.16}$$

$$\sigma(s, (u, v)) = (u, v) \quad \text{for all } s \in [0,1] \text{ and } (u, v) \in (E \setminus J^{\beta+\overline{\varepsilon}}) \cup J^{\beta-\overline{\varepsilon}}. \tag{4.17}$$

We choose $\gamma \in \Gamma$ such that $\max_{s \in [0,1]} J(\gamma(s)) \le \beta + \varepsilon$ and set

$$\widehat{\gamma}(s) = \sigma(1, \gamma(s)).$$

We can easily see from (4.15)–(4.17) that $\widehat{\gamma} \in \Gamma$ and $\widehat{\gamma}([0,1]) \subset J^{\beta-\varepsilon} \cup O$. We set

$$s_+ = \sup\{s \in [0,1] : \widehat{\gamma}(s) \notin J^{\beta-\varepsilon}\}, \quad s_- = \inf\{s \in [0,1] : \widehat{\gamma}(s) \notin J^{\beta-\varepsilon}\}.$$

We have

$$\widehat{\gamma}(s_\pm) \in O \cap J^{\beta-\varepsilon}, \quad \widehat{\gamma}([0,1] \setminus [s_-, s_+]) \subset J^{\beta-\varepsilon}.$$

Since $J(u,v) \ge I(u,v)$ on E, we have

$$\widehat{\gamma}(s_\pm) \in O \cap I^{\beta-\varepsilon} \subset O \cap \mathring{I}^\beta.$$

Since $O \cap \mathring{I}^\beta$ is path-connected, there exists a path $\nu : [s_-, s_+] \to E$ such that

$$\nu(s_\pm) = \widehat{\gamma}(s_\pm), \quad \nu([s_-, s_+]) \subset O \cap \mathring{I}^\beta.$$

We define a path $\widetilde{\gamma}(s)$ by

$$\widetilde{\gamma}(s) = \begin{cases} \nu(s) & \text{if } s \in [s_-, s_+], \\ \widehat{\gamma}(s) & \text{otherwise.} \end{cases}$$

Then $\widetilde{\gamma} \in \Gamma$ and

$$\max_{s\in[0,1]} I(\widetilde{\gamma}(s)) < \beta.$$

But this contradicts definition (2.4) of β. Therefore, (u_*, v_*) is of mountain pass type with respect to $I(u,v)$. □

Hofer [H2], [H3] proved the following.

Proposition 4.8 (Hofer [H2], [H3]). *Let X be a Hilbert space and $\Phi \in C^2(X, \mathbb{R})$. Assume that the gradient Φ' has the form identity–compact. Further assume that*

(Φ) *for a critical point $x_0 \in X$ the first* (*smallest*) *eigenvalue λ_1 of linearization $\Phi''(x_0)$ at x_0 is simple provided $\lambda_1 = 0$.*

Then for an isolated critical point $x_0 \in X$ of mountain pass type with respect to Φ

$$\deg_{\mathrm{loc}}(\Phi', x_0) = -1.$$

To verify condition (Φ) for $I(u,v)$, we use assumption (V4). To derive (Φ) from (V4), we need the following

Proposition 4.9. *Let $a(x)$, $b(x)$, $c(x) \in C(\overline{\Omega}, \mathbb{R})$ and $b(x) > 0$ in $\overline{\Omega}$. Then the first eigenvalue of the following eigenvalue problem is simple.*

$$\begin{aligned} A\begin{bmatrix} u \\ v \end{bmatrix} &= \lambda \begin{bmatrix} u \\ v \end{bmatrix} \quad && \text{in } \Omega, \\ \frac{\partial}{\partial n}\begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} && \text{on } \partial\Omega, \end{aligned}$$

where

$$A = \begin{bmatrix} -k_1\Delta & 0 \\ 0 & -k_2\Delta \end{bmatrix} + \begin{bmatrix} a(x) & b(x) \\ b(x) & c(x) \end{bmatrix}.$$

Proof. It suffices to show that the first eigenvalue of $A + \ell \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is simple for sufficiently large $\ell > 1$. We define a cone $C \subset E$ by

$$C = \{(u,v) \in E : u(x) \ge 0 \text{ and } v(x) \le 0 \text{ in } \overline{\Omega}\}.$$

The desired result follows from Krein–Rutman theory (c.f. Schaefer [Sc]) if we verify for large $\ell > 1$

(4.18) $\left(A + \ell \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1}$ is well-defined and it is a compact operator.

(4.19) $\left(A + \ell \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1} C \subset C.$

(4.18) is clear. We will check (4.19). Let $(f, g) \in C$ and define $(u, v) \in E$ by

$$\begin{bmatrix} u \\ v \end{bmatrix} = \left(A + \ell \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} f \\ g \end{bmatrix}.$$

We write the above equation as

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -k_1\Delta + a(x) + \ell & 0 \\ 0 & -k_2\Delta + c(x) + \ell \end{bmatrix}^{-1} \cdot \left(\begin{bmatrix} 0 & -b(x) \\ -b(x) & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix} \right).$$

We denote the right hand side by $B\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right)$. For a sufficiently large $\ell > 1$ it is easily seen that $B : E \to E$ defines a contraction mapping on E, and (u, v) can be obtained as a limit

$$\begin{bmatrix} u \\ v \end{bmatrix} = \lim_{n\to\infty} B^n \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} \right).$$

We can also see that $B(C) \subset C$ under the condition $b(x) > 0$ in $\overline{\Omega}$. Thus $(u, v) \in C$. Therefore (4.19) is verified. $\square$

Now we can prove (D.4).

Proof of (D.4). We can see

$$\begin{aligned} &\langle\, I''(u_*, v_*)(\phi, \psi)\,,\, (\phi, \psi)\,\rangle \\ &= \int_\Omega \left(\left(\begin{bmatrix} -k_1\Delta & 0 \\ 0 & -k_2\Delta \end{bmatrix} - \begin{bmatrix} V_{uu}(u_*, v_*) & V_{uv}(u_*, v_*) \\ V_{uv}(u_*, v_*) & V_{vv}(u_*, v_*) \end{bmatrix} \right) \begin{bmatrix} \phi \\ \psi \end{bmatrix}, \begin{bmatrix} \phi \\ \psi \end{bmatrix} \right) dx. \end{aligned}$$

By (V4), we see

$$-V_{uv}(u_*(x), v_*(x)) > 0 \quad \text{in } \overline{\Omega}.$$

Thus we can apply Proposition 4.9 and the first eigenvalue of $I''(u_*, v_*)$ is simple. Thus by Proposition 4.7 and Theorem 4.8, we get

$$\deg_{\mathrm{loc}}(I', (u_*, v_*)) = -1.$$

Applying Lemma 4.5 with $(\overline{u}, \overline{v}) = (u_*, v_*)$, we see

$$\deg_{\mathrm{loc}}(J', (u_*, v_*)) = \deg_{\mathrm{loc}}(I', (u_*, v_*)) = -1. \qquad \square$$

In the above subsections, we proved (D.1)–(D.4) and completed the proof of Proposition 3.2.

4.6. Proof of Proposition 3.3. In this section, we prove (D.5), that is, $\deg_{\mathrm{loc}}(J', 0, B_R(0,0)) = 1$ for sufficiently large $R > 1$. We consider the following homotopy:

$$\begin{aligned} G_\theta(u,v) = \int_\Omega \bigg[& \frac{k_1}{2}|\nabla u|^2 + \frac{k_2}{2}|\nabla v|^2 \\ & + (1-\theta)\bigg(-V(u,v) + \frac{A}{2}(u^2 - u_+^2 + v^2 - v_+^2)\bigg) + \theta\frac{A}{2}(u^2+v^2)\bigg]\,dx. \end{aligned}$$

First, we have

Lemma 4.10. *For sufficiently large $R > 0$,*

$$G'_\theta(u,v) \neq 0 \quad \textit{for all } \theta \in [0,1] \textit{ and } (u,v) \in \partial B_R(0,0). \tag{4.20}$$

Proof. Suppose that $G'_\theta(u,v) = 0$ for some $\theta \in [0,1]$. We have

$$\begin{aligned} \int_\Omega [& k_1|\nabla u|^2 + k_2|\nabla v|^2 + (1-\theta)(-V_u(u,v)u - V_v(u,v)v \\ & + A(u^2 - u_+^2 + v^2 - v_+^2)) + \theta A(u^2+v^2)]\,dx = 0, \end{aligned} \tag{4.21}$$

since $\langle\, G'_\theta(u,v)\,,\,(u,v)\,\rangle = 0$. By (V5), we have for some constant $C_1 > 0$

$$\begin{aligned} (1-\theta)(-V_u(u,v)u - V_v(u,v)v + A(u^2 - u_+^2 + v^2 - v_+^2)) + \theta A(u^2+v^2) & \\ \geq \min\{1, A\}(u^2+v^2) - C_1 \quad \text{for all } (u,v) \in \mathbb{R}^2 \text{ and } \theta \in [0,1]. & \end{aligned} \tag{4.22}$$

Thus by (4.21)–(4.22)

$$\int_\Omega [k_1|\nabla u|^2 + k_2|\nabla v|^2 + \min\{1,A\}(u^2+v^2)]\,dx \leq C_1|\Omega|.$$

Therefore, we can find a constant $R_1 > 0$ independent of $\theta \in [0,1]$ such that

$$\|(u,v)\|_E \leq R_1.$$

Thus, for $R > R_1$ we get (4.20). □

Proof of Proposition 3.3. Let $R > 0$ be a sufficiently large constant so that (4.20) holds. By the homotopy invariance of Leray–Schauder degree, we have

$$\deg(J', 0, B_R(0,0)) = \deg(G'_0, 0, B_R(0,0)) = \deg(G'_1, 0, B_R(0,0)).$$

Since the unique critical point of $G_1(u,v)$ is $(0,0)$ and $\operatorname{index} G'_1(0,0) = 0$,

$$\deg(G'_1, 0, B_R(0,0)) = 1.$$

Thus we get (D.5). □

5. The case $N = 1$

In this section we deal with the case $N = 1$ and prove Theorem 0.2. In what follows, we assume (V0)–(V3) and $\Omega = (0,1)$. We consider the following problem:

$$k_1 u'' + V_u(u,v) = 0 \quad \text{in } (0,1), \tag{5.1}$$
$$k_2 v'' + V_v(u,v) = 0 \quad \text{in } (0,1), \tag{5.2}$$
$$u'(0) = u'(1) = v'(0) = v'(1) = 0 \tag{5.3}$$
$$u(x) > 0,\ v(x) > 0, \quad \text{in } (0,1), \tag{5.4}$$

A key of the proof of Theorem 0.2 is the following lemma.

Lemma 5.1. *Suppose that* $(\overline{u},\overline{v}) \in E$ *is a non-constant positive solution of* (5.1)–(5.4) *satisfying*

$$\overline{u}'(x_0) = \overline{v}'(x_0) = 0 \quad \textit{for some } x_0 \in (0,1).$$

Then $\operatorname{index} I''(\overline{u},\overline{v}) \geq 2$.

To prove the above lemma, we need the following notation:

$$\mu_j(\overline{u},\overline{v}) = \inf_{\substack{S\subset E \text{ is a subspace} \\ \dim S = j}} \max_{\substack{(\phi,\psi)\in S \\ \|(\phi,\psi)\|_E \leq 1}} \langle I''(\overline{u},\overline{v})(\phi,\psi)\,,\,(\phi,\psi)\rangle,$$
$$\mu_j^D(\overline{u},\overline{v}) = \inf_{\substack{S\subset H_0^1(0,1)\times H_0^1(0,1) \text{ is a subspace} \\ \dim S = j}} \max_{\substack{(\phi,\psi)\in S \\ \|(\phi,\psi)\|_E \leq 1}} \langle I''(\overline{u},\overline{v})(\phi,\psi)\,,\,(\phi,\psi)\rangle.$$

Clearly, we have

$$\mu_j(\overline{u},\overline{v}) < \mu_j^D(\overline{u},\overline{v}), \tag{5.5}$$
$$\mu_j(\overline{u},\overline{v}) < 0 \quad \text{if and only if} \quad \operatorname{index} I''(\overline{u},\overline{v}) \geq j. \tag{5.6}$$

Proof of Lemma 5.1. Suppose that $(\overline{u},\overline{v})$ is a non-constant positive solution of (5.1)–(5.4) satisfying $\overline{u}'(x_0) = \overline{v}'(x_0) = 0$ for some $x_0 \in (0,1)$.

Differentiating (5.1)–(5.2) we get

$$k_1\overline{u}''' + V_{uu}(\overline{u},\overline{v})\overline{u}' + V_{uv}(\overline{u},\overline{v})\overline{v}' = 0,$$
$$k_2\overline{v}''' + V_{uv}(\overline{u},\overline{v})\overline{u}' + V_{vv}(\overline{u},\overline{v})\overline{v}' = 0.$$

Setting

$$\phi_0(x) = \chi_{[0,x_0]}(x)\overline{u}'(x), \qquad \psi_0(x) = \chi_{[0,x_0]}(x)\overline{v}'(x),$$
$$\phi_1(x) = \chi_{[x_0,1]}(x)\overline{u}'(x), \qquad \psi_1(x) = \chi_{[x_0,1]}(x)\overline{v}'(x),$$

where

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a,b], \\ 0 & \text{otherwise,} \end{cases}$$

we find (ϕ_0, ψ_0), $(\phi_1, \psi_1) \in H^1_0(0,1) \times H^1_0(0,1)$ and

$$\langle\, I''(\overline{u}, \overline{v})(\phi, \psi)\,,\, (\phi, \psi)\,\rangle = 0 \quad \text{for all } (\phi, \psi) \in \operatorname{span}\{(\phi_0, \psi_0), (\phi_1, \psi_1)\}.$$

Therefore $\mu^D_2(\overline{u}, \overline{v}) \le 0$. Thus by (5.5), $\mu_2(\overline{u}, \overline{v}) < 0$. That is, $\operatorname{index} I''(\overline{u}, \overline{v}) \ge 2$ by (5.6). □

Combining with the result of Section 2, we get.

Proposition 5.2. *Assume* (0.6)*, i.e.,* $\operatorname{index} I''(u_0, v_0) \ge 2$*. Let* (u_*, v_*) *be the non-constant positive solution obtained in Section* 2*. Then we have*

(i) $(u'_*(x), v'_*(x)) \ne (0,0)$ *for all* $x \in (0,1)$*,*
(ii) $(u_*(1-x), v_*(1-x)) \not\equiv (u_*(x), v_*(x))$*.*

Proof. (i) By Proposition 2.1, $\operatorname{index} I''(u_*, v_*) \le 1$. Thus the desired result follows from Lemma 5.1.

(ii) Suppose that $(u_*(1-x), v_*(1-x)) \equiv (u_*(x), v_*(x))$. Then we have $(u'_*(1/2), v'_*(1/2)) = (0,0)$. However, this contradicts (i). □

We remark that if $(u_*(x), v_*(x))$ is a solution of (5.1)–(5.4) then $(u_*(1-x), v_*(1-x))$ is also a solution. Thus we have

Corollary 5.3. *Suppose* (0.6)*. Then* (5.1)–(5.4) *have at least* 2 *non-constant positive solutions.*

To find more non-constant positive solutions under the assumption (0.8), we consider (5.1)–(5.2) in subinterval $(0, 1/\ell)$. That is, for $\ell \in \mathbb{N}$ we consider

$$k_1 u'' + V_u(u,v) = 0 \qquad \text{in } (0, 1/\ell), \tag{5.7}$$

$$k_2 v'' + V_v(u,v) = 0 \qquad \text{in } (0, 1/\ell), \tag{5.8}$$

$$u'(0) = u'(1/\ell) = v'(0) = v'(1/\ell) = 0 \tag{5.9}$$

$$u(x) > 0,\ v(x) > 0 \qquad \text{in } (0, 1/\ell), \tag{5.10}$$

We remark that if (u,v) satisfies (5.7)–(5.10), we can extend it to $(0,1)$ by reflection. More precisely, we set

$$(\widehat{u}(x), \widehat{v}(x)) = \begin{cases} (u(x), v(x)) & \text{for } x \in [0, 1/\ell], \\ (u(1/\ell - x), v(1/\ell - x)) & \text{for } x \in (1/\ell, 2/\ell], \\ (u(x - 2/\ell), v(x - 2/\ell)) & \text{for } x \in (2/\ell, 3/\ell], \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

then we can see $(\widehat{u}, \widehat{v})$ is a solution of (5.1)–(5.4).

We denote by $\mathrm{index}_{(0,1/\ell)} I''(u_0,v_0)$ the Morse index at (u_0,v_0) for the problem (5.7)–(5.10):

$$\operatorname*{index}_{(0,1/\ell)} I''(u_0,v_0) = \max\bigg\{ \dim V : V \subset H^1(0,1/\ell)\times H^1(0,1/\ell) \text{ is a subspace}$$
$$\text{such that } \int_0^{1/\ell} [k_1|\phi'|^2 + k_2|\psi'|^2 - (D^2V(u_0,v_0)(\phi,\psi),(\phi,\psi))]\,dx < 0$$
$$\text{for all } (\phi,\psi)\in V\setminus\{(0,0)\}\bigg\}.$$

We remark that the original index $I''(u_0,v_0)$ is equal to $\mathrm{index}_{(0,1)} I''(u_0,v_0)$.

Lemma 5.3. *Let* $m = \mathrm{index}\, I''(u_0,v_0)$. *Then*

$$\begin{aligned}\operatorname*{index}_{(0,1/\ell)} I''(u_0,v_0) &\geq 2 \quad \textit{for } \ell \leq m-1,\\ \operatorname*{index}_{(0,1/\ell)} I''(u_0,v_0) &= 1 \quad \textit{for } \ell \geq m.\end{aligned}$$

Proof. By (ii) of Corollary 1.8, $\operatorname*{index}_{(0,1/\ell)} I''(u_0,v_0)$ can be represented as

$$\operatorname*{index}_{(0,1/\ell)} I''(u_0,v_0) = \max\bigg\{ j\in\mathbb{N} : \det\bigg(\lambda_j^{(\ell)}\begin{bmatrix} k_1 & 0\\ 0 & k_2\end{bmatrix} - D^2V(u_0,v_0)\bigg) < 0\bigg\}, \tag{5.11}$$

where $\lambda_j^{(\ell)}$ is the jth eigenvalue of $-d^2/dx^2$ in $(0,1/\ell)$ under Neumann boundary conditions. Since $\lambda_j^{(\ell)} = (\pi^2/\ell^2)(j-1)^2$, we can get the desired result easily from (5.11). □

Using this property, we can give a proof of Theorem 0.2.

Proof of Theorem 0.2. Suppose that $m = \mathrm{index}\, I''(u_0,v_0) \geq 2$. By Lemma 5.3, we have

$$\operatorname*{index}_{(0,1/\ell)} I''(u_0,v_0) \geq 2 \quad \text{for } \ell = 1,\dots,m-1.$$

Using Corollary 5.2 in $(0,1/\ell)$, we get the existence of 2 non-constant positive solutions $(u_{\ell,1},v_{\ell,1})$ and $(u_{\ell,2},v_{\ell,2})$ of (5.7)–(5.10). We extend them to $(0,1)$ by reflection — we denote the extended solutions also by $(u_{\ell,1},v_{\ell,1})$, $(u_{\ell,2},v_{\ell,2})$. These solutions have the following properties:

$$(u'_{\ell,i}(1/\ell), v'_{\ell,i}(1/\ell)) = (0,0) \quad \text{for } i=1,2, \tag{5.12}$$
$$(u'_{\ell,i}(x), v'_{\ell,i}(x)) \neq (0,0) \quad \text{in } (0,1/\ell) \text{ for } i=1,2. \tag{5.13}$$

By (5.12) and (5.13), it is easy to see that $u_{\ell,i}(x)$ ($\ell=1,\dots,m-1$, $i=1,2$) give different solutions and (5.1)–(5.4) have at least $2(m-1)$ non-constant positive solutions. □

Remark 5.4. For a nonlinear Strum–Liouville problem, a similar relation between the Morse indices and the number of zeros of solutions is studied in Berestycki [B] and Tanaka [T].

Remark 5.5. Nakashima [N] has obtained a similar existence result to our Theorem 0.2 recently. She considered the system (0.9)–(0.12) in case $N = 1$ and studied under the competition condition:

$$f_v(u,v) < 0 \quad \text{and} \quad g_u(u,v) < 0,$$

instead of variational structure, and her result can be applied to competition Lotka–Volterra models. She used a version of Leray–Schauder degree theory in the cone, which was developed by Dancer [D2].

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