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HETEROCLINICS FOR A HAMILTONIAN SYSTEM OF DOUBLE PENDULUM TYPE

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1. Introduction

Consider

$$\ddot{q} + V'(q) = 0 \tag{HS}$$

where $q \in \mathbb{R}^2$ and V satisfies

(V$_1$) $V \in C^2(\mathbb{R}^2, \mathbb{R})$ and $V(x)$ is T_i-periodic in x_i, $i = 1, 2$.

This system arises as a simpler model of the double pendulum. Indeed, the Lagrangian associated with (HS) is

$$\mathcal{L}(q) = \tfrac{1}{2}|\dot{q}|^2 - V(q).$$

The actual double pendulum has a related Lagrangian of the form

$$\mathcal{L}_1(q) = \sum_{i,j=1}^{2} a_{ij}(q)\dot{q}_i\dot{q}_j - V(q)$$

where the matrix $(a_{ij}(x))$ is positive definite and periodic in the components of x with the same periods as in (V$_1$). The existence results obtained for (HS) can also be obtained for the Hamiltonian system associated with $\mathcal{L}_1(q)$ but we

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prefer to deal with the simpler case. It will also be assumed for convenience that $T_1 = 1 = T_2$.

The maximum of the potential V occurs on a lattice of points, say $\mathbb{Z}^2$. Further assume

$$(\mathrm{V}_2) \qquad V(x) < V(0) = 0, \quad x \in \mathbb{R}^2 \setminus \mathbb{Z}^2.$$

By (V_2), each $x \in \mathbb{Z}^2$ is an equilibrium solution of (HS). It is known that for each such x, there is a heteroclinic solution of (HS) joining x to $\mathbb{Z}^2 \setminus \{x\}$ (see [12]). Because of (V_1), $\mathbb{R}^2$ can be viewed as the covering space of T^2 and these heteroclinics can be interpreted as homoclinic solutions of (HS) on T^2. Furthermore, as is well known, straightforward minimization arguments show that for each $T > 0$ and each $k \in \mathbb{Z}^2 \setminus \{0\}$, (HS) has a solution p satisfying

$$p(t + mT) = p(t) + mk \tag{1.1}$$

for all $m \in \mathbb{Z}$. Viewed on T^2, p is a T-periodic solution of (HS) of homotopy type k. Actually the above cited results are true for the analogous more general $\mathbb{R}^n$ setting.

The main goal of this paper is to establish the existence of orbits of (HS) which viewed on T^2 are heteroclinic to 0 and to one of the periodic orbits p mentioned above. Before formulating a theorem, a more precise description must be given of the heteroclinics of [12] and above periodics. These solutions are obtained by variational arguments. Set

$$I(q) = \int_{\mathbb{R}} \mathcal{L}(q)\, dt.$$

Let $k \in \mathbb{Z}^2 \setminus \{0\}$,

$$G_k = \{q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2) \mid q(-\infty) = 0,\ q(\infty) = k\}, \quad G = \bigcup_{k \in \mathbb{Z}^2 \setminus \{0\}} G_k,$$

and

$$c_k = \inf_{G_k} I. \tag{1.2}$$

Further, set

$$c_0 = \inf_{G} I = \inf_{k \in \mathbb{Z}^2 \setminus \{0\}} c_k. \tag{1.3}$$

It was shown in [12] that there is a $q_0 \in G$ such that $I(q_0) = c_0$ and q_0 is a heteroclinic solution of (HS) with $q_0(-\infty) = 0$ and $q_0(\infty) = k_0 \in \mathbb{Z}^2 \setminus \{0\}$. Moreover (see [13–14]), for each $k \in \mathbb{Z}^2 \setminus \{0\}$, there is a heteroclinic chain of solutions of (HS) joining 0 and k and

$$c_k = \sum_{i=1}^{j} I(q_i)$$

where the functions q_i are heteroclinic solutions of (HS) joining $q_{i-1}(\infty) = q_i(-\infty)$ to $q_i(\infty) = q_{i+1}(-\infty)$ with $q_1(-\infty) = 0$ and $q_j(\infty) = k$. In fact, $q_i - q_i(-\infty)$ minimizes I on G_{k_i} where $k_i = q_i(\infty) - q_i(-\infty)$.

To get the solutions of (HS) corresponding to (1.1), for each $k \in \mathbb{Z}^2 \setminus \{0\}$, let

$$F_k = \bigcup_{T>0} \{q \in W^{1,2}_{\mathrm{loc}} \mid q(t+T) = q(t) + k\} \equiv \bigcup_{T>0} F_{k,T}.$$

As we noted earlier, viewed on T^2, $q \in F_k$ is a closed $W^{1,2}$ curve of homotopy type k. For $q \in F_k$, set

$$I^*(q) = \int_{-T/2}^{T/2} \mathcal{L}(q)\, dt.$$

Let

$$c^*_{k,T} = \inf_{F_{k,T}} I^*. \tag{1.4}$$

Then there is a $q_{k,T} \in F_{k,T}$ satisfying (HS) with $I^*(q_{k,T}) = c^*_{k,T}$. Let

$$c^*_k = \inf_{T>0} c^*_{k,T}. \tag{1.5}$$

Since any $q \in G_k$ can be obtained as a $W^{1,2}_{\mathrm{loc}}$ limit of elements of F_k, it follows that

$$c^*_k \le c_k. \tag{1.6}$$

Moreover, elementary arguments as in [12] show that if along some minimizing sequence for (1.5), (T_m) is bounded, then along a subsequence, $T_m \to T^*_k > 0$, and q_{k,T_m} converges to a solution q^*_k of (HS) with $I^*(q^*_k) = c^*_k$. On the other hand, if (T_m) is unbounded, q_{k,T_m} will "converge" to a heteroclinic chain of solutions of (HS) corresponding to c^*_k as in [12–14].

Let

$$c^* = \inf_{k \in \mathbb{Z}^2 \setminus \{0\}} c^*_k. \tag{1.7}$$

Then there is a $k^* \in \mathbb{Z}^2 \setminus \{0\}$ such that $c^* = c^*_{k^*}$ and by (1.6) and (1.3),

$$c^* \le c_0. \tag{1.8}$$

Assume that

$$c^* < c_0. \tag{1.9}$$

Since $c_0 = c_k$ for some $k \in \mathbb{Z}^2 \setminus \{0\}$, a sufficient condition for (1.9) to hold is that there is a $T > 0$ and $q \in F_{k,T}$ satisfying $I^*(q) < c_0$. It is not difficult to impose conditions on V so that this is the case.

By the remarks following (1.6), there are $T^* > 0$ and $p^* \in F_{k^*,T^*}$ such that

$$c^*_{k^*} = I^*(p^*). \tag{1.10}$$

Certainly p^* is not unique. For $\theta \in \mathbb{R}$, the time translates

$$\tau_\theta q(t) = q(t-\theta)$$

satisfy $I^*(\tau_\theta p^*) = I^*(p^*)$. Note also that $I^*(p^*+j) = I^*(p^*)$ for all $j \in \mathbb{Z}^2$. Moreover, since (HS) is time reversible, $I^*(p^*(-t)) = c^*$ with $p^*(-t) \in F_{-k^*}$ and it is possible that there are other values of T, k, and $p \in F_{k,T}$ with $I^*(p) = c^*$. Let

$$P_k = \{p \in F_k \mid I^*(p) = c_k^*\}.$$

Then we have

THEOREM 1.11. *Suppose that V satisfies* (V_1), (V_2), *and* (1.9) *holds. Then there is a $p \in P_{k^*}$ and a solution Q of* (HS) *such that $Q(-\infty) = 0$ and Q is asymptotic to p as $t \to \infty$.*

In fact, the construction that gives q also shows there is a second such solution asymptotic to $p(-t) \in P_{-k^*}$ and a second pair of such solutions asymptotic to another $\overline{p}(t)$, $\overline{p}(-t)$ where $\overline{p} \in P_{k^*}$ and is adjacent to p^* with 0 lying between the curves p and $\overline{p}$. These results are special cases of a more general theorem. Choose $k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{0\}$ with k_1, k_2 relatively prime. Then there are associated minimization values c_k^* and c_k.

THEOREM 1.12. *Suppose that V satisfies* (V_1)–(V_2) *and*

$$c_k^* < c_k \tag{1.13}$$

where $k = (k_1, k_2)$ with k_1, k_2 relatively prime. Then (HS) *has two solutions Q_k^-, Q_k^+ which are heteroclinic to* 0 *and an adjacent pair $p_-, p_+ \in P_k$ and two solutions Q_{-k}^-, Q_{-k}^+ which are heteroclinic to* 0 *and to $p_-(-t)$, $p_+(-t)$.*

More precise geometrical information on the nature of these heteroclinics is given in conjunction with their proofs. Of course (1.13) is more difficult to verify than (1.9). Hypothesis (1.13) has another interesting consequence. Namely it implies for some $\beta \in \mathbb{N}$, $\beta \geq 2$, that (HS) has a pair of solutions q_m^-, q_m^+ heteroclinic to 0 and to mk for all $m \geq \beta$. This is in the spirit of a related kind of result in the setting of a singular Hamiltonian system due to Caldiroli and Jeanjean [6]. Moreover, certain monotonicity properties of $q_m^\pm$ permit us to give a further characterization of $Q_k^\pm$ as a limit of the functions $q_m^\pm$.

Theorems 1.11–1.12 are reminiscent of old results of Morse [11] and Hedlund [8] on the existence of geodesics heteroclinic to an adjacent pair of periodic geodesics in a given homotopy class in T^2. See also the interesting paper of Bangert [2] for a more modern view of this work and its connections to several other problems such as the work of Aubry and LeDaeron [1] and of Mather [9]. After completing this paper, we learned of the related work of Bolotin and Negrini [5] who, among other things, also consider (HS) on T^2 and establish an analogue of Theorem 1.12 using tools from Riemannian geometry in the spirit of [8, 11]. The primary concern of [5] is a variational criterion for the nonintegrability of (HS) when V is analytic and (1.13) holds. The current paper has some ideas in common with [5, 11] although our approach is rather different and our motivation to study the problem came from [12] and [15]. (See also Bolotin [3–4].)

Variational arguments will be used to establish the existence of the heteroclinic orbits. Indeed, one feature of the argument presented here is that it yields a direct variational characterization of the heteroclinic, Q_k^+, joining 0 to some $p_+ \in P_k$, rather than requiring approximation arguments because of the difficulties of dealing with a geodesic of infinite length in the Jacobi metric and with the points where the metric degenerates. Furthermore, in a natural way, it gives more geometrical information about the solutions than the approaches using the Jacobi metric. In §2, some preliminaries concerning the properties of P_k will be carried out. It will also be shown that there is a heteroclinic orbit joining 0 and k corresponding to c_k, i.e. when (1.13) holds, the heteroclinic chain joining 0 and k consists of a single orbit. Then in §3 the variational problem used to find the heteroclinic orbit, $Q = Q_k^+$, asymptotic to 0 and some $p \in F_k$ will be formulated. This entails introducing both an appropriate class of functions, Γ, and an associated functional, J, and seeking Q as the infimum of J over Γ. That J has a minimizer in Γ will be established in §4 and that Q is a solution of (HS) will be proved in §5. Some further properties of Q will also be obtained in §5. Lastly, in §6, it is shown that (1.13) implies the existence of heteroclinic solutions of (HS) in G_{mk} for all m large enough and these solutions possess certain monotonicity properties with respect to each other that lead to a new characterization of $Q_k^\pm$ as their limit.

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2. Some preliminaries

This section is devoted to some properties of P_k and the existence of a heteroclinic connection between 0 and k. It is always assumed for what follows that V satisfies (V_1)–(V_2) and (1.13) holds.

PROPOSITION 2.1. *Let $p \in F_{k,T}$ minimize $I^*|_{F_k}$.*

(a) *$p|_0^T$ is a simple curve.*
(b) *If $k = k^*$, then $p|_0^{T^*}/\mathbb{Z}^2$ is a simple curve, i.e. p is a simple curve on T^2.*
(c) *If $k = (k_1, k_2)$ with k_1, k_2 relatively prime, then*

$$c^*_{mk} = mc^*_k. \tag{2.2}$$

(d) (Minimality property) *If* k *is as in* (c), *for any* $a < b \in \mathbb{R}$, *and any* $q \in W^{1,2}[\overline{a}, \overline{b}]$ *where* $q(\overline{a}) = p(a)$, $q(\overline{b}) = q(b)$, *then*

$$\int_a^b \mathcal{L}(p)\,dt \le \int_{\overline{a}}^{\overline{b}} \mathcal{L}(q)\,dt \tag{2.3}$$

with equality only if $\overline{b} - \overline{a} = b - a$ *and* $q(\overline{a} + t) = p(a + t)$ *where defined.*

(e) *If* k *is as in* (c), *then* p *is a simple curve on* $\mathbb{R}$.

(f) *If* $q \in F_k$ *and minimizes* I^* *on* F_k, *then either* $p(\mathbb{R}) \cap q(\mathbb{R}) = \emptyset$ *or* $q = \tau_\theta p$ *for some* $\theta \in \mathbb{R}$.

PROOF. (a) If not, there exists $0 \le \sigma < s < T$ such that $p(\sigma) = p(s)$. Excising the closed loop $p([\sigma, s))$ from $p|_0^T$ yields a new curve $\overline{p} \in F_k$ with $I^*(\overline{p}) < I^*(p) = c_k^*$, contrary to (1.5).

(b) If not, there are σ, s as in (a) and $p \in \mathbb{Z}^2 \setminus \{0\}$ such that $p(\sigma) = p(s) + j$. But then $q = p|_\sigma^s \in F_{-j,s-\sigma}$ with $I^*(q) < I^*(p)$, contrary to (1.7).

(c) To prove (2.2), first the case of $m = 2$ will be verified. Let $\varepsilon > 0$ and $q \in F_{2k}$ be such that

$$I^*(q) \le c_{2k}^* + \varepsilon. \tag{2.4}$$

Let $\overline{L}$, $\underline{L}$ be lines of slope $2k$ such that q lies between $\underline{L}$ and $\overline{L}$ and is tangent to each line. Without loss of generality, $q(0) \in \underline{L}$ and if $q \in F_{2k,\overline{T}}$, there is a smallest $\overline{t} \in (0, \overline{T}\,)$ such that $q(\overline{t}\,) \in \overline{L}$. If $q(\sigma) + k = q(s)$ for some $0 \le \sigma < s \le \overline{T}$, then $q|_\sigma^s, q|_s^{\overline{T}+\sigma} \in F_k$ so

$$c_{2k}^* + \varepsilon \ge \int_\sigma^{\overline{T}+\sigma} \mathcal{L}(q)\,dt = \int_\sigma^s \mathcal{L}(q)\,dt + \int_s^{\overline{T}+\sigma} \mathcal{L}(q)\,dt \ge 2c_k^*. \tag{2.5}$$

On the other hand, $p|_0^{2T} \in F_{2k}$ so

$$2c_k^* = \int_0^{2T} \mathcal{L}(p)\,dt \ge c_{2k}^*. \tag{2.6}$$

Since ε is arbitrary, combining (2.5)–(2.6) yields (2.2) for $m = 2$. To obtain σ and s, consider $q(t) + k$ for $t \in [0, \overline{t}\,]$. If $q(0) + k = q(s)$ for some $s \in (0, \overline{T}\,)$ or $q(\overline{t}\,) + k = q(s)$ for some $s \in (\overline{t}, \overline{T}\,]$, we are through. Otherwise for small $t > 0$, $q(t) + k$ lies inside the region bounded by $\underline{L}$ and $q|_0^{\overline{T}}$ while for $t < \overline{t}$ and near $\overline{t}$, $q(t) + k$ lies outside this region. Hence there is a $\sigma \in (0, \overline{t})$ such that $q(\sigma) + k = q(s)$ and the case of $m = 2$ is complete.

For the general case, suppose that (2.2) holds for $m - 1$. Again, let $\varepsilon > 0$ and $q \in F_{mk}$ be such that

$$I^*(q) \le c_{mk}^* + \varepsilon.$$

The argument producing σ and s works in exactly the same fashion. Hence $q(\overline{T}+\sigma)-q(s)=(m-1)k$ so the analogue of (2.5) is

$$c^*_{m_k}+\varepsilon \geq \int_\sigma^s \mathcal{L}(q)\,dt + \int_s^{\overline{T}+\sigma} \mathcal{L}(q)\,dt \geq c^*_k + c^*_{(m-1)k} = mc^*_k$$

and (2.2) follows as for $m=2$.

(d) Suppose there is a $q \in W^{1,2}[\overline{a},\overline{b}]$ such that

$$\int_{\overline{a}}^{\overline{b}} \mathcal{L}(q)\,dt < \int_a^b \mathcal{L}(p)\,dt. \tag{2.7}$$

Choose $j,l \in \mathbb{Z}$ such that $jT<\overline{a}<\overline{b}<lT$. Define $r(t)$ via

$$r(t)=\begin{cases} q(t), & t\in[\overline{a},\overline{b}],\\ p(t), & \text{otherwise}.\end{cases} \tag{2.8}$$

Then $r|_{jT}^{lT} \in F_{(l-j)k}$ but

$$c^*_{(l-j)k} \leq \int_{jT}^{lT} \mathcal{L}(r)\,dt < \int_{jT}^{lT} \mathcal{L}(p)\,dt = (l-j)c^*_k \tag{2.9}$$

by (c). Hence the first assertion of (d) follows.

Choose $\alpha < s$ so that $p(b)-p(\alpha)=lk$ for some $l\in\mathbb{N}$. Extend q to $[\overline{a}-(a-\alpha),\overline{b}]$ via $q(\overline{a}-s)=p(a-s)$ for $s\in[0,a-\alpha]$. Thus p and q so extended lie in F_{lk} and $I^*(p)=c^*_{lk}=lc^*_k=I^*(q)$. Hence $q\in P_{lk}$ and is a solution of (HS) which coincides with p on an interval. Therefore $q(\overline{a}+t)=p(a+t)$ for all $t\in[\overline{a}-(a-\alpha),\overline{b}-\overline{a}\,]$.

(e) Given (d), a self-intersection of p leads to a contradiction as in (a).

(f) Suppose that $p(\mathbb{R})\cap q(\mathbb{R})\neq\emptyset$. Then $p(0)=q(s)$ for some $s\in\mathbb{R}$. Set $r=\tau_{-s}q$ so $r\in F_k$ and $r(0)=p(0)$. Suppose $r\in F_{k,\overline{T}}$. Set $\varphi(t)=r(t)$ on $[-\overline{T},0)$ and $\varphi(t)=p(t)$ on $[0,T]$. Then $\varphi\in F_{2k}$ and $I^*(\varphi)=2c^*_k=c^*_{2k}$ via (c). Hence φ minimizes I^* on F_{2k} and therefore is a solution of (HS). Due to the definition of φ, $r(t)\equiv p(t)=\tau_{-s}q(t)$ or $q=\tau_s p$.

Remark. See also [11] and [8] for closely related results in their setting.

Since (HS) is a Hamiltonian system, for any solution p of (HS),

$$\tfrac{1}{2}|\dot{p}(t)|^2+V(p(t))\equiv \text{constant}\equiv \alpha_p. \tag{2.10}$$

Proposition 2.11. *If $k\in\mathbb{Z}^2\setminus\{0\}$ and $p\in P_k$, $\alpha_p=0$.*

Proof. This simple result is probably well known. See e.g. T. Maxwell [10] for a more general result. Since the proof of this special case is brief we give the details.

Let $q \in F_{k,T}$. There is a corresponding $u \in W^{1,2}[0,1]$ via $q(t) = u(t/T) \equiv u(s)$ and

$$\int_0^T \mathcal{L}(q)\,dt = \int_0^1 \left(\frac{1}{2T}\left|\frac{du}{ds}\right|^2 - TV(u)\right)dt \equiv \Phi(T,u).$$

Hence

$$c_k^* = \inf\{\Phi(T,u) \mid T > 0 \text{ and } u \in W^{1,2}[0,1] \text{ with } u(1) - u(0) = k\}.$$

In particular, at the pair T, $p = u(t/T)$, the Fréchet derivative of Φ with respect to (T,u) vanishes:

$$\begin{aligned}\Phi'(T,u)(\sigma,\varphi) = 0 = &\int_0^1 \left(\frac{1}{2}\dot{u}\cdot\dot{\varphi} - TV'(u)\cdot\varphi\right)ds \\ &- \sigma\int_0^1 \left(\frac{1}{2T^2}|\dot{u}|^2 + V(u)\right)ds.\end{aligned} \tag{2.12}$$

The first term on the right in (2.12) vanishes since p is a solution of (HS). Hence

$$\int_0^1 \left(\frac{1}{2T^2}|\dot{u}|^2 + V(u)\right)ds = 0 = T\int_0^T \left(\frac{1}{2}|\dot{p}|^2 + V(p)\right)dt = T^2\alpha_p$$

so $\alpha_p = 0$.

COROLLARY 2.13. $p(\mathbb{R}) \cap \mathbb{Z}^2 = \emptyset$.

PROOF. If $p(\sigma) = j \in \mathbb{Z}^2$ for some $\sigma \in \mathbb{R}$, then (2.10) with $t = \sigma$, Proposition 2.11, and (V_2) imply $\dot{p}(\sigma) = 0$. But then the uniqueness of solutions of the initial value problem for (HS) implies $p(t) \equiv j$. Since $j \notin F_k$, we have a contradiction.

By Corollary 2.13, $0 \in \mathbb{R}^2 \setminus P_k$. Let p_-, p_+ denote the curves in P_k which are the boundaries of the component, $\mathcal{C}$, of $\mathbb{R}^2 \setminus P_k$ to which 0 belongs. (Geometrically for what follows, we think of p_- as being to the left and p_+ to the right of 0.)

PROPOSITION 2.14. $\mathcal{C} \cap \mathbb{Z}^2 = \mathbb{Z}k$.

PROOF. Certainly $\mathbb{Z}k \subset \mathcal{C}$. Suppose $j \in \mathbb{Z}^2 \setminus \mathbb{Z}k$ with $j \in \mathcal{C}$. Consider the straight line segment joining 0 and j. Its endpoints lie in $\mathcal{C}$. Hence the straight line extension of the segment intersects p_- and p_+. Suppose e.g. the extension in the direction of j intersects p_+. Then $p_+ - j \in P_k$ with 0 to the left of $p_+ - j$. By Proposition 2.1(f), either $(p_+(\mathbb{R}) - j) \cap p_-(\mathbb{R}) = \emptyset$ in which case 0 lies between p_- and $p_+ - j$, contrary to the choice of p_+, or $p_+(\mathbb{R}) - j = \tau_\theta p_-(\mathbb{R}) = p_-(\mathbb{R})$, which again is impossible since 0 lies to the left of $p_+ - j$. Hence there is no such j.

As was mentioned in the introduction, there is a heteroclinic chain of solutions of (HS) joining 0 and k. The next result shows that in fact for the current setting, there is a single heteroclinic orbit joining 0 and k.

Theorem 2.15. *Suppose V satisfies* (V_1)–(V_2), $k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{0\}$ *with* k_1, k_2 *relatively prime, and* (1.13) *holds. Then* (HS) *has a heteroclinic solution,* q^*, *with* $q^*(-\infty) = 0$, $q^*(\infty) = k$ *and* $q^*(t) \in \mathcal{C}$ *for all* $t \in \mathbb{R}$.

Proof. It was shown in [12] and [13] that any appropriately normalized minimizing sequence (q_m) for (1.2) "converges" to a heteroclinic chain of solutions of (HS) joining 0 and k and if $Q_1, \ldots, Q_j$ are the "links" in the chain, then

$$c_k = \sum_{i=1}^{j} I(Q_i). \tag{2.16}$$

We claim each q_m can be assumed to lie in $\overline{\mathcal{C}}$. Assuming this for the moment, the arguments of [12], [13] show $Q_i \in \overline{\mathcal{C}}$ and therefore by Proposition 2.14, $Q_i(\pm\infty) \in \mathbb{Z}k$. If $j > 1$ and any Q_i is such that $Q_i(\infty) - Q_i(-\infty) = \pm k$, then translation and, if necessary, time reversal yield a $\widehat{Q} \in G_k$ such that

$$I(\widehat{Q}) = \int_{\mathbb{R}} \mathcal{L}(Q_i)\, dt < c_k$$

contrary to the definition of c_k. Thus each Q_i has $Q_i(\infty) - Q_i(-\infty) = \pm lk$ for some $l \in \mathbb{N} \setminus \{1\}$. Choose i so that

$$\int_{\mathbb{R}} \mathcal{L}(Q_i)\, dt = \min_{1 \le n \le j} \int_{\mathbb{R}} \mathcal{L}(Q_n)\, dt. \tag{2.17}$$

Again translation and time reversal yield $\widehat{Q}$ such that $\widehat{Q}(-\infty) = 0$, $\widehat{Q}(\infty) = lk$, and

$$\int_{\mathbb{R}} \mathcal{L}(\widehat{Q})\, dt = \int_{\mathbb{R}} \mathcal{L}(Q_i)\, dt.$$

The argument of Proposition 2.1(c) shows $\widehat{Q}$ and $\widehat{Q} + k$ intersect. Suppose that $\widehat{Q}(\sigma) = \widehat{Q}(s) + k$. Let

$$\widetilde{Q}(t) = \begin{cases} \widehat{Q}(t), & -\infty \le t \le \sigma, \\ \widehat{Q}(s + \sigma - t) + k, & \sigma \le t \le \infty. \end{cases} \tag{2.18}$$

Then $\widetilde{Q} \in G_k$ and

$$I(\widetilde{Q}) < 2I(\widehat{Q}) \le \sum_{i=1}^{j} I(Q_i) = c_k, \tag{2.19}$$

a contradiction. Hence $j = 1$ and $q^* = Q_1$. To see that $q^*(\mathbb{R}) \subset \mathcal{C}$, suppose $q^*(\mathbb{R}) \cap P_k \neq \emptyset$. Then q^* and p have a point in common where $p = p_-$ or p_+. Say $q^*(\sigma) = p(s)$. Moreover, p is tangent to q^* at that point. Since both p and q^* satisfy

$$\tfrac{1}{2}|\dot{q}(t)|^2 + V(q(t)) \equiv 0, \quad t \in \mathbb{R}, \tag{2.20}$$

it follows that $q^*(t) \equiv p(t)$ or $q^*(t) \equiv p(-t)$. But then $q^* \notin G_k$. Hence q^* lies in C.

Finally, to show that the minimizing sequence (q_m) can be assumed to be in $\overline{\mathcal{C}}$, suppose this is not the case. Then for some m and p as above, there are numbers $\sigma_1 < s_1$ and $\sigma < s$ such that $q_m(\sigma_1) = p(\sigma)$, $q_m(s_1) = p(s)$ and $q_m(t) \notin \overline{\mathcal{C}}$ for $t \in (\sigma_1, s_1)$. But then by Proposition 2.1(d),

$$\int_{\sigma_1}^{s_1} \mathcal{L}(q_m)\,dt > \int_{\sigma}^{s} \mathcal{L}(p)\,dt. \tag{2.21}$$

Therefore replacing $q_m|_{\sigma_1}^{s_1}$ by $p|_{\sigma}^{s}$ and doing the same for any other such interval where q_m lies outside $\overline{\mathcal{C}}$ yields a new function $\widehat{q}_m \in G_k$ with $I(\widehat{q}_m) < I(q_m)$.

This section concludes with the construction of an orbit joining 0 and p_+. Let

$$\Lambda = \{q \in W^{1,2}_{\mathrm{loc}} \mid q(0) \in p_+(\mathbb{R}) \text{ and } q(\infty) = 0\}.$$

Consider the problem of minimizing

$$\int_0^\infty \mathcal{L}(q)\,dt \tag{2.22}$$

for $q \in \Lambda$. A straightforward minimization argument as in [17] produces $z_0^+ \in \Lambda$, a solution of (HS), which minimizes the functional in (2.22). Moreover, as in the proof of Theorem 2.15, z_0^+ intersects p_+ only at $z_0^+(0)$ and intersects $\{q^* + mk \mid m \in \mathbb{Z}\}$ only at $z_0^+(\infty) = 0$. It is possible that z_0^+ is not unique. However, if $z^+ \in \Lambda$ with

$$\int_0^\infty \mathcal{L}(z^+)\,dt = \int_0^\infty \mathcal{L}(z_0^+)\,dt,$$

then as above, z^+ and z_0^+ intersect only at their endpoints.

For $j \in \mathbb{Z}$, set $z_j^+ = z_0^+ + jk$. Then the curves z_{j-1}^+, z_j^+, p_+, and $q^* + (j-1)k$ bound a "rectangle" $\mathcal{R}_{j-1}$. Set $\mathcal{R} = \bigcup_{i \in \mathbb{Z}} \mathcal{R}_i$.

REMARK 2.23. By the above arguments, if z_0^+ is not a unique minimizer in Λ, any other minimizer z^+ cannot cross z_j^+ and if z^+ and z_j^+ touch for $j \neq 0$, they must be identical as in Corollary 2.13.

3. Formulation of a variational problem

In this section a class of curves, Γ, will be introduced. These curves start at $t = -\infty$ at 0, lie in $\mathcal{C}$, and are asymptotic to p_+ as $t \to \infty$. A functional, J, will be defined on Γ, and the first heteroclinic solution of (HS) that we seek will be obtained by minimizing J over Γ.

To begin, let

$$\Gamma = \{q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2) \mid (\Gamma_1)\text{–}(\Gamma_5) \text{ hold}\}$$

where

(Γ_1) $q(-\infty) = 0$,
(Γ_2) q lies in $\mathcal{R}$, and
(Γ_3) q intersects z_i^+ for all $i \in \mathbb{N}$.

To state (Γ_4)–(Γ_5), some further remarks are necessary. For $i \in \mathbb{N}$ and q satisfying (Γ_1)–(Γ_3), let $t_i(q)$ denote the smallest value of t such that q intersects z_i^+. When the choice of q is evident, we just write t_i. Let $s_0(q) = \infty$ and for $i \in \mathbb{N}$, define $s_i(q) \in [0, \infty]$ via $z_i^+(s_i(q)) = q(t_i(q))$. The final requirements for $q \in \Gamma$ are

(Γ_4) $q(t) \in \mathcal{R}_0$ for $t \in [-\infty, t_1]$ and for $i \in \mathbb{N}$, $q(t) \in \mathcal{R}_i$ for $t \in [t_i, t_{i+1}]$,
(Γ_5) $s_{i+1}(q) \le s_i(q)$ for all $i \in \mathbb{N}$.

At this point fix the time scale for p_+ by requiring that $p_+(0) = z_0^+(0)$. Therefore $p_+(iT) = z_i^+(0)$ for all $i \in \mathbb{N}$ and (Γ_5) is equivalent to

$$|q(t_{i+1}) - p_+((i+1)T)| \le |q(t_i) - p_+(iT)|.$$

For $q \in \Gamma$, define

$$a_1(q) = \int_{-\infty}^{t_1(q)} \mathcal{L}(q)\, dt - c_k^* \tag{3.1}$$

and for $i \ge 2$,

$$a_2(q) = \int_{t_{i-1}(q)}^{t_i(q)} \mathcal{L}(q)\, dt - c_k^*. \tag{3.2}$$

Now for $q \in \Gamma$, set

$$J(q) = \sum_{i=1}^{\infty} a_i(q) \tag{3.3}$$

and define

$$c = \inf_{\Gamma} J. \tag{3.4}$$

Theorem 3.5. *If V satisfies* (V_1)–(V_2), *$k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{0\}$ with k_1, k_2 relatively prime, and* (1.13) *holds, then there is a $q \in \Gamma$ such that $I(Q) = c$. Moreover, Q is a solution of* (HS) *with $Q(-\infty) = 0$ and $Q(t) - p_+(t) \to 0$ as $t \to \infty$.*

The proof of Theorem 3.5 will be accomplished in §§3–5. Some properties of J will be established next. Let $K = \int_0^\infty \mathcal{L}(z_0^+)\, dt$.

PROPOSITION 3.6.

(a) *For each* $q \in \Gamma$, $J(q) \geq -K$.

(b) $-K \leq c \leq K$.

(c) *If* $q \in \Gamma$ *and* $J(q) \leq M$, *then*

$$\sum_{i=1}^{\infty} |a_i(q)| \leq M + 2K.$$

PROOF. (a) For $i \geq 2$, let ψ_i denote the curve obtained by gluing together $z_{i-1}^+|_{s_i}^{s_{i-1}}$ and $q|_{t_{i-1}}^{t_i}$. Then ψ_i extends to $\mathbb{R}$ with $T_i \equiv s_{i-1} - s_i + t_i - t_{i-1}$ via (1.1) and belongs to F_k. Therefore by (1.5),

$$I^*(\psi_i) \geq c_k^* \tag{3.7}$$

or

$$a_i(q) \geq -\int_{s_i}^{s_{i-1}} \mathcal{L}(z_0^+)\,dt. \tag{3.8}$$

The same inequality holds for $i = 1$ via an approximation argument. Adding these inequalities then yields (a).

(b) The lower bound is immediate from (a) and (3.4). For the upper bound, set $q(t) = z_0^+(-t)$, $-\infty \leq t \leq 0$ and $q(t) = p_+(t)$, $t \geq 0$. Then $q \in \Gamma$ and $J(q) = K$.

(c) Set $N^-(q) = \{l \in \mathbb{N} \mid a_l(q) < 0\}$ and

$$J^-(q) = \sum_{i \in N^-(q)} a_i(q).$$

Then by (3.8),

$$-J^-(q) \leq K. \tag{3.9}$$

Hence

$$J^+(q) \equiv J(q) - J^-(q) \leq M + K \tag{3.10}$$

and

$$\sum_{i=1}^{\infty} |a_i(q)| \leq M + 2K. \tag{3.11}$$

The next result establishes the asymptotic behavior of $q \in \Gamma$ when $J(q) < \infty$. Recall that $p_+ \in F_{k,T}$.

Proposition 3.12. *If* $q \in \Gamma$ *and* $J(q) \le M$, *then as* $i \to \infty$,

(a) $t_{i+1}(q) - t_i(q) \to T$, *and*
(b) $\|q - p_+\|_{L^\infty[t_i,t_{i+1}]} \to 0$.

Proof. Since $J(q) \le M$, $a_i(q) \to 0$ as $i \to \infty$. Therefore

$$\int_{t_{i-1}}^{t_i} \mathcal{L}(q)\, dt \to c_k^* \tag{3.13}$$

as $i \to \infty$. Consider the functions ψ_i defined in the proof of Proposition 3.6. Translating time and subtracting $(i-1)k$ from ψ_i, it can be assumed that $\psi_i(0) = z_0^+(s_i)$ and $\psi_i(T_i) = z_1^+(s_i)$. For large i,

$$\begin{aligned} 0 < |k| = |z_1^+(s_i) - z_0^+(s_i)| &= \left| \int_0^{T_i} \psi_i \, dt \right| \\ &\le T_i^{1/2} \left(\int_0^{T_i} |\psi_i|^2 \, dt \right)^{1/2} \le T_i^{1/2} (c_k^* + 1)^{1/2} \end{aligned} \tag{3.14}$$

via (3.13). By (Γ_5), s_i decreases monotonically to $s^* \ge 0$. Hence (3.14) shows that $t_i - t_{i-1}$ is bounded from below by a positive constant and in particular cannot approach 0.

The L^2 bounds for $\dot{\psi}_i$ given by (3.13) and L^∞ bound for ψ_i imply ψ_i is bounded in $W^{1,2}_{\mathrm{loc}}$ and therefore converges along a subsequence weakly in $W^{1,2}_{\mathrm{loc}}$ and strongly in L^∞_{loc} as $i \to \infty$ to $\psi \in W^{1,2}_{\mathrm{loc}}$. If $(t_i - t_{i-1})$ is bounded, there is a $\overline{T} > 0$ such that $t_i - t_{i-1} \to \overline{T}$ along a subsequence and

$$\int_0^{\overline{T}} \mathcal{L}(\psi)\, dt \le \varliminf_{i\to\infty} \int_{t_{i-1}}^{t_i} \mathcal{L}(\psi_i)\, dt \le c_k^*. \tag{3.15}$$

Moreover, $\psi(0) = z_0^+(s^*)$ and $\psi(\overline{T}\,) = z_1^+(s^*)$. Therefore $\psi \in F_k$ so

$$\int_0^{\overline{T}} \mathcal{L}(\psi)\, dt \ge c_k^*. \tag{3.16}$$

Hence equality holds in (3.16) so $\psi \in P_k$. Moreover, the construction of $\mathcal{R}$ then implies $\overline{T} = T$, $s^* = 0$, and $\psi = p_+$. Thus ψ_i converges to p_+ in $L^\infty[0,T]$ along the subsequence. The uniqueness of the limit implies the entire sequence converges to p_+.

To complete the proof, it remains to show that $(t_i - t_{i-1})$ is bounded. Thus suppose that $t_i - t_{i-1} \to \infty$ as $i \to \infty$ along a subsequence. Let $\varepsilon > 0$. For i sufficiently large,

$$\int_0^{T_i} \mathcal{L}(\psi_i)\, dt \le c_k^* + \varepsilon. \tag{3.17}$$

Let $B_\delta(x)$ denote an open ball about x of radius δ in $\mathbb{R}^2$. For any $\delta > 0$ and all large i, ψ_i intersects $B_\delta(\mathbb{Z}^2) \cap \mathcal{R}_0$, for otherwise by (V_2), there is a $\gamma = \gamma(\delta) > 0$ such that $-V(\psi_i(t)) \geq \gamma$ for all $t \in [0, T_i]$. Therefore

$$\int_0^{T_i} \mathcal{L}(\psi_i)\, dt \geq \gamma(t_i - t_{i-1}) \to \infty \tag{3.18}$$

as $i \to \infty$. Thus for some $y_i \in [0, T_i]$, $\psi_i(y_i) \in B_\delta(\{0\} \cup \{k\})$. The same argument applies in either event so suppose $\psi_i(y_i) \in B_\delta(0)$. Append a straight line segment, S_i, run back and forth from $\psi_i(y_i)$ to 0 to $\psi_i(y_i)$ and call the resulting curve χ_i. It can be assumed that $|\chi_i'(t)| = |\psi_i(y_i)|$ for $t \in S_i$ and that χ_i spends time $2L_i$ traversing S_i back and forth. Therefore

$$\left| \int_0^{T_i} \mathcal{L}(\psi_i)\, dt - \int_0^{T_i + 2L_i} \mathcal{L}(\chi_i)\, dt \right| = \left| \int_{S_i} \mathcal{L}(\chi_i)\, dt \right| = o(1) \tag{3.19}$$

as $\delta \to 0$ uniformly for large i. Define

$$\varphi_i(t) = \begin{cases} \chi_i(t + y_i + L_i), & 0 \leq t \leq T_i - y_i + L_i, \\ k + \chi_i(t - T_i + y_i - L_i), & T_i - y_i + L_i \leq t \leq T_i + 2L_i, \end{cases} \tag{3.20}$$

i.e. φ_i follows χ_i from 0 to $\psi_i(y_i)$ along S_i, then follows ψ_i from $\psi_i(y_i)$ to $z_1(s^*)$, and then follows $k + \psi_i(y)$ from $z_1^+(s^*)$ to k.

Further, extend φ_i to $\mathbb{R}$ via $\varphi_i(t) = 0$ for $t \leq 0$ and $\varphi_i(t) = k$ for $t \geq T_i + 2L_i$. Then $\varphi_i \in G_k$ and by (3.19),

$$\int_0^{T_i + 2L_i} \mathcal{L}(\chi_i)\, dt = \int_{\mathbb{R}} \mathcal{L}(\varphi_i)\, dt = \int_0^{T_i} \mathcal{L}(\psi_i)\, dt + o(1) \quad \text{as } \delta \to 0. \tag{3.21}$$

Since $\varphi_i \in G_k$, by (3.17) and (3.21),

$$c_k \leq \int_{\mathbb{R}} \mathcal{L}(\varphi_i)\, dt \leq c_k^* + \varepsilon + o(1) \quad \text{as } \delta \to 0. \tag{3.22}$$

Choose e.g. $\varepsilon = \frac{1}{3}(c_k - c_k^*)$ and δ so small that $o(1) \leq \varepsilon$. Then (3.22) shows

$$c_k < c_k^*, \tag{3.23}$$

contrary to (1.13). The proof is complete.

Remark 3.24. By Proposition 3.12, once it is shown that there is $Q \in \Gamma$ such that $J(Q) = c$, it follows that Q has the desired asymptotic behavior as $t \to \infty$.

4. The minimization argument

The goal of this section is to establish the existence of $Q \in \Gamma$ such that $J(Q) = c$. Although the ideas are elementary, the details are lengthy and technical. Observe first that if $q \in \Gamma$, then $\tau_\theta q \in \Gamma$ for all $\theta \in \mathbb{R}$ and $t_i(\tau_\theta q) = t_i(q) + \theta$. Moreover,

$$J(\tau_\theta(q)) = J(q). \tag{4.1}$$

Now let (q_m) be a minimizing sequence for (3.4). A normalization can be made for (q_m). Let $\delta > 0$ be so small that $B_\delta(0) \cap \mathbb{Z}^2 = \{0\}$ and $B_\delta(0) \cap z_1^+ = \emptyset$. Then there is a smallest value of t, $t_0 = t_0(q)$, such that $q(t_0) \in \partial B_\delta(0)$ and $q(t) \in B_\delta(0)$ for $t \in [-\infty, t_0)$. By (4.1), the normalization that $t_0(q_m) = 0$ for all $m \in \mathbb{N}$ can be made.

The remainder of this section is divided into three main steps, namely proving that

(A) q_m converges, along a subsequence, weakly in $W^{1,2}_{\mathrm{loc}}$ and strongly in L^∞_{loc}, to a function $Q \in W^{1,2}_{\mathrm{loc}}$,
(B) $Q \in \Gamma$, and
(C) $J(Q) = c$.

The proof of (A) requires two preliminaries:

LEMMA 4.2. *There is a $\beta = \beta(M) > 0$ such that if $q \in \Gamma$ and $J(q) \le M$, then for all $i \in \mathbb{N}$, $t_i(q) - t_{i-1}(q) \ge \beta$.*

PROOF. By (3.11), for $i \ge 2$,

$$\int_{t_{i-1}}^{t_i} \mathcal{L}(q)\,dt \le M + 2K + c_k^* \equiv M_1 \tag{4.3}$$

and

$$\int_{t_{i-1}}^{t_i} |\dot q|^2\,dt \le 2M_1 \equiv M_2^2. \tag{4.4}$$

The lower bound then follows as in (3.14) for any $\beta \le (|k|/M_2)^2$. A similar argument gives the lower bound for $i = 1$.

LEMMA 4.5. *(q_m) is bounded in $W^{1,2}_{\mathrm{loc}}$.*

PROOF. Since $q_m(0)$ lies on $\partial B_\delta(0) \cap \mathcal{R}_0$, to prove the result it suffices to find a bound for $\dot q_m$ in $L^2[-l, l]$ for each $l > 0$. Choose i_m so that

$$t_{i_m}(q_m) \le l < t_{i_m+1}(q_m). \tag{4.6}$$

Therefore

$$(4.7)\qquad \int_{-l}^{l} |\dot q_m|^2\,dt \le 2\int_{-\infty}^{t_{i_m+1}} \mathcal{L}(q_m)\,dt \le 2\Big(\sum_{i=1}^{i_m+1} a_i(q_m) + (i_m+1)c_k^*\Big) \le 2(M+2K+(i_m+1)c_k^*)$$

via (3.11). By Lemma 4.2,

$$(4.8)\qquad t_{i_m}(q_m) = \sum_{i=1}^{i_m}(t_i(q_m) - t_{i-1}(q_m)) \ge i_m\beta$$

so (4.6)–(4.8) yield the bound for $\|\dot q_m\|_{L^2[-l,l]}$.

Now by Lemma 4.5, a subsequence of (q_m) which can be taken to be the entire sequence converges weakly in $W^{1,2}_{\rm loc}$ and strongly in $L^\infty_{\rm loc}$ to $Q \in W^{1,2}_{\rm loc}$ with $Q(0) \in \partial B_\delta(0) \cap \mathcal{R}_0$ and (A) has been established.

To prove (B), (Γ_1)–(Γ_5) must be verified for Q. By (3.11),

$$(4.9)\qquad \int_{-\infty}^{0} \mathcal{L}(q_m)\,dt \le M_1.$$

The convergence of q_m to Q obtained in (A) and standard weak lower semicontinuity arguments (see e.g. [12]) then imply

$$(4.10)\qquad \int_{-\infty}^{0} \mathcal{L}(Q)\,dt \le M_1.$$

The form of $\mathcal{L}$ and (4.10) show Q has a limit as $t \to -\infty$ and $Q(-\infty) \in V^{-1}(0) = \mathbb{Z}^2$. But for $t \le 0$, $Q(t) \in B_\delta(0)$ and $B_\delta(0) \cap \mathbb{Z}^2 = \{0\}$. Hence $Q(-\infty) = 0$ and Q satisfies (Γ_1). The $L^\infty_{\rm loc}$ convergence of q_m to Q further implies that (Γ_2) holds for Q. Properties (Γ_3)–(Γ_5) for Q will follow from the next result.

Proposition 4.11. *For each $i \in \mathbb{N}$, there is an $A_i > 0$ such that $t_i(q_m) \le A_i$ for all $m \in \mathbb{N}$.*

Assuming Proposition 4.11 for the moment, the uniform m-independent bounds for $(t_i(q_m))$ and the $L^\infty_{\rm loc}$ convergence of (q_m) imply (Γ_3) for Q. It can be assumed that $t_i(q_m) \to \overline{t}_i = \overline{t}_i(Q)$ as $m \to \infty$ for all $i \in \mathbb{N}$. Hence by (Γ_3) for q_m, $Q(t) \in \mathcal{R}_i$ for $t \in (\overline{t}_{i-1}, \overline{t}_i]$ (and $Q(t) \in \mathcal{R}_0$ for $t \in [-\infty, \overline{t}_1)$). Moreover $Q(t) \in \mathcal{R}_i$ for $t = t_{i-1}(Q)$, $t_i(Q)$. Therefore $Q(t) \in \mathcal{R}_{i-1} \cap \mathcal{R}_i = z_i^+$ for $t \in [t_i(Q), \overline{t}_i(\overline{Q})]$. Consequently, Q satisfies (Γ_4). That (Γ_5) holds for Q will be established in the course of the proof of Proposition 4.11.

Proof of Proposition 4.11. Arguing indirectly, suppose there is a smallest $j \in \mathbb{N}$ such that $t_j(q_m) \to \infty$ along a subsequence.

Case 1: $j = 1$. By (1.11),

$$\int_{-\infty}^{t_1(q_m)} \mathcal{L}(q_m)\,dt \le M + 2K + c_k^* \equiv M_3.$$

Hence for any $r > 0$,

$$\int_{-\infty}^{r} \mathcal{L}(q_m)\,dt \le M_3 \tag{4.12}$$

and (4.12) implies

$$\int_{\mathbb{R}} \mathcal{L}(Q)\,dt \le M_3. \tag{4.13}$$

Since $q_m(t) \in \mathcal{R}_0$ for all $t \in (-\infty, t_1(q_m))$, $Q(t) \in \mathcal{R}_0$ for all $t \in \mathbb{R}$ so either (a) $Q(\infty) = 0$, or (b) $Q(\infty) = k$. It will be shown that each of (a) and (b) leads to the conclusion that (q_m) is not a minimizing sequence.

(a) Recall that $q_m(0)$ and therefore $Q(0)$ lie on $\partial B_\delta(0) \cap \mathcal{R}_0$. Let $\varrho < \delta/4$. Then there is an $S = S(\varrho) \in \mathbb{R}$ such that $Q(t) \in B_\varrho(0)$ for $t \ge S$. Therefore for m large, $q_m(S) \in B_{2\varrho}(0)$. As in the construction of z_0^+, there is a curve $q_m^*(t)$ such that $q_m^*(-\infty) = 0$, $q_m^*(S) = q_m(S)$, and $q_m^* \in \mathcal{R}_0$ for $t \in [-\infty, S]$. Extend q_m^* to $\mathbb{R}$ via $q_m^*(t) = q_m(t)$ for $t \ge S$. Then $q_m^* \in \Gamma$ and

$$J(q_m) - J(q_m^*) = \int_{-\infty}^{S} \mathcal{L}(q_m)\,dt - \int_{-\infty}^{S} \mathcal{L}(q_m^*)\,dt. \tag{4.14}$$

As $\varrho \to 0$, the second integral in (4.14) approaches 0. Since for $t \in [-\infty, S]$, q_m emanates from 0, intersects $\partial B_\delta(0)$, and returns to $\partial B_\varrho(0)$, there is a $\gamma = \gamma(\delta) > 0$ such that

$$\int_{-\infty}^{S} \mathcal{L}(q_m)\,dt \ge \gamma. \tag{4.15}$$

But then (4.14)–(4.15) show (q_m) is not a minimizing sequence for (3.4) so (a) is impossible.

(b) Since $Q(\infty) = k$, as in (a), for all $\varrho > 0$, there is an $S = S(\varrho) > 0$ such that $Q(t) \in B_\varrho(k)$ for $t \ge S$. Therefore $q_m(S) \in B_{2\varrho}(k)$ for all m large. As in (a), there is a $q_m^*(t)$ such that $q_m^*(t) = q_m(t)$ for $t \le S$, $q_m^*(\infty) = k$, $q_m^* \in \mathcal{R}_0$, and

$$\int_{S}^{\infty} \mathcal{L}(q_m^*)\,dt \to 0$$

as $\varrho \to 0$. Note that $q_m^* \in G_k$. Now

$$
\begin{aligned}
(4.16)\quad J(q_m^*) &= \int_{-\infty}^{S} \mathcal{L}(q_m)\,dt + \int_S^{t_1(q_m)} \mathcal{L}(q_m)\,dt - c_k^* + \sum_{i=2}^{\infty} a_i(q_m) \\
&= \int_{\mathbb{R}} \mathcal{L}(q_m^*)\,dt - \int_S^{\infty} \mathcal{L}(q_m^*)\,dt + \int_S^{t_1(q_m)} \mathcal{L}(q_m)\,dt \\
&\quad - c_k^* + \sum_{i=2}^{\infty} a_i(q_m) \\
&\ge c_k - c_k^* - \int_S^{\infty} \mathcal{L}(q_m^*)\,dt + \int_S^{t_1(q_m)} \mathcal{L}(q_m)\,dt + \sum_{i=2}^{\infty} a_i(q_m).
\end{aligned}
$$

Observe that

$$
(4.17)\qquad \int_{s_1(q_m)}^{\infty} \mathcal{L}(z_1^*)\,dt \le \int_S^{\infty} \mathcal{L}(q_m^*)\,dt + \int_S^{t_1(q_m)} \mathcal{L}(q_m)\,dt.
$$

Combining (4.16)–(4.17) gives

$$
(4.18)\qquad J(q_m^*) \ge c_k - c_k^* - 2\int_S^{\infty} \mathcal{L}(q_m^*)\,dt + \int_{s_1(q_m)}^{\infty} \mathcal{L}(z_1^*)\,dt + \sum_{i=2}^{\infty} a_i(q_m).
$$

Define

$$
(4.19)\qquad \widehat{q}_m(t) = \begin{cases} z_0^+(-t), & -\infty < t < -s_1(q_m), \\ q_m(t + s_1(q_m) + t_1(q_m)) - k, & t \ge -s_1(q_m). \end{cases}
$$

Then $\widehat{q}_m \in \Gamma$ and

$$
(4.20)\qquad J(\widehat{q}_m) = \int_{s_1(q_m)}^{\infty} \mathcal{L}(z_1^+)\,dt + \sum_{i=2}^{\infty} a_i(q_m).
$$

Therefore by (4.18) and (4.20),

$$
(4.21)\qquad J(q_m) \ge J(\widehat{q}_m) + c_k - c_k^* + o(1)
$$

as $\varrho \to 0$. Consequently, by (1.13), (q_m) is not a minimizing sequence for (3.4) and (b) is not possible.

REMARK 4.22. By Case 1, $t_1(Q) \le \bar{t}_1(Q) < \infty$. Moreover, $s_1(Q) < s_0(Q) = \infty$. Indeed, if $s_1(Q) = \infty$, then $Q(t_1(Q)) = k$ and a slightly simpler version of the argument of (b) shows (q_m) is not a minimizing sequence.

CASE 2: $j > 1$. Then $t_i(Q) \le \bar{t}_i(Q) < \infty$, $1 \le i \le j-1$, and $Q(t) \in \mathcal{R}_{j-1}$ for $t > t_{j-1}(Q)$ so (a) $Q(\infty) = (j-1)k$ or (b) $Q(\infty) = jk$. It will again be shown that (a) and (b) are impossible as for Case 1. To do so we require

PROPOSITION 4.23. $s_{i-1}(Q) \le s_i(Q)$, $1 \le i \le j-1$.

PROOF. Define $\overline{s}_i = \overline{s}_i(Q)$ via

$$Q(\overline{t}_i) = z_i^+(\overline{s}_i). \tag{4.24}$$

Since

$$Q(\overline{t}_i) = \lim_{m\to\infty} q_m(t_i(q_m)) = \lim_{m\to\infty} z_i^+(s_i(q_m)), \tag{4.25}$$

it follows that

$$\overline{s}_i = \lim_{m\to\infty} s_i(q_m). \tag{4.26}$$

Hence by (Γ_5) for (q_m),

$$\overline{s}_i \le \overline{s}_{i-1}. \tag{4.27}$$

Therefore if $\overline{t}_l(Q) = t_l(Q)$, $l = i-1, i$, then

$$s_i \le s_{i-1}. \tag{4.28}$$

Thus suppose that $\overline{t}_l(Q) \ne t_l(Q)$ for at least one of $i-1, i$. The worst case in which inequality holds for both $i-1$ and i will be treated. The remaining cases are handled similarly. Let $L_n = Q([t_n(Q), \overline{t}_n(Q)])$, $n = i-1, i$. By earlier remarks, L_{i-1} and $L_i - k$ lie on z_{i-1}^+. If the intersection of these curves is empty or a single point, since $Q(\overline{t}_{i-1}) \in L_i$ and $Q(\overline{t}_i) \in L_{i+1}$, (4.27) implies L_{i-1} lies to the left of $L_i - k$ on z_{i-1}^+. Hence $Q(t_{i-1}(Q)) = z_{i-1}(s_{i-1}(Q))$ is to the left of or equals $z_i^+(s_i(Q)) - k = Q(t_i(Q)) - k$, i.e. $s_i(Q) \le s_{i-1}(Q)$. Thus suppose the intersection of L_{i-1} and $L_i - k$ is a nontrivial curve. Then there are $\sigma_{i-1} \in (t_{i-1}(Q), \overline{t}_{i-1}(Q))$ and $\sigma_i \in (t_i(Q), \overline{t}_i(Q))$ such that $Q(\sigma_i) = Q(\sigma_{i-1}) + k$ and $Q(\sigma_{i-1}) \notin p_+$.

Proceeding formally for the moment, suppose $Q \in \Gamma_i$ and minimizes J. Define

$$\widehat{Q}(t) = \begin{cases} Q(t), & t \le \sigma_{i-1}, \\ Q(t + \sigma_i - \sigma_{i-1}) - k, & t \ge \sigma_{i-1}. \end{cases} \tag{4.29}$$

Then $\widehat{Q} \in \Gamma$ and since $Q|_{\sigma_{i-1}}^{\sigma_i} \in F_k$,

$$J(Q) - J(\widehat{Q}) = \int_{\sigma_{i-1}}^{\sigma_i} \mathcal{L}(Q)\, dt - c_k^* > 0 \tag{4.30}$$

unless Q coincides with a translate of p_+ on $[\sigma_{i-1}, \sigma_i]$. But $Q(\sigma_{i-1}) \notin p_+$ excludes the latter possibility. Hence

$$J(Q) - J(\widehat{Q}) \ge 2\gamma > 0. \tag{4.31}$$

so Q would not be a minimizer of J on Γ. Now replacing q_m by $\widehat{q}_m$ in the spirit of (4.29) but with an extra small modification and using the convergence of q_m to p_+ shows

$$J(\widehat{q}_m) \le J(q_m) - \gamma, \tag{4.32}$$

again contradicting that (q_m) is a minimizing sequence.

REMARK 4.33. It will be seen in §5 that $t_j(Q) = \overline{t}_j(Q)$.

COMPLETION OF PROOF OF PROPOSITION 4.11.

CASE 2(a): $Q(\infty) = (j-1)k$. As earlier, there is an $S = S(\varrho)$ such that $Q(t) \in B_\varrho((j-1)k)$ for $t \ge S$. Define $\widehat{s}$ by $z^+_{j-1}(\widehat{s}) \in \partial B_\varrho((j-1)k) \cap z^+_{j-1}$ and let $\varphi_m(t)$, $t \in [0,1]$, be a curve in $\mathcal{R}_{j-1}$ joining $q_m(S)$ to $z^+_{j-1}(\widehat{s})$ with

$$\int_0^1 \mathcal{L}(\varphi_m)\,dt = o(1) \quad \text{as } \varrho \to 0.$$

Note that

$$\int_{s_{j-1}(q_m)}^{\overline{s}} \mathcal{L}(z^+_{j-1})\,dt \le \int_0^1 \mathcal{L}(\varphi_m)\,dt + \int_{t_{j-1}(q_m)}^{S} \mathcal{L}(q_m)\,dt. \tag{4.34}$$

Define

$$q^*_m(t) = \begin{cases} q_m(t), & t \le t_{j-1}(q_m), \\ z^+_{j-1}(t - t_{j-1}(q_m) + s_{j-1}(q_m)), & \\ \qquad\qquad t_{j-1}(q_m) \le t \le t_{j-1}(q_m) + \overline{s} - s_{j-1}(q_m) \equiv S_1, & \\ \varphi_m(t - S_1), & S_1 \le t \le 1 + S_1, \\ q_m(t + S - (1 + S_1)), & t \ge 1 + S_1. \end{cases} \tag{4.35}$$

Then $q^*_m \in \Gamma$ and

$$\begin{aligned} J(q_m) - J(q^*_m) &= \int_{t_{j-1}(q_m)}^{S} \mathcal{L}(q_m)\,dt \\ &\quad - \int_{s_{j-1}(q_m)}^{S} \mathcal{L}(q^*_{j-1})\,dt - \int_0^1 \mathcal{L}(\varphi_m)\,dt \\ &\ge -2\int_0^1 \mathcal{L}(\varphi_m)\,dt \end{aligned} \tag{4.36}$$

via (4.34). Let $\widehat{q}_m \in \Gamma$ be the function obtained from q^*_m by setting $\widehat{q}_m(t) = q^*_m(t)$, $t \le t_{j-2}(q_m)$, excising $q_m|_{t_{j-2}}^{t_{j-1}}$ and $z^+_{j-1}|_{s_{j-1}}^{s_{j-2}}$ from q^*_m, and shifting the remainder of q^*_m by $-k$. Then

$$J(q^*_m) - J(\widehat{q}_m) = \int_{t_{j-2}(q_m)}^{t_{j-1}(q_m)} \mathcal{L}(q_m)\,dt + \int_{s_{j-1}(q_m)}^{s_{j-2}(q_m)} \mathcal{L}(z^+_{j-1})\,dt - c^*_k. \tag{4.37}$$

Since the excised portion of q^*_m lies in F_k, the right hand side of (4.37) exceeds $2\gamma > 0$ independently of m unless the integrals on the left in (4.37) converge to

$\int_{(j-2)T}^{(j-1)T} \mathcal{L}(p_+)\,dt$ and in particular $q_m|_{t_{j-2}}^{t_{j-1}} \to p_+|_{(j-2)T}^{(j-1)T}$ and $s_{j-1}(q_m) \to 0$. But then $s_j(q_m) \to 0$ and the contribution to $J(q_m)$ from $a_j(q_m)$ will be bounded from below by some $\gamma > 0$ independently of m so a modification of q_m in $[t_{j-1}(q_m), t_j(q_m)]$ shows (q_m) is not a minimizing sequence. Combining (4.36)–(4.37) shows (q_m) is not a minimizing sequence.

Case 2(b): $Q(\infty) = jk$. This follows the same lines as (a) so we will be brief. Let φ_m join $q_m(S)$ to $z_j^+(\widehat{s})$ with $\widehat{s}$ as in (a). Let $q_m^* = q_m$ up to S, then follow φ_m to $z_j^+(\widehat{s})$, follow z_j^+ to $z_j^+(s_j(q_m))$ and then follow q_m. Obtain $\widehat{q}_m$ from q_m^* by excising $q_m|_{t_{j-1}}^{S}$, φ_m, and $z_j^+|_{s_{j-1}}^{\widehat{s}}$ and shifting the remaining portion of $\widehat{q}_m$ by $-k$ so $\widehat{q}_m \in \Gamma$. Again $J(q_m) - J(\widehat{q}_m) \geq \gamma > 0$.

With the completion of Proposition 4.11, it has been verified that $Q \in \Gamma$.

Lemma 4.38. $\min(s_j(Q), \overline{s}_j(Q)) \geq \max(s_{j+1}(Q), \overline{s}_{j+1}(Q))$.

Proof. If the inequality fails, $\widehat{Q}$ can be defined as in (4.29). Since $Q \in \Gamma$, we have $\widehat{Q} \in \Gamma$ and the reasoning of Proposition 4.23 again shows (q_m) is not a minimizing sequence.

To complete this section, it will be shown that

(C) $$J(Q) = c.$$

The proof of (C) will be carried out in 3 steps. For each $i \in \mathbb{N}$, set

$$J_i(q) = \sum_{l=1}^{i} a_l(q).$$

Lemma 4.39. *For each* $i \in \mathbb{N}$,

$$J_i(Q) \leq \varliminf_{m\to\infty} J_i(q_m).$$

Proof. Observe that

(4.40) $$J_i(q_m) = \int_{-\infty}^{t_i(q_m)} \mathcal{L}(q_m)\,dt - ic_k^*.$$

For any $\varepsilon > 0$,

(4.41) $$\lim_{m\to\infty} t_i(q_m) = \overline{t}_i \geq t_i(Q) > t_i(Q) - \varepsilon.$$

Hence

(4.42) $$\begin{aligned} J_i(Q) &= \lim_{\varepsilon\to 0} \int_{-\infty}^{t_i(Q)-\varepsilon} \mathcal{L}(Q)\,dt - ic_k^* \\ &\leq \lim_{\varepsilon\to 0} \varliminf_{m\to\infty} \int_{-\infty}^{t_i(Q)-\varepsilon} \mathcal{L}(q_m)\,dt - ic_k^* \\ &\leq \lim_{\varepsilon\to 0} \varliminf_{m\to\infty} \int_{-\infty}^{t_i(q_m)} \mathcal{L}(q_m)\,dt - ic_k^* = \varliminf_{m\to\infty} J_i(q_m). \end{aligned}$$

LEMMA 4.43. $-K \le J(Q) \le 5K$.

PROOF. The lower bound follows from Proposition 3.6(a). By Lemma 4.39,

$$J_i(Q) \le \varliminf_{m\to\infty} \sum_{l=1}^{i} a_l(q_m) \le \varliminf_{m\to\infty} \sum_{l=1}^{\infty} |a_l(q_m)|. \tag{4.44}$$

By Proposition 3.6(b), it can be assumed that $J(q_m) \le K$. Hence by (4.44) and Proposition 3.6(c),

$$J_i(Q) \le 3K \tag{4.45}$$

independently of i. Let

$$J_i^-(q) = \sum_{l\le i,\, l\in N^-(q)} a_l(q), \quad J_i^+(q) = \sum_{l\le i,\, l\in N^+(q)} a_l(q)$$

where $N^+(q) = \mathbb{N} \setminus N^-(q)$. Then as in Proposition 3.6(a),

$$-J_i^-(Q) \le -J^-(Q) \le K \tag{4.46}$$

independently of i. Therefore by (4.45), (4.46),

$$J_i^+(Q) = J_i(Q) - J_i^-(Q) \le 4K \tag{4.47}$$

and

$$\sum_{l=1}^{i} |a_l(Q)| \le 5K \tag{4.48}$$

independently of i, from which the lemma follows.

Finally, we have

PROPOSITION 4.49. $J(Q) = c$.

PROOF. Let $\varepsilon > 0$. Since $Q \in \Gamma$, $s_i(Q) \to 0$ as $i \to \infty$ via Proposition 3.12. Choose $l = l(\varepsilon)$ so that

$$s_i(Q) < \varepsilon \quad \text{for } i \ge l. \tag{4.50}$$

By Lemma 4.43, the series $J(Q)$ converges. Thus it can be further assumed that for $i \ge l$,

$$J(Q) \le J_i(Q) + \varepsilon. \tag{4.51}$$

By Lemma 4.39, there is an $n = n(\varepsilon)$ such that for all $m \ge n$,

$$J_l(Q) \le J_l(q_m) + \varepsilon. \tag{4.52}$$

Now

$$J_l(q_m) = J(q_m) - \sum_{l<i} a_i(q_m) \le J(q_m) - \sum_{l<i\in N^-(q_m)} a_i(q_m). \tag{4.53}$$

Since (q_m) is a minimizing sequence for (3.5), there is an $\overline{n} = \overline{n}(\varepsilon)$ such that for all $m \geq \overline{n}$,

$$J(q_m) \leq c + \varepsilon. \tag{4.54}$$

Hence by (4.51)–(4.54) and (3.8) for $m \geq \max(\overline{n}, n)$,

$$\begin{aligned} J(Q) &\leq c + 3\varepsilon + \sum_{l<i} \int_{s_i(q_m)}^{s_{i-1}(q_m)} \mathcal{L}(z_0^+)\, dt \\ &= c + 3\varepsilon + \int_0^{s_l(q_m)} \mathcal{L}(z_0^+)\, dt. \end{aligned} \tag{4.55}$$

Now $s_l(q_m) \to \overline{s}_l(Q)$ as $m \to \infty$ so further requiring $m \geq n_1(\varepsilon)$,

$$s_l(q_m) \leq \overline{s}_l(Q) + \varepsilon. \tag{4.56}$$

By Lemma 4.38, $\overline{s}_l(Q) \to 0$ as $l \to \infty$. Hence for l large enough,

$$J(Q) \leq c + 3\varepsilon + \int_0^{2\varepsilon} \mathcal{L}(z_0^+)\, dt. \tag{4.57}$$

Since ε is arbitrary and $Q \in \Gamma$, it now follows that $J(Q) = c$.

5. Q is a solution of (HS)

The main goal of this section is to prove that Q is a solution of (HS). In the process, some further qualitative properties of Q will be obtained. It will also be shown that (HS) has three additional solutions of heteroclinic type.

LEMMA 5.1. *Q is a simple curve.*

PROOF. By (Γ_4), it suffices to show that Q restricted to $[-\infty, t_1]$ and to $[t_i, t_{i+1}]$ for $i \in \mathbb{N}$ are simple curves. But by the usual "curve shortening" argument, if any of these curves were not simple, slicing off a closed loop yields $\widehat{Q} \in \Gamma$ with $J(\widehat{Q}) < J(Q) = c$, contrary to (3.5).

LEMMA 5.2. *$s_{i+1}(Q) < s_i(Q)$ for all $i \in \mathbb{N}$ unless $s_i(Q) = 0$ in which case $Q(t) = p_+(t)$ for $t \geq t_i(Q)$.*

PROOF. If $s_i(Q) = s_{i+1}(Q)$, as in (4.29), set

$$\widehat{Q}(t) = \begin{cases} Q(t), & t \leq t_i(Q), \\ Q(t + t_{i+1} - t_i) - k, & t \geq t_i(Q). \end{cases} \tag{5.3}$$

Then $\widehat{Q} \in \Gamma$ and $J(\widehat{Q}) < J(Q)$ unless $Q(t) = p_+(t)$ for $t \in [t_i, t_{i+1}]$. But then $s_i(Q) = 0$ and by (Γ_5), $s_j(Q) = 0$ for all $j > i$. Hence the argument just given implies $Q|_{t_j}^{t_{j+1}} = p_+|_{jT}^{(j+1)T}$ for all $j > i$.

PROPOSITION 5.4. *Q is a solution of* (HS).

PROOF. A local argument will be used to show Q is a solution of (HS) near each $t^* \in \mathbb{R}$. Note that by Lemma 5.2, $Q(t^*) \notin \mathbb{Z}^2$. Let $\mathcal{Q} = \bigcup_{j\in\mathbb{N}\cup\{0\}}(q^*(\mathbb{R})+j)$ and $\mathcal{Z} = \bigcup_{i\in\mathbb{N}\cup\{0\}} z_i^+(0,\infty]$. Set

$$\mathcal{T} = \{t \in \mathbb{R} \mid Q(t) \in p_+(\mathbb{R}) \cup \mathcal{Q} \cup \mathcal{Z}\}.$$

Choose $\sigma < t^* < s$ with σ and s near t^*. A standard variational argument yields a solution p of (HS) joining $Q(\sigma)$ and $Q(s)$ and lying near $Q(t^*)$. Indeed, it is obtained by minimizing $\int \mathcal{L}(q)\,dt$ over all $W^{1,2}$ curves joining $Q(\sigma)$ and $Q(s)$. Now either $Q|_\sigma^s$ is a minimizer of this problem and therefore Q is a solution of (HS) on (σ, s) or there is a minimizer p with

$$\int \mathcal{L}(p)\,dt < \int_\sigma^s \mathcal{L}(Q)\,dt. \tag{5.5}$$

If, further, $t^* \notin \mathcal{T}$, then replacing $Q|_\sigma^s$ by p produces $\widehat{Q} \in \Gamma$ such that $J(\widehat{Q}) < J(Q)$, which is impossible. Hence for all $t^* \notin \mathcal{T}$, Q satisfies (HS) for t near t^*.

It remains to study $t^* \in \mathcal{T}$.

CASE 1: $Q(t^*) \in \mathcal{Q}$. Let $\sigma < t^* < s$ as above. If $Q|_\sigma^s$ is not a local minimizer, there is a p as in (5.5). If p lies in $\mathcal{R}$, the above argument shows $Q|_\sigma^s$ satisfies (HS). Thus suppose p does not lie in $\mathcal{R}$. Then p has a subarc $p(\sigma_1, s_1)$ such that $p(\sigma_1)$, $p(s_1) \in \mathcal{Q}$ and $p(\sigma_1, s_1) \notin \mathcal{R}$. Now $p(\sigma_1) = q^*(\alpha) + mk$, $p(s_1) = q^*(\beta) + mk$, the same value of m appearing since $Q(t^*) \notin \mathbb{Z}^2$ implies $Q(t^*)$ is not at an intersection of the heteroclinics that constitute $\mathcal{Q}$. The minimality properties of q^* and p now imply $p = q^*|_\alpha^\beta$ so in fact p lies in $\mathcal{R}$ and Q satisfies (HS) for this case.

CASE 2: $Q(t^*) \in \mathcal{Z}$. If $Q(t^*) = z_0^+(s^*)$, the minimality property of z_0^* implies Q coincides with $z_0^*|_{s^*}^\infty$. A priori it is possible that $s^* = 0$, which will be discussed in Case 3 below. Thus suppose that $Q(t^*) \neq p_+(0)$. Then for $\sigma < t^* < s$ we are in the setting of Case 1 with z_0^+ replacing q^* and the argument given there shows $Q|_\sigma^s$ satisfies (HS).

Next suppose that $Q(t^*) \in z_i^+$ for some $i \in \mathbb{N}$ and $Q(t^*) \neq z_i^+(0)$. Again arguments as above using the minimality property of z_i^+ show $Q|_\sigma^s$ satisfies (HS).

CASE 3: $Q(t^*) \in p_+$. Choose $\sigma < t^* < s$ and p as earlier allowing for the possibility that $Q(\sigma) \in z_i^+$ and $Q(s) \in p_+$. Once again familiar arguments using the minimality properties of z_i^+ and p_+ show Q satisfies (HS) on (σ, s).

COROLLARY 5.6. $\frac{1}{2}|\dot{Q}(t)|^2 + V(Q(t)) = 0$ *for all* $t \in \mathbb{R}$.

PROOF. Since Q is a solution of (HS), its energy is constant. Moreover, $Q(-\infty) = 0$, so $V(Q(-\infty)) = 0$. Thus to prove the result, it suffices to show

$|\dot{Q}(t)| \to 0$ as $t \to -\infty$. By (HS) and (V_2), $|\ddot{Q}(t)| \to 0$ as $t \to -\infty$. Let $\varepsilon > 0$. By a standard interpolation inequality (see e.g. [7]) there is a $K(\varepsilon)$ so that

$$\|\dot{q}\|_{L^\infty[a,b]} \le \varepsilon \|\ddot{q}\|_{L^\infty[a,b]} + K(\varepsilon)\|q\|_{L^\infty[a,b]}$$

where K also depends on $|b-a|$. Taking $[a,b] = [n, n+1]$, $q = Q$, and letting $n \to -\infty$ shows $\dot{Q}(-\infty) = 0$ and the corollary is proved.

Now that it is known that Q is a solution of (HS) with energy 0, some of the possibilities encountered in the proof of Proposition 5.4 and earlier can be excluded.

Corollary 5.7. *For $t \in \mathbb{R}$, $Q(t) \cap (\mathcal{Q} \cup p_+(\mathbb{R}) \cup z_0^*(\mathbb{R}^+)) = \emptyset$. Moreover, $Q(t) \cap z_i^+ = Q(t_i)$ for $i \in \mathbb{N}$. In particular, $t_i(Q) = \overline{t}_i(Q)$.*

Proof. For $t \in \mathbb{R}$, if $Q(t) \in \mathcal{Q}$, $p_+(\mathbb{R})$ or $z_0^+(\mathbb{R}^+)$, by Corollary 5.6, it must be tangent to the corresponding curve and therefore by an earlier argument must coincide with it, an impossibility. Similarly if $Q(t)$ intersects z_i^+ at more than one point, it must coincide with this curve, which is impossible.

Corollary 5.8. *$s_{i+1}(Q) < s_i(Q)$ for all $i \in \mathbb{N} \cup \{0\}$.*

Proof. $Q \in \Gamma$ implies $s_{i+1}(Q) \le s_i(Q)$. Corollary 5.7 and the argument of (4.29)–(4.31) show equality is not possible.

The next result and its corollary give some further qualitative information on Q.

Proposition 5.9. *$Q/\mathbb{Z}^2$ is simple, i.e. Q is a simple curve on T^2.*

Proof. If not there are numbers $\sigma < s$ and $j \in \mathbb{Z}^2$ such that $Q(\sigma) = Q(s) + j$. Since Q lies between p_- and p_+, this is only possible if $j = mk$ for some $m \in \mathbb{N} \cup \{0\}$. Proposition 5.1 implies $m \ne 0$, $\sigma \in (-\infty, t_1]$ or $[t_i, t_{i+1}]$ for some $i \ge 1$, and $s \in (t_l, t_{l+1})$ for some $l > i$. Now $q = Q|_\sigma^s \in F_{mk}$ so

$$\int_\sigma^s \mathcal{L}(Q)\,dt > c^*_{mk} = mc^*_k. \tag{5.10}$$

The inequality in (5.10) is strict via Corollary 5.7. Now by a familiar argument excising $Q|_\sigma^s$ from Q leads to $\widehat{Q} \in \Gamma$ with $J(\widehat{Q}) < J(Q)$, a contradiction. Hence Q is simple on T^2.

The next result shows Q approaches p_0 monotonically in an appropriate sense. Set $Q_1 = Q|_{-\infty}^{t_1}$ and for $i > 1$, set $Q_i = Q|_{t_{i-1}}^{t_i}$. Let $p_i = p_0|_{(i-1)T}^{iT}$.

Corollary 5.11. *For all $i \in \mathbb{N}$, $Q_{i+1} - k$ lies between Q_i and p_i.*

Proof. Q_{i+1} lies to the left of p_{i+1} by construction. Hence $Q_{i+1} - k$ lies to the left of $p_{i+1} - k = p_i$. By Corollary 5.8, the endpoints of Q_i are to the left of those of $Q_{i+1} - k$. Moreover, Q_i and $Q_{i+1} - k$ cannot intersect via Proposition 5.9 and the corollary follows.

Finally, observe that the construction employed to get a solution of (HS) lying between $\mathcal{Q}$ and p_+ and heteroclinic to 0 and p_+ works equally well to get a second solution lying between $\mathcal{Q}$ and p_- and heteroclinic to 0 and p_-. The same reasoning also yields a pair of solutions of (HS) heteroclinic to 0 and to $p_+(-t)$, $p_-(-t)$ respectively. Denoting these solutions by $Q_k^+, Q_k^-, Q_{-k}^+, Q_{-k}^-$ respectively, this completes the proof of Theorem 1.12.

Remark 5.12. The heteroclinic solution Q_k^+ of (HS) (and similarly for its three relatives) may not be unique in Γ. However, the usual minimization argument as e.g. in Lemma 5.1 implies that any two such minimizers Q and P in Γ intersect only at 0. Therefore one of these solutions, e.g. P, lies between Q and p_+. Hence there is a unique Q_k^+ such that every other minimizer P lies between Q_k^+ and p_+, i.e. Q_k^+ is the minimizer farthest to the left of p_+. It is this solution that will be further characterized as a limit of homoclinics (on T^2) in §6.

6. More homoclinics on T^2

In this final section it will be shown that (HS) has two additional families of heteroclinic solutions $q_m^\pm \in G_{mk}$ with q_m^+ and q_m^- lying on opposite sides of $\mathcal{Q}$. These solutions are homoclinic to 0 on T^2. Moreover, q_m^+ (resp. q_m^-) converges in a monotone sense that will be made precise below to $Q_k^+ \cup Q_{-k}^+$ (resp. $Q_k^- \cup Q_{-k}^-$), thus finishing an additional characterization of these solutions. Related but more restrictive results in another setting were obtained by Caldiroli and Jeanjean [6]. Their analogue q_m of $q_m^\pm$ is characterized by the number of times it winds around a singularity. Some of their estimates play a role in obtaining $q_m^\pm$ below.

To begin, observe that the same argument establishing the existence of z_0^+ yields z_0^-, a solution of (HS) with $z_0^- \in p_-$ and $z_0^-(\infty) = 0$. For $j \in \mathbb{N}$, let $z_j^- = z_0^- + jk$. Now set

$$G_{mk}^\pm = \{g \in G_{mk} \mid q \text{ lies between } \mathcal{Q} \text{ and } p^\pm \text{ and } q \text{ lies above } z_0^\pm\} \tag{6.1}$$

Define

$$c_{mk}^\pm = \inf_{q \in G_{mk}^\pm} I(q). \tag*{(6.2)$^\pm$}$$

Taking e.g. the + case and arguing again as in [13] or [16], any minimizing sequence for $(6.2)^+$ converges to a chain of functions $u_1, \dots, u_j$ with $j = j(m)$ lying between $\mathcal{Q}$ and p_+ and above z_0^+ with $u_1(-\infty) = 0$, $u_j(\infty) = mk, u_i(-\infty) = u_{i-1}(\infty), u_i(\infty) - u_i(-\infty) = w_i k$ with $w_i \in \mathbb{Z}$, $\sum_{i=1}^j w_i = m$, and

$$c_{m,k}^+ = \sum_{i=1}^j I(u_i) \tag{6.3}$$

Now (6.3), i.e. the minimality of the chain, and the argument of Proposition 5.4 show u_i is a solution of (HS), $1 \le i \le j$, and therefore u_i does not intersect p_+ or z_0^+ (except for $t = -\infty$). Moreover, $u_i(t) \in \mathcal{Q}$ for $t \ne \pm\infty$ implies $u_i \subset \mathcal{Q}$, in which case $|w_i| = 1$.

It will be shown next that for m sufficiently large, $j(m) = 1$ and $u_1 = q_m^\pm \in G_{mk}^\pm$. More precisely:

Theorem 6.4. *Let V satisfy* (V_1)–(V_2) *and suppose* (1.13) *holds. Then there are numbers $\beta^\pm \ge 2$ such that for each $m \ge \beta^\pm$, there is a solution $q_m^\pm \in G_{mk}^\pm$ of* (HS) *with $I(q_m^\pm) = c_{mk}^\pm$. Moreover, for $l > m$, $q_l^\pm$ lies between $q_m^\pm$ and $p_\pm$.*

Remark 6.5. The minimizers $q_m^\pm \in G_{mk}^\pm$ of $(6.2)^\pm$ need not be unique. The second assertion of Theorem 6.4 applies to any pair of minimizers $q_l^\pm, q_m^\pm$ with $l > m$.

The first step in the proof of Theorem 6.4 is

Lemma 6.6. *For m large, $c_{mk}^\pm < mc_k$.*

Proof. Let $m \in \mathbb{N}$. Define $q \in G_{mk}^+$ via

$$q(t) = \begin{cases} z_0^+(-t), & -\infty \le t \le 0, \\ p_+(t), & 0 \le t \le mT, \\ \tau_{mT} z_m^+(t), & mT \le t \le \infty. \end{cases} \tag{6.5}$$

Then by (1.13), for m sufficiently large,

$$I(q) = 2\int_0^\infty \mathcal{L}(z_0^+)\,dt + mc_k^* < mc_k. \tag{6.6}$$

Therefore

$$c_{mk}^+ < mc_k. \tag{6.7}$$

A similar argument shows $c_{mk}^- < mc_k$. Let $\beta^\pm$ be the smallest value of m for which $c_{mk}^\pm < mc_k$.

Proposition 6.8. *There is a solution $q_{\beta^\pm}^\pm \in G_{\beta^\pm k}^\pm$ of* (HS) *with $I(q_{\beta^\pm}^\pm) = c_{\beta^\pm k}^\pm$.*

Proof. Dropping $\pm$, by the above remarks, a minimizing sequence for $c_{\beta k}$ converges to the chain $u_1, \dots, u_j$. If $|w_i| < \beta$ for all i, then $I(u_i) = |w_i| c_k$ and by (6.3),

$$c_{\beta k} = \sum_{i=1}^j I(u_i) = \left(\sum_{i=1}^j |w_i|\right) c_k \ge \left(\sum_{i=1}^j w_i\right) c_k = \beta c_k, \tag{6.9}$$

contrary to the choice of β. Therefore $|w_i| > \beta$ for some i. If $|w_i| > \beta$, without loss of generality, $w_i > 0$ and the argument of Proposition 2.1(c) shows u_i and $u_i + k$ intersect. Repeated application of this fact and the excision of an appropriate portion of u_i yields $u \in G_{\beta k}$ with $I(u) < I(u_i) \le c_{\beta k}$, contrary to the definition of $c_{\beta k}$. Therefore $|w_i| = \beta$ and again it can be assumed that $w_i = |w_i|$. Moreover, $i = 1$ and $u_i = q_\beta$. The proposition is proved.

REMARK 6.10. For the sequel, note that by the choice of β (dropping $\pm$ for β, C, and G),

$$\frac{c_{\beta k}}{\beta} < \frac{c_{(\beta-1)k}}{\beta - 1} = c_k. \tag{6.11}$$

Moreover, the argument of Proposition 2.1(c) yields numbers $\sigma_\beta < s_\beta$ such that $q_\beta(s) = q_\beta(\sigma) + k$. Set

$$\varphi_{\beta-1}(t) = \begin{cases} q_\beta(t), & t \le \sigma_\beta, \\ q_\beta(t - \sigma_\beta + s_\beta), & t \ge \sigma_\beta, \end{cases} \tag{6.12}$$

and

$$\varphi_{\beta+1}(t) = \begin{cases} q_\beta(t), & t \le s_\beta, \\ q_\beta(t - s_\beta + \sigma_\beta) & t \ge s_\beta. \end{cases} \tag{6.13}$$

Then $\varphi_{\beta\pm1} \in G_{(\beta\pm1)k}$ and therefore

$$c_{(\beta\pm1)k} = I(\varphi_\beta) \pm \int_{\sigma_\beta}^{s_\beta} \mathcal{L}(q_\beta)\,dt = c_{\beta k} \pm \int_{\sigma_\beta}^{s_\beta} \mathcal{L}(q_\beta)\,dt. \tag{6.14}$$

Hence

$$c_{(\beta+1)k} - c_{\beta k} \le \int_{\sigma_\beta}^{s_\beta} \mathcal{L}(q_\beta)\,dt \le c_{\beta k} - c_{(\beta-1)k}. \tag{6.15}$$

PROOF OF THEOREM 6.4. Again dropping $\pm$ as in Remark 6.10, suppose solutions q_i of (HS) have been obtained with $q_i \in G_{ik}$, $I(q_i) = c_{ik}$,

$$\frac{c_{ik}}{i} < \frac{c_{(i-1)k}}{i-1} \tag{6.16}$$

and

$$c_{(i+1)k} - c_{ik} \le \int_{\sigma_i}^{s_i} \mathcal{L}(q_i)\,dt \le c_{ik} - c_{(i-1)k}, \tag{6.17}$$

where $\beta \le i \le m$. Note that these conditions hold for $i = \beta$ via Remark 6.10. To obtain the existence of q_{m+1} satisfying (6.16)–(6.17) for $i = m+1$, note first that by (6.16)–(6.17) for $i = m$,

$$c_{(m+1)k} \le c_{mk} + c_{mk} - c_{(m-1)k} < c_{mk} + \frac{c_{mk}}{m} \tag{6.18}$$

and (6.18) is equivalent to (6.16) for $i = m+1$. As earlier, there is a heteroclinic chain $u_1, \dots, u_j$ of solutions of (HS) joining 0 and $(m+1)k$ with

$$c_{(m+1)k} = \sum_{i=1}^{j} I(u_i) \tag{6.19}$$

and

$$I(u_i) = \sum_{i=1}^{j} c_{w_i k} \tag{6.20}$$

with w_i as in the proof of Proposition 6.8. If $j = 1$, then $u_1 \equiv q_{m+1}$ is the desired solution. If $j > 1$ and $w_i < m+1$ for all i, then by (6.16) (for $\beta \le i \le m+1$),

$$c_{(m+1)k} = \sum_{i=1}^{j} c_{w_i k} > \sum_{i=1}^{j} \frac{w_i}{m+1} c_{(m+1)k} \ge c_{(m+1)k} \tag{6.22}$$

since $\sum_{i=1}^{j} w_i \ge (m+1)k$. Hence $w_i \ge m+1$ for some i. If $w_i = m+1$, then

$$I(u_i) < \sum_{l=1}^{j} I(u_l) = c_{(m+1)k} \tag{6.23}$$

contrary to the definition of $c_{(m+1)k}$. If $w_i > m+1$, repeated excisions of u_i as in (2.18) yield a $v_i \in G_{(m+1)k}$ with

$$I(v_i) < I(u_i) < c_{(m+1)k}, \tag{6.24}$$

again a contradiction. Therefore $j = 1$ and the existence of q_{m+1} has been established. Finally, the argument of (6.14)–(6.15) gives these inequalities and hence (6.17) for $m+1$. This proves the first assertion of Theorem 6.4.

To get the second assertion of the theorem, observe that since q_m is a minimizer for (6.2), as is q_l for the associated problem with m replaced by l, $q_m \cap q_l = \{0\}$, i.e. the function cannot intersect except when $t = -\infty$. Therefore if $l > m$, then $q_l^{\pm}$ must be between $q_m^{\pm}$ and $p_{\pm}$.

Remark 6.25. The above reasoning also shows that for $l > m, q_l^{\pm}$ lies between $q_m^{\pm}$ and $Q_k^{\pm}$. This monotonicity of $q_m^{\pm}$ with respect to m suggests that $Q_k^{\pm}$ may in some sense be the limit of $(q_m^{\pm})$. This possibility will be studied next.

Normalize $q_m^{\pm}$ in the same fashion as $Q_k^{\pm}$, i.e. since $\tau_\theta q_m^{\pm}$ also minimizes I in $G_{km}^{\pm}$, it can be assumed that $q_m^{\pm}(0) \in \partial B_\delta(0)$ for all $m \in \mathbb{N}$ and $q_m^{\pm}(t) \in B_\delta(\delta)$ for all $t < 0$ where δ is as in §4. Now the functions $(q_m^{\pm})$ lie between $\mathcal{Q}$ and $Q_k^{\pm}$ and $(q_m^{\pm})$ are solutions of (HS). Therefore they are bounded in $C^2_{\rm loc}$. Hence $(q_m^{\pm})$ converges along a subsequence in $C^2_{\rm loc}$ to $Q^{\pm}$, where $Q^{\pm}$ is a solution of (HS) with $Q^{\pm}(0) \in \partial B_\delta(0)$, $Q^{\pm}(t) \in B_\delta(0)$ for $t < 0$, and $Q^{\pm}$ lies between $\mathcal{Q}$ and $Q_k^{\pm}$. Moreover, $Q^{\pm}$ does not touch $\mathcal{Q}$ or $Q_k^{\pm}$ except at 0 unless it coincides with $Q_k^{\pm}$.

Finally, observe that since $q_m^\pm$ lies between $q_{m-1}^\pm$ and $q_{m+1}^\pm$, the entire sequence converges to $Q^\pm$.

It will be shown next that $Q^\pm = Q_k^\pm$. Note that a set of functions Γ^- can be defined in an analogous fashion to $\Gamma \equiv \Gamma^+$.

PROPOSITION 6.26. *$Q^\pm \in \Gamma^\pm$.*

PROOF. This will be proved for Q^+. For notational simplicity, the $\pm$'s will be dropped below. Suppose that there is an $M > 0$ such that

$$\int_{-\infty}^0 \mathcal{L}(q_m^+)\,dt \le M. \tag{6.27}$$

Then as in [12] or [17], the functions (q_m^+) are bounded in

$$E = \{u \in W^{1,2}_{\mathrm{loc}} \mid u(-\infty) = 0\}$$

under the norm

$$\left(\int_{-\infty}^0 |\dot u|^2\,dt + |u(0)|^2\right)^{1/2}.$$

Then as in (4.10),

$$\int_{-\infty}^0 \mathcal{L}(Q^+)\,dt \le M \tag{6.28}$$

and $Q^+(-\infty) = 0$. Thus Q^+ satisfies (Γ_1).

To get the estimate (6.27), arguing essentially as in [17], for each $b \in B_\delta(0)$, there is a solution u_x of (HS) such that $u_x(-\infty) = 0, u_x(0) = x$, and

$$\int_{-\infty}^0 \mathcal{L}(u_x)\,dt = \inf_{u \in E_x} \int_{-\infty}^0 \mathcal{L}(u)\,dt \tag{6.29}$$

where $E_x = \{u \in E \mid u(0) = x\}$. Now (6.29) and a straightforward comparison argument show there is an M as in (6.27).

Since $(q_m^\pm) \subset \mathcal{R}$ and converge pointwise to $Q^+, (\Gamma_2)$ is satisfied by Q^+. For verification of (Γ_3), it suffices to show that for each $j \in \mathbb{N}, t_j(q_m^+)$ is bounded from above independently of m. Then the L^∞_{loc} convergence of q_m^+ to Q^+ yields (Γ_3). To get the bounds for $t_j(q_m^+)$, note first that

$$\begin{aligned} \int_{-\infty}^{t_j(q_m^+)} \mathcal{L}(q_m^+)\,dt &\le \int_{-\infty}^{\infty} \mathcal{L}(q_j^+)\,dt + \int_{s_j(q_m)}^{\infty} \mathcal{L}(z_0^+)\,dt \\ &\le c_{jk} + \int_0^\infty \mathcal{L}(z_0^+)\,dt. \end{aligned} \tag{6.30}$$

Now (6.30) gives an m-independent bound for $\dot q_m^+$ in $L^2(-\infty, t_j(q_m^+))$. Since

$$\tfrac{1}{2}|\dot q_m^+(t)|^2 + V(q_m^+(t)) \equiv 0, \tag{6.31}$$

it follows that

$$\int_0^{t_j(q_m^+)} \tfrac{1}{2}|\dot{q}_m^+|^2\,dt = -\int_0^{t_j(q_m^+)} V(q_m^+(t))\,dt. \tag{6.32}$$

Now by the argument of Proposition 4.11 (Case 1(a)), $q_m^+(t)$ cannot get too close to 0 for $t > 0$ (independently of m). Therefore there is a $\varrho > 0$ such that $|q_m^+(t)| \geq \varrho$ for $t > 0$. Moreover, $|q_m^+(t) - \mathbb{Z}^2| \geq \varrho$ for $t \in (0, t_j(q_m^+))$. Hence there is a $\gamma(\varrho) > 0$ such that

$$-\int_0^{t_j(q_m^+)} V(q_m^+)\,dt \geq \gamma t_j(q_m^+). \tag{6.33}$$

Combining (6.30), (6.32)–(6.33) gives the upper bounds for $t_j(q_m^+)$ and (Γ_3). Since (q_m^+) satisfy (Γ_4), the bounds for $t_j(q_m^+)$ readily imply (Γ_4) for Q^+.

It remains only to show that Q^+ satisfies (Γ_5). By (Γ_1), $s_0(Q^+) = \infty > s_1(Q^+)$. Suppose there is some i such that $s_{i+1}(Q^+) > s_i(Q^+)$. Then by a familiar argument, there is a $\sigma_{i-1} \in (t_{i-1}(Q^+), t_i(Q^+)]$ (or $(-\infty, t_1(Q^+))$ if $i = 1$) and $\sigma_i \in (t_i(Q^+), t_{i+1}(Q^+))$ such that $Q^+(\sigma_i) - Q^+(\sigma_{i-1}) = k$. Moreover, since $Q^+|_{\sigma_{i-1}}^{\sigma_i} \in F_k$ and Q^+ is not a translate of p^+, there is a $\gamma > 0$ such that

$$\int_{\sigma_{i-1}}^{\sigma_i} \mathcal{L}(Q^+)\,dt \geq c_k^* + 2\gamma. \tag{6.34}$$

Excise $q_m^+|_{\sigma_{i-1}}^{\sigma_i}$ from q_m^+, shift $q_m^+|_{\sigma_i}^{\infty}$ by $-k$, and join $q_m^+(\sigma_{i-1})$ to $q_m^+(\sigma_i) - k$ by a straight line segment L_i. Let $\overline{q}_m$ denote the resulting function. Then $\overline{q}_m \in G_{(m-1)k}$ and since $q_m|_{\sigma_{i-1}}^{\sigma_i} \to Q^+$ in C^2 as $m \to \infty$, it can be assumed that

$$I(q_m^+) - I(\overline{q}_m) \geq c_k^* + \gamma. \tag{6.35}$$

Let $a_m = c_{mk}^+ - mc_k^* = I(q_m^+) - mc_k^*$. Then for all large m,

$$a_{m+1} = I(q_{m+1}^+) - (m+1)c_k^* \geq I(\overline{q}_{m+1}) + c_k^+ + \gamma - (m+1)c_k^* \tag{6.36}$$

so (a_m) is an unbounded sequence. On the other hand, as in (6.6),

$$a_m \leq 2\int_0^{\infty} \mathcal{L}(z_0^+)\,dt \tag{6.37}$$

so (a_m) is bounded from above. Thus $s_{i+1}(Q^+) > s_i(Q^+)$ is impossible and (Γ_5) holds for Q^+.

Since Q^+ lies in Γ, an immediate consequence of (Γ_1) and Proposition 3.12 is:

Corollary 6.38. *Q^+ is heteroclinic to* 0 *and* p^+.

Now finally it can be shown that Q^+ coincides with Q_k^+.

Proposition 6.39. $Q^+ = Q_k^+$.

Proof. If not, then Q^+ lies between $\mathcal{Q}$ and Q_k^+. Moreover, since Q_k^+ is, by definition, the minimizer of J in Γ farthest to the left of p_+, there is a $\varrho > 0$ such that

$$J(Q^+) \geq J(Q_k^+) + \varrho. \tag{6.40}$$

This inequality will lead to a contradiction.

First observe that the functions (q_m^+) can be used to produce another heteroclinic solution of (HS) lying between the natural extension of $\mathcal{Q}$ below z_0^+ and Q_{-k}^+. Indeed, supplement $\mathcal{Q}$ by $\bigcup_{m=-\infty}^{-1}(q^*(\mathbb{R}) + mk)$, still denoting the extension by $\mathcal{Q}$. Let $\psi_m(t) = q_m^+(-t) - mk$. Then ψ_m is a solution of (HS) heteroclinic to 0 and $-mk$ and lying between $\mathcal{Q}$ and Q_{-k}^+. Choose $\theta_m > 0$ such that $\varphi_m(t) = \tau_{\theta_m}\psi_m(t) \in B_\delta(0)$ for $t < 0$ and $\varphi_m(0) \in \partial B_\delta(0)$. Then the arguments given above for Q^+ show φ_m converges to P^+, a solution of (HS) heteroclinic to 0 and $p_+(-t)$ and lying between $\mathcal{Q}$ and Q_{-k}^+. Let Γ_+ be the analogue of $\Gamma = \Gamma^+$ for functions lying between $\mathcal{Q}$ and $p_+(-t)$ and set

$$J_+(q) = \sum_{i=-\infty}^{-1} a_i(q) \tag{6.41}$$

Therefore Q_{-k}^+ minimizes J_+ on Γ_+ and

$$J_+(P^+) \geq J_+(Q_{-k}^+). \tag{6.42}$$

Let

$$0 < \varepsilon < \varrho/11. \tag{6.43}$$

Then there is an $n_0 = n_0(\varepsilon)$ such that for $n \geq n_0$,

$$-\varepsilon + \sum_{i=1}^{n} a_i(Q_k^+) \leq J(Q_k^+) \tag{6.44}$$

and

$$-\varepsilon + \sum_{i=-n}^{-1} a_i(Q_{-k}^+) \leq J_+(Q_{-k}^+). \tag{6.45}$$

Note that

$$Q_{-k}^+(t_{-n}(Q_{-k}^+)) + 2nk = z_n^+(s_{-n}(Q_{-k}^+)).$$

Define a function $u_n(t) \in G_{2nk}^+$ as follows. Set

$$u_n(t) = Q_k^+(t), \quad t < t_n(Q_k^+).$$

Glue to $Q_k^+(t_n(Q_k^+))$ the portion of z_n^+ joining $z_n^+(s_n(Q_k^+))$ to $z_n^+(s_{-n}(Q_{-k}^+))$. Finally, join to this the function $Q_{-k}^+(-t)$ for $t \geq t_{-n}(Q_{-k}^+)$. For n_0 sufficiently large, both $s_n(Q_k^+)$ and $s_{-n}(Q_{-k}^+)$ are near 0 and

$$\left| \int_{s_n(Q_k^+)}^{s_{-n}(Q_{-k}^+)} \mathcal{L}(z_n^+)\, dt \right| < \varepsilon. \tag{6.46}$$

Therefore by (6.44)–(6.46),

$$\sum_{i=1}^{n} a_i(Q_k^+) + \sum_{i=-n}^{-1} a_i(Q_{-k}^+) \geq I(u_n) - 2nc_k^+ - \varepsilon \geq c_{2nk}^+ - 2nc_k^+ - \varepsilon. \tag{6.47}$$

Combining (6.40), (6.42), (6.44)–(6.45) and (6.47) yields

$$J(Q^+) + J_+(P^+) \geq c_{2nk}^+ - 2nc_k^* - 3\varepsilon + \varrho. \tag{6.48}$$

It remains to get an appropriate upper bound for the right hand side of (6.48). As above for $l \geq l_1(\varepsilon)$,

$$J(Q^+) \leq \sum_{i=1}^{l} a_i(Q^+) + \varepsilon \tag{6.49}$$

and

$$J_+(P^+) \leq \sum_{i=-l}^{-1} a_i(P^+) + \varepsilon. \tag{6.50}$$

Hence

$$J(Q^+) + J_+(P^+) \leq \int_{-\infty}^{t_l(Q^+)} \mathcal{L}(Q^+)\, dt + \int_{-\infty}^{t_{-l}(P^+)} \mathcal{L}(P^+)\, dt - 2lc_k^* + 2\varepsilon. \tag{6.51}$$

Now the C^2_{loc} convergence of q_m^+ to Q^+ and transversal crossing of z_l^+ by q_m^+ and Q^+ implies

$$t_l(Q^+) = \lim_{m\to\infty} t_l(q_m^+). \tag{6.52}$$

Similarly

$$t_{-l}(P^+) = \lim_{m\to\infty} t_{-l}(\varphi_m). \tag{6.53}$$

Therefore

$$\int_{-\infty}^{t_l(Q^+)} \mathcal{L}(Q^+)\, dt \leq \varliminf_{m\to\infty} \int_{-\infty}^{t_l(q_m^+)} \mathcal{L}(q_m^+)\, dt \tag{6.54}$$

and

$$\int_{-\infty}^{t_{-l}(P^+)} \mathcal{L}(P^+)\, dt \leq \varliminf_{m\to\infty} \int_{-\infty}^{t_{-l}(\varphi_m)} \mathcal{L}(\varphi_m)\, dt \tag{6.55}$$

Hence for all $m \ge m_0(l,\varepsilon)$, along an appropriate sequence of m's,

$$\int_{-\infty}^{t_l(Q^+)} \mathcal{L}(Q^+)\,dt + \int_{-\infty}^{t_{-l}(P+)} \mathcal{L}(P^+)\,dt \le 2\varepsilon + \int_{-\infty}^{t_l(q_m^+)} \mathcal{L}(q_m^+)\,dt + \int_{-\infty}^{t_{-l}(\varphi_m)} \mathcal{L}(\varphi_m)\,dt. \tag{6.56}$$

Now

$$\begin{aligned}\int_{-\infty}^{t_{-l}(\varphi_m)} \mathcal{L}(\varphi_m)\,dt &= \int_{-\infty}^{t_{-l}(\psi_m)+\theta_m} \mathcal{L}(\psi_m(t-\theta_m))\,dt \\ &= \int_{-\infty}^{t_{-l}(q_m^+(-t)-mk)} \mathcal{L}(q_m^+(-t)-mk)\,dt \\ &= \int_{t_{m-l}(q_m^+)}^{\infty} \mathcal{L}(q_m^+)\,dt.\end{aligned} \tag{6.57}$$

Therefore by (6.56)–(6.57),

$$\begin{aligned}\int_{-\infty}^{t_l(Q^+)} \mathcal{L}(Q^+)\,dt + \int_{-\infty}^{t_{-l}(P^+)} \mathcal{L}(P^+)\,dt &\le 2\varepsilon + \int_{-\infty}^{t_l(q_m^+)} \mathcal{L}(q_m^+)\,dt + \int_{t_{m-l}(q_m^+)}^{\infty} \mathcal{L}(q_m^+)\,dt \\ &= 2\varepsilon + c_{mk}^+ - \int_{t_l(q_m^+)}^{t_{m-l}(q_m^+)} \mathcal{L}(q_m^+)\,dt.\end{aligned} \tag{6.58}$$

For $l \ge l_2(\varepsilon)$,

$$\int_0^{s_l(Q^+)} \mathcal{L}(z_l^+)\,dt \le \varepsilon \tag{6.59}$$

and

$$\int_0^{s_{-l}(P^+)} \mathcal{L}(z_{-l}^+)\,dt \le \varepsilon. \tag{6.60}$$

By (6.52)–(6.53),

$$s_l(Q^+) = \lim_{m\to\infty} s_l(q_m^+) \tag{6.61}$$

and

$$s_l(P^+) = \lim_{m\to\infty} s_{-l}(Q_m). \tag{6.62}$$

Therefore for $m \ge m_1(l,\varepsilon)$,

$$\int_0^{s_l(q_m^+)} \mathcal{L}(z_m^+)\,dt \le 2\varepsilon \tag{6.63}$$

and

$$\int_0^{s_{-l}(\varphi_m)} \mathcal{L}(z_{-l}^+)\,dt \le 2\varepsilon. \tag{6.64}$$

Choose $l(\varepsilon) = \max(l_1(\varepsilon), l_2(\varepsilon))$ and with this choice of l, let $m \ge \max m_0(l(\varepsilon), \varepsilon)$, $m_1(l(\varepsilon), \varepsilon))$. Then

$$\int_{t_l(q_m^+)}^{t_{m-l}(q_m^+)} \mathcal{L}(q_m^+)\,dt + \int_0^{s_l(q_m^+)} \mathcal{L}(z_l^+)\,dt + \int_0^{s_l(q_m^+)} \mathcal{L}(z_{m-l}^+)\,dt \ge (m-2l)c_k^* \tag{6.65}$$

Combining (6.51), (6.58), (6.63)–(6.65) yields

$$J(Q^+) + J_+(P^+) \le 8\varepsilon + c_{mk}^+ - mc_k^*. \tag{6.66}$$

Choosing $m = 2n$ and comparing (6.48) and (6.68) shows

$$11\varepsilon \ge \varrho, \tag{6.67}$$

contrary to (6.43). The proof is complete.

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