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SENSITIVITY PROBLEMS FOR SOME SHELLS WITH EDGES

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Dedicated to O. A. Ladyzhenskaya

1. Introduction

1.1. We wish to present a family of two-dimensional boundary value problems which are linear but exhibit some strong instabilities, so as to make in particular numerical computation impossible.

Although very singular, the problems considered here have a physical origin. We begin by explaining that origin; we then proceed in this introduction to give a more general idea of what are the instabilities mentioned above.

1.2. Physical origin of the problems. We are dealing with a class of slightly curved shells that we now describe. Let Ω be a 2-dimensional domain, bounded, with smooth boundary $\partial\Omega$, simply connected or not. Consider a surface S defined by a function

$$(x_1, x_2) \to \theta(x_1, x_2) \tag{1.1}$$

from $\overline{\Omega}$ to $\mathbb{R}^2$ where θ is of the form

$$\theta(x_1, x_2) = (x_1, x_2, \psi(x_1, x_2)) \tag{1.2}$$

where ψ denotes a smooth function $\overline{\Omega} \to \mathbb{R}$ satisfying

$$\psi(x_1, x_2) > 0 \quad \text{for } (x_1, x_2) \in \Omega, \tag{1.3}$$

$$\psi(x_1, x_2) = 0 \quad \text{for } (x_1, x_2) \in \partial\Omega. \tag{1.4}$$

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Note that this is a very particular kind of surface, which is intersected only once by straight lines parallel to the x_3 axis.

In order to define what we mean by "slightly curved shells", we "flatten" the surface S. More precisely, we consider the family of surfaces S_δ

$$x_3 = \delta\psi(x_1, x_2) \tag{1.5}$$

where $\delta > 0$ is a small parameter.

Of course, physically the shell has some thickness so that we have to introduce a new small parameter ε and consider the set

$$\delta\psi(x_1, x_2) - \varepsilon < x_3 < \delta\psi(x_1, x_2) + \varepsilon. \tag{1.6}$$

We shall consider the case when the small parameters δ, ε satisfy

$$\varepsilon \ll \delta \ll 1, \tag{1.7}$$

which *defines the slightly curved shells.*

REMARK 1.1. The case when δ is of order 1 (with $\varepsilon \ll \delta$) is that of a "classical shell". The case $\varepsilon \approx \delta \ll 1$ constitutes the "shallow shells".

In order to consider edges, we introduce a variant of the above framework by considering the surface made of the two half-surfaces $S_\delta^\pm$ defined by

$$x_3 = \pm\delta\psi(x_1, x_2). \tag{1.8}$$

Let $\widehat{u} = \{\widehat{u}_1, \widehat{u}_2, \widehat{u}_3\}$ denote the displacements of the shell under the action of appropriate forces. They are obviously functions of $\varepsilon, \delta, x_1, x_2, x_3$ defined for small ε and δ (satisfying (1.7), $(x_1, x_2) \in \Omega$, x_3 satisfying (1.6)). Under appropriate hypotheses (in particular *hypothesis* H below), some formal asymptotic expansions (Sanchez-Palencia [1994]) show that the leading terms (when ε and δ tend to zero) have the form

$$\begin{aligned} \widehat{u}_\alpha &= \frac{\varepsilon^2}{\delta}\left(u_\alpha - \frac{\partial u_3}{\partial x_\alpha}\psi\right) + \dots, \quad \alpha = 1, 2, \\ \widehat{u}_3 &= \frac{\varepsilon^2}{\delta^2} u_3 + \dots \end{aligned} \tag{1.9}$$

where the dots denote lower order terms, and u_1, u_2, u_3 *are functions independent of* δ, ε, x_3, *defined for* $(x_1, x_2) \in \Omega$. Of course, in the context of (1.8), we will have two expressions (1.9) for the $S_\delta^\pm$.

We shall *only consider symmetric displacement fields*, i.e. satisfying

$$\widehat{u}_\alpha^+(x_1, x_2) = \widehat{u}_\alpha^-(x_1, x_2), \quad \alpha = 1, 2, \tag{1.10}$$

$$\widehat{u}_3^+(x_1, x_2) = -\widehat{u}_3^-(x_1, x_2). \tag{1.11}$$

Obviously $\widehat{u}^+$ and $\widehat{u}^-$ must coincide on the edge, so that

$$\widehat{u}_3^+(x_1,x_2) = \widehat{u}_3^-(x_1,x_2) = 0 \quad \text{for } (x_1,x_2) \in \partial\Omega. \tag{1.12}$$

Thus the symmetric displacements will be characterized by $\widehat{u}_1^+, \widehat{u}_2^+, \widehat{u}_3^+$ satisfying $\widehat{u}_3^+ = 0$ on $\partial\Omega$ where $\psi = 0$. According to (1.9), they are expressed in terms of u_1, u_2, u_3 *defined on* Ω *and satisfying*

$$u_3(x_1,x_2) = 0 \quad \text{for } (x_1,x_2) \in \partial\Omega, \tag{1.13}$$

and no other boundary conditions. Note, in particular, that any u_1, u_2 on the boundary are consistent with a symmetric displacement.

The problem is now to define a boundary value problem which characterizes u_1, u_2, u_3.

1.3. Boundary value problem. We introduce the membrane elasticity coefficients $A^{\alpha\beta\gamma\delta}$ satisfying the classical symmetry and positivity conditions:

$$A^{\alpha\beta\gamma\delta} = A^{\beta\alpha\delta\gamma} = A^{\gamma\delta\alpha\beta}, \tag{1.14}$$

$$A^{\alpha\beta\gamma\delta}\xi_{\gamma\delta}\xi_{\alpha\beta} \geq C\xi_{\alpha\beta}\xi_{\alpha\beta}, \quad \forall\xi_{\alpha\beta} \text{ symmetric.} \tag{1.15}$$

The expressions $\gamma_{\alpha\beta}(v)$ which describe the variations of the first fundamental form of the surface S_δ^+ are given by

$$\gamma_{\alpha\beta}(v) = \tfrac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - \psi\partial_\alpha\partial_\beta v_3, \quad \alpha,\beta = 1,2. \tag{1.16}$$

The deformation u *of the shell* (in fact, the physical deformation $\widehat{u}$ is given by (1.9)) *is the function which minimizes*

$$\frac{1}{2}\int_\Omega A^{\alpha\beta\gamma\delta}\gamma_{\gamma\delta}(v)\gamma_{\alpha\beta}(v) - \int_\Omega f^i v_i \tag{1.17}$$

over all functions v *which are such that*

$$\gamma_{\alpha\beta}(v) \in L^2(\Omega) \tag{1.18}$$

and which satisfy the boundary condition (1.13).

In (1.17), the f^i are properly scaled forces applied to the shell.

We then define the "configuration space of smooth functions" $\mathcal{V}$ (which will later be completed to define a Hilbert space)

$$\mathcal{V} = \{v = (v_1,v_2,v_3) \text{ smooth } \overline{\Omega} \to \mathbb{R}, \text{ satisfying } v_3 = 0 \text{ on } \partial\Omega\}. \tag{1.19}$$

At this level, we make an important hypothesis:

Hypothesis (H). Let $v \in \mathcal{V}$ (defined in (1.19)). Then $\gamma_{\alpha\beta}(v) = 0, \alpha, \beta = 1, 2$, implies $v = \{v_1, v_2, 0\}$, and $\{v_1, v_2\}$ corresponds to a rigid displacement in $\mathbb{R}^2$.

Remark 1.2. (H) is a hypothesis concerning the *geometric rigidity* of the surface. Notice that this hypothesis is independent of δ. If it is not satisfied, the shell is called *noninhibited*, and the deformation has a structure different from (1.9) (see Sanchez-Palencia [1994]), which will not be considered here.

1.4. Generalities on sensitivity. In the classical treatment of problems of mechanics, linear theories are concerned with "small solutions". Instabilities in the linear theory lead to "large solutions", which clearly need a nonlinear treatment. This paper is concerned with a kind of highly unstable linear problems called *"sensitive problems"* (Lions and Sanchez-Palencia [1994, 1996, 1997]), first encountered in the theory of thin elastic shells.

Roughly speaking, the modelling of a physical problem defines a space $\mathcal{V}$ of reasonably smooth solutions, a norm on this space (often an "energy norm") and a natural duality product, usually associated with the L^2 scalar product (or equivalently, duality between the space $\mathcal{D}$ of infinitely differentiable functions with compact support and the space $\mathcal{D}'$ of distributions). In order to handle such a problem in the variational framework, we construct the Hilbert spaces V and H obtained as the completions of $\mathcal{V}$ with the energy norm and the L^2-norm respectively. The usual situation is such that $V \subset H$, and we are led to the classical situation

$$V \subset H \equiv H' \subset V' \tag{1.20}$$

where the prime denotes the dual space. But in certain cases, we have

$$V \not\subset H \equiv H' \Leftrightarrow H \not\subset V'. \tag{1.21}$$

The F. Riesz theorem defines an isomorphism between the "data space" V' and the "solution space" V and there are "given forces" $f \in H$ such that there is no solution u in V. This situation is often encountered when the origin belongs to the essential spectrum of the operator considered. Examples of this situation appear in the theory of thin shells with folds (Geymonat and Sanchez-Palencia [1995]).

A somewhat more pathological situation appears when

$$V \not\subset \mathcal{D}' \Leftrightarrow \mathcal{D} \not\subset V'. \tag{1.22}$$

The problem is then said to be *sensitive*, as (1.3) implies that there are "forces" $f \in \mathcal{D}$ such that there is no solution u (or at least u is not in V). In other words, *there are* (*scaled*) *forces f which are infinitely small, smooth and with compact support, such that the corresponding "solution" goes out of the energy space V* (which is "very large"…!).

Clearly, the existence of a scaled function $f \in \mathcal{D} \setminus V'$ implies a *highly unstable phenomenon* when the system is acted upon with that f. Moreover, *the* *"number" of such f* (i.e. the "number of situations" leading to such an instability phenomenon) *may be considered as a "measure" of the instability of the system.* We shall prove in this paper that, for the problems under consideration, $\mathcal{D} \cap V'$ is a somewhat "small set" in the sense that it is contained in a subspace of L^2 with infinite codimension.

It should be pointed out that (fortunately!) not every shell problem is sensitive. In fact, sensitivity is only concerned with some cases of the limit behavior of shells when the parameter ε (which describes the ratio of the thickness to the other of the shell) tends to zero. This limit behavior may be of two different kinds, according to the geometric rigidity or lack of rigidity of the middle surface, submitted to the kinetic boundary conditions. In the case of geometric ridigity (also called *"inhibited case"* as the pure bendings of the surface are inhibited), the limit problem is concerned with the membrane approximation. The corresponding system is of elliptic or hyperbolic type at elliptic or hyperbolic points of the surface, respectively. Then sensitivity is mainly concerned with elliptic problems with a part of the boundary free of kinematic boundary conditions as well as with certain surfaces with edges (see for instance Bernadou and Ciarlet [1976] for a general framework and Sanchez-Palencia [1992] for the limit behavior).

As for physical specific shells, it clearly appears that sensitivity-like instabilities (and the corresponding difficulties of numerical computation) appear for very small values of ε when the limit problem is sensitive.

At this point, it is worthwhile to point out some analogy with fluid mechanics. Although the complete (i.e. including dissipative terms) equations of fluid dynamics in the steady flow case are of elliptic type (at least in the incompressible case, for which we refer to the classical works of J. Leray [1933, 1934] and O. A. Ladyzhenskaya [1963]), the limit behavior as the dissipative terms tend to zero is elliptic or hyperbolic at subsonic or supersonic points of the flow respectively (Courant and Friedrichs [1948], Sect. 105, or von Mises [1958], Sect. IV.16.3).

We now present, in a more precise fashion, the mathematical problems which correspond to the above considerations. We study (Section 2) the elliptic case, some generalizations being presented in Section 3. We conclude in Section 4 with some further remarks.

2. Elliptic case

2.1. General remarks. We keep the notations of Section 1. We denote by $v = \{v_1, v_2, v_3\}$ functions defined in Ω with values in $\mathbb{R}^3$. For the time being they are assumed to be smooth in $\overline{\Omega}$. We define

$$\gamma(v) = \{\gamma_{11}(v), \gamma_{22}(v), \gamma_{12}(v)\}, \tag{2.1}$$

$$\begin{aligned} \gamma_{11}(v) &= \partial_1 v_1 - \psi \partial_1^2 v_3, \\ \gamma_{22}(v) &= \partial_2 v_2 - \psi \partial_2^2 v_3, \\ \gamma_{12}(v) &= \gamma_{21}(v) = \tfrac{1}{2}(\partial_1 v_2 + \partial_2 v_1) - \psi \partial_1 \partial_2 v_3. \end{aligned} \tag{2.2}$$

The function ψ is given and smooth in $\overline{\Omega}$. More precisely, we shall assume that

$$\psi \in C^3(\overline{\Omega}), \quad \psi = 0 \quad \text{on } \partial\Omega. \tag{2.3}$$

Remark 2.1. We have made no attempt to weaken hypothesis (2.3). But $\psi \in C^2(\overline{\Omega})$ would be enough. Whether it is possible to consider weaker assumptions on ψ is not clear.

Remark 2.2. Later on in this section, we shall make an "elliptic hypothesis" on ψ. For the time being no further hypothesis is made. The remarks made here will be used in Section 3.

We shall use the following result:

Lemma 2.1. *The set of vectors v such that*

$$v = \{v_1, v_2, 0\}, \quad \gamma(v) = 0 \tag{2.4}$$

is given by

$$v_1 = \alpha_1 + \beta x_2, \quad v_2 = \alpha_2 - \beta x_1, \quad \alpha_i, \beta \in \mathbb{R}. \tag{2.5}$$

We shall denote by $\mathcal{R}$ this set. [*It is the set of rigid displacements.*]

Proof. This is physically obvious, as well as mathematically, since (2.4) is equivalent to $\partial_1 v_1 = 0, \partial_2 v_2 = 0$ and $\partial_1 v_2 + \partial_2 v_1 = 0$, hence (2.5) follows. □

As we said in Section 1, in order to solve the equilibrium problem, we have to introduce the functional (which is equivalent to the quadratic part of (1.17))

$$\mathcal{E}(v) = \|\gamma_{11}(v)\|^2 + \|\gamma_{22}(v)\|^2 + \|\gamma_{12}(v)\|^2 \tag{2.6}$$

where $\|f\| = (\int_\Omega f^2\, dx)^{1/2}$.

We now study the kernel of $\mathcal{E}(v)$.

2.2. The kernel of $\mathcal{E}(v)$. We introduce the following notations:

$$a_1 = \partial_1^2 \psi, \quad a_2 = \partial_2^2 \psi, \quad a_3 = \partial_1 \partial_2 \psi, \quad a = \{a_1, a_2, a_3\}, \tag{2.7}$$

$$Pw = -\partial_1(a_2 \partial_1 w) - \partial_2(a_1 \partial_2 w) + \partial_1(a_3 \partial_2 w) + \partial_2(a_3 \partial_1 w). \tag{2.8}$$

Remark 2.3. One can write P in the nonvariational form

$$Pw = -a_2 \partial_1^2 w - a_1 \partial_2^2 w + 2a_3 \partial_1 \partial_2 w. \tag{2.9}$$

We have

LEMMA 2.2. *Let v be a smooth vector function in $\overline{\Omega}$ such that*

$$\mathcal{E}(v) = 0. \tag{2.10}$$

Then

$$Pv_3 = 0. \tag{2.11}$$

PROOF. Of course, (2.10) is equivalent to

$$\gamma_{11}(v) = \gamma_{22}(v) = \gamma_{12}(v) = 0. \tag{2.12}$$

We can eliminate the components v_1, v_2 by considering the combination

$$\partial_2^2\gamma_{11}(v) + \partial_1^2\gamma_{22}(v) - 2\partial_1\partial_2\gamma_{12}(v),$$

which equals 0 if (2.12) holds true. One easily verifies that

$$\partial_2^2\gamma_{11}(v) + \partial_1^2\gamma_{22}(v) - 2\partial_1\partial_2\gamma_{12}(v) = Pv_3, \tag{2.13}$$

hence (2.11) follows. □

2.3. Boundary conditions on v_3 and ellipticity hypothesis. We shall assume in the next part of this section that

$$P \text{ is an elliptic operator, in the sense that } a_1a_2 - a_3^2 \geq \gamma > 0 \text{ in } \overline{\Omega}. \tag{2.14}$$

We now introduce (according to what has been said in Section 1)

$$\mathcal{V} = \{v \text{ smooth } \overline{\Omega} \to \mathbb{R}^3 \text{ such that } v_3 = 0 \text{ on } \partial\Omega\}. \tag{2.15}$$

We have

LEMMA 2.3. *We assume that P is elliptic* (see (2.14)). *Then, for $v \in \mathcal{V}$,*

$$\mathcal{E}(v) = 0 \quad \textit{is equivalent to} \quad v \in \mathcal{R}. \tag{2.16}$$

[We recall that $\mathcal{R}$ is defined in Lemma 2.1 (rigid displacements).]

PROOF. According to Lemma 2.2, if $\mathcal{E}(v) = 0$ then $Pv_3 = 0$. Since P as defined in (2.8) is elliptic and $v_3 = 0$ on $\partial\Omega$, by the maximum principle, $v_3 = 0$ in Ω.

Thus $\gamma(v) = 0$ is equivalent to $\partial_1 v_1 = \partial_2 v_2 = \partial_1 v_2 + \partial_2 v_1 = 0$ and hence the result follows. □

We then introduce

$$\|v\|_V = \mathcal{E}(v)^{1/2} \tag{2.17}$$

in the quotient space $\mathcal{V}/\mathcal{R}$ and we define in this way a *norm* (a pre-Hilbertian norm).

We then introduce the *Hilbert space* V defined by

$$V = \text{completion of } \mathcal{V}/\mathcal{R} \text{ for the norm (2.17).} \tag{2.18}$$

The main point is now to study the structure of V. Before starting this study, a few remarks are in order.

Remark 2.4. The key point of Lemma 2.3 is the following uniqueness property:

$$v \in \mathcal{V},\ Pv_3 = 0 \quad \text{implies} \quad v_3 = 0. \tag{2.19}$$

The hypotheses made in Lemma 2.3 are that P is elliptic and that $v_3 = 0$ on $\partial\Omega$. Then (2.19) is true. But of course one can have a similar conclusion under completely different hypotheses. We shall return to this in Section 3.

Remark 2.5. There is a natural example of different structure where (2.19) is still true (cf. Section 3). We assume again that P is elliptic and that

$$\mathcal{V} = \{\text{smooth vectors } v \text{ such that } v_3 = \partial v_3/\partial n = 0 \text{ on } \Gamma_0 \subset \partial\Omega\}. \tag{2.20}$$

Then (2.19) holds true according to the Cauchy Uniqueness Theorem.

2.4. Continuous linear forms on V. As we said in the Introduction, the equilibrium of the thin shell is given by the solution of

$$\inf\left(\tfrac{1}{2}\mathcal{E}(v) - \langle\varphi, v\rangle\right) \tag{2.21}$$

where $v \to \langle\varphi, v\rangle$ is a continuous linear form on V, i.e. on $\mathcal{V}/\mathcal{R}$ provided with the norm $\mathcal{E}(v)^{1/2}$.

We then introduce the space

$$\{\mathcal{D}(\Omega)^3, \mathcal{R}\} = \{\varphi \mid \varphi \in \mathcal{D}(\Omega)^3,\ \langle\varphi, r\rangle = 0,\ \forall r \in \mathcal{R}\}, \tag{2.22}$$

i.e.

$$\langle\varphi_1, 1\rangle = \langle\varphi_2, 1\rangle = 0, \quad \langle\varphi_1, x_2\rangle - \langle\varphi_2, x_1\rangle = 0. \tag{2.23}$$

We are now going to prove the following result:

Theorem 2.1. *Assume that P is elliptic (i.e. (2.14) holds true). Let φ be given in $\{\mathcal{D}(\Omega)^3, \mathcal{R}\}$ (defined in (2.22)). Consider the linear form*

$$v \to \langle\varphi, v\rangle \text{ defined on } \mathcal{V}/\mathcal{R}. \tag{2.24}$$

In order that this form be continuous for the norm $\mathcal{E}(v)^{1/2}$ it is necessary that

$$\langle\varphi, \sigma\rangle = 0 \tag{2.25}$$

for an infinite number of independent functions σ in $L^2(\Omega)^3$.

REMARK 2.6. Condition (2.25) will be made more explicit in the proof to follow. The condition that we obtain seems to be sufficient.

PROOF OF THEOREM 2.1.

STEP 1. We consider the mapping $v \to \gamma(v)$ (defined in (2.1)) from $\mathcal{V}/\mathcal{R}$ to $(L^2(\Omega))^3$. By extension by continuity it defines an isometry from V to a closed subspace of $L^2(\Omega)^3 = \gamma(V)$. Now, if the linear form (2.24) is continuous then

$$\omega \to \langle \gamma^{-1}(\omega), \varphi \rangle$$

is continuous on $\gamma(V) \subset L^2(\Omega)^3$. If $\mathcal{O} \in L^2(\Omega)^3$ is an extension of this mapping, we have

$$\langle \gamma^{-1}(\omega), \varphi \rangle = (\mathcal{O}, \omega)_{L^2(\Omega)^3},$$

i.e. *there exists* $\mathcal{O} \in L^2(\Omega)^3$ *such that*

$$(\mathcal{O}, \gamma(v)) = \langle \varphi, v \rangle \quad \forall v \in \mathcal{V}/\mathcal{R}. \tag{2.26}$$

We are now going to show that if $\mathcal{O}$ exists then φ necessarily satisfies an infinite number of conditions (2.25).

STEP 2. We choose first $v = \{v_1, 0, 0\} \in \mathcal{V}$. Then (2.26) should hold true for $v + r, r \in \mathcal{R}$, and it reduces to

$$(\mathcal{O}_1, \partial_1 v_1) + \big(\mathcal{O}_3, \tfrac{1}{2}\partial_2 v_1\big) = (\varphi_1, v_1). \tag{2.27}$$

Therefore

$$-\partial_1 \mathcal{O}_1 - \tfrac{1}{2}\partial_2 \mathcal{O}_3 = \varphi_1 \quad \text{in } \Omega. \tag{2.28}$$

If $n = \{n_1, n_2\}$ denotes the unit normal to $\partial\Omega$ oriented towards the exterior of Ω, it follows from $\mathcal{O} \in L^2(\Omega)^3$ and (2.28) that $n_1\mathcal{O}_1 + \frac{1}{2}n_2\mathcal{O}_3|_{\partial\Omega}$ is defined (in $H^{-1/2}(\partial\Omega)$; cf. J. L. Lions and E. Magenes [1968]). Since in (2.27), v_1 is arbitrary in $\overline{\Omega}$, it follows that

$$n_1\mathcal{O}_1 + \tfrac{1}{2}n_2\mathcal{O}_3 = 0 \quad \text{on } \partial\Omega. \tag{2.29}$$

In a similar way, we take $v = \{0, v_2, 0\} \in \mathcal{V}$ in (2.26), which leads to

$$-\partial_2 \mathcal{O}_2 - \tfrac{1}{2}\partial_1 \mathcal{O}_3 = \varphi_2 \quad \text{in } \Omega, \tag{2.30}$$

$$n_2\mathcal{O}_2 + \tfrac{1}{2}n_1\mathcal{O}_3 = 0 \quad \text{on } \partial\Omega. \tag{2.31}$$

We now take in (2.26)

$$\begin{aligned} &v = \{0, 0, v_3\}, \quad v_3 \text{ smooth in } \overline{\Omega}, \\ &v_3 = 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.32}$$

We obtain

$$-\partial_1^2(\psi\mathcal{O}_1) - \partial_2^2(\psi\mathcal{O}_2) - \partial_1\partial_2(\psi\mathcal{O}_3) = \varphi_3. \tag{2.33}$$

We transform (2.33) as follows. It can be written equivalently as

$$
\begin{aligned}
(2.34)\qquad -\partial_1\big(\psi(\partial_1\mathcal{O}_1+\tfrac12\partial_2\mathcal{O}_3)\big)-\partial_2\big(\psi(\partial_2\mathcal{O}_2+\tfrac12\partial_1\mathcal{O}_3)\big)&\\
-(\partial_1\psi)\big(\partial_1\mathcal{O}_1+\tfrac12\partial_2\mathcal{O}_3\big)&\\
-(\partial_2\psi)\big(\partial_2\mathcal{O}_2+\tfrac12\partial_1\mathcal{O}_3\big)&\\
-\mathcal{O}_1\partial_1^2\psi-\mathcal{O}_2\partial_2^2\psi-\mathcal{O}_3\partial_1\partial_2\psi=\varphi_3.&
\end{aligned}
$$

Using (2.28), (2.30) and the notation (2.7), (2.34) gives

$$\partial_1(\psi\varphi_1)+\partial_2(\psi\varphi_2)+(\partial_1\psi)\varphi_1+(\partial_2\psi)\varphi_2-(a_1\mathcal{O}_1+a_2\mathcal{O}_2+a_3\mathcal{O}_3)=\varphi_3.$$

We set

$$\varphi_3^*=\varphi_3-\partial_1(\psi\varphi_1)-\partial_2(\psi\varphi_2)-(\partial_1\psi)\varphi_1-(\partial_2\psi)\varphi_2. \tag{2.35}$$

Then

$$a\mathcal{O}=-\varphi_3^* \quad\text{where}\quad a\mathcal{O}=a_1\mathcal{O}_1+a_2\mathcal{O}_2+a_3\mathcal{O}_3. \tag{2.36}$$

No surface integral appears because of (2.29), (2.31) and of $v_3=0$ on $\partial\Omega$.

Therefore, if (2.24) is continuous for $\|v\|_V$, then there exists $\mathcal{O}\in L^2(\Omega)^3$ such that (2.28), (2.30), (2.36) hold true with the boundary conditions (2.29), (2.31).

We are now going to show in the following two steps that this is in general impossible, i.e. it is not true for every $\varphi\in\{\mathcal{D}(\Omega)^3,\mathcal{R}\}$.

STEP 3. We introduce $\varrho=\{\varrho_1,\varrho_2,\varrho_3\}$ smooth in $\overline{\Omega}$ such that

$$
\begin{aligned}
&\partial_1\varrho_1+a_1\varrho_3=0,\\
&\partial_2\varrho_2+a_2\varrho_3=0,\\
&\tfrac12(\partial_2\varrho_1+\partial_1\varrho_2)+a_3\varrho_3=0\quad\text{in }\Omega.
\end{aligned} \tag{2.37}
$$

We shall show in Step 4 that such a choice is possible in *infinitely many ways.*

We multiply (2.28), (2.30), (2.36) by $\varrho_1,\varrho_2,\varrho_3$ respectively.

Let us set for the moment

$$\lambda=(\varphi_1,\varrho_1)+(\varphi_2,\varrho_2)-(\varphi_3^*,\varrho_3). \tag{2.38}$$

Then

$$\big(-\partial_1\mathcal{O}_1-\tfrac12\partial_2\mathcal{O}_3,\varrho_1\big)+\big(-\partial_2\mathcal{O}_2-\tfrac12\partial_1\mathcal{O}_3,\varrho_2\big)+(a_1\mathcal{O}_1+a_2\mathcal{O}_2+a_3\mathcal{O}_3,\varrho_3)=\lambda.$$

Using the boundary conditions (2.29), (2.31) and the properties (2.37) it follows that $\lambda=0$. Therefore the linear form (2.24) *may* be continuous for the norm $\|v\|_V$ only if $\varphi\in\{\mathcal{D}(\Omega)^3,\mathcal{R}\}$ satisfies $\lambda=0$, i.e.

$$(\varphi_1,\varrho_1)+(\varphi_2,\varrho_2)-(\varphi_3^*,\varrho_3)=0\quad\forall\varrho\text{ satisfying (2.37)}. \tag{2.39}$$

STEP 4. We introduce the space

$$\mathcal{Z}=\{w\mid w\in H^1(\Omega),\ Pw=0\}. \tag{2.40}$$

We define in this way an *infinite-dimensional* space. Indeed, given $g \in H^{1/2}(\partial\Omega)$, we can solve

$$Pw = 0, \quad w = g \quad \text{on } \partial\Omega \tag{2.41}$$

which admits a unique solution in $H^1(\Omega)$. Therefore

$$\mathcal{Z} \textit{ is isomorphic to } H^{1/2}(\partial\Omega) \tag{2.42}$$

(the isomorphism being defined through (2.41)).

We now choose

$$\varrho_3 = w \in \mathcal{Z}. \tag{2.43}$$

We then choose ϱ_2 such that

$$\partial_2\varrho_2 = -a_2\varrho_3, \quad \partial_1^2\varrho_2 = \partial_2(a_1\varrho_3) - 2\partial_2(a_3\varrho_3). \tag{2.44}$$

This is possible since $\partial_1^2(-a_2\varrho_3) = \partial_2(\partial_2(a_1\varrho_3) - 2\partial_1(a_3\varrho_3))$, which is equivalent to $P\varrho_3 = 0$. Then we choose ϱ_1 such that

$$\partial_1\varrho_1 + a_1\varrho_3 = 0, \quad \tfrac{1}{2}(\partial_2\varrho_1 + \partial_1\varrho_2) + a_3\varrho_3 = 0.$$

This is possible since $\partial_2(a_1\varrho_3) = \partial_1(2a_3\varrho_3 + \partial_1\varrho_2)$, which is $(2.44)_2$.

Therefore,

(2.45) *for any* $g \in H^{1/2}(\partial\Omega)$ *there exists* $\varrho \in (H^1(\Omega))^3$ *such that* (2.37) *holds.*

(In fact, we have a little more regularity on ϱ than being in $H^1(\Omega)^3$ but this is not useful.)

We can now conclude the proof. We write (2.39) as (2.25) where

$$\begin{aligned}
\sigma &= \{\sigma_1, \sigma_2, \sigma_3\},\\
\sigma_1 &= \varrho_1 - \psi\partial_1\varrho_3 + (\partial_1\psi)\varrho_3,\\
\sigma_2 &= \varrho_2 - \psi\partial_2\varrho_3 + (\partial_2\psi)\varrho_3,\\
\sigma_3 &= \varrho_3,
\end{aligned}$$

where ϱ_3 spans the infinite-dimensional space $\mathcal{Z}$ (cf. (2.40), (2.42)).

The proof is complete. □

3. Some generalizations

3.1. Abstract result. We introduce

$$\mathcal{V}_B = \{v \mid v \text{ smooth vectors } \overline{\Omega} \to \mathbb{R}^3 \text{ which satisfy the boundary conditions } Bv_3 = 0 \text{ on } \partial\Omega\} \tag{3.1}$$

where

(3.2) B is a linear operator from smooth functions in $\overline{\Omega}$ to $C(\partial\Omega)$ or $C(\partial\Omega)^2$, where $C(\partial\Omega)$ = space of continuous functions on $\partial\Omega$.

We notice that the boundary conditions involve *only* v_3.

We shall assume (using the notations (2.7), (2.8))

$$Pv_3 = 0, Bv_3 = 0 \quad \text{implies} \quad v_3 = 0 \tag{3.3}$$

and

(3.4) *the space* $\{w \mid w$ *smooth in* $\overline{\Omega}, Pw = 0$ *in* $\Omega\}$ *is infinite-dimensional.*

We can verify, following the steps of the proof of Theorem 2.1, that the result of Theorem 2.1 still holds true, namely

Theorem 3.1. *Assume that* (3.3) *and* (3.4) *hold true. Denote by* V_B *the completion of* $\mathcal{V}_B/\mathcal{R}$ *for the norm* (2.17). *Consider the linear form*

$$v \to \langle \varphi, v\rangle \quad \textit{defined on } \mathcal{V}/\mathcal{R}.$$

In order that this form be continuous for the norm $\mathcal{E}(v)^{1/2}$, *it is necessary that*

$$\langle \varphi, \sigma\rangle = 0$$

for an infinite number of independent functions σ *in* $L^2(\Omega)^3$.

3.2. Applications of Theorem 3.1

Example 3.1. We assume as in Section 2 that P is elliptic (strongly elliptic in the sense of (2.14)) and that

$$Bw = \{w, \partial w/\partial n\} \quad \text{on } \Gamma_0 \subset \partial\Omega. \tag{3.5}$$

Then (3.3), (3.4) hold true.

Remark 3.1. Example 3.1 has a physical interpretation. It corresponds, after symmetrization (see (1.10), (1.11)), to the case when

- we impose on a part Γ_0 of the edge the condition that the angle of S^+ and S^- remains invariant under the deformation (in addition to the classical condition $\widehat{u}^+ = \widehat{u}^-$),
- the two lips of the edge are free on $\partial\Omega \setminus \Gamma_0$. In this case, (1.4) is only assumed to hold on Γ_0.

4. Concluding remarks

Remark 4.1. System (1.16) (or (2.2), which is more explicit) is non-Kovalevskian. It appears that the principal symbol obtained by the substitution $\partial_1 \to i\xi_1, \partial_2 \to i\xi_2$, with $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{O\}$, has a determinant which vanishes identically. At each point (x_1, x_2), the characteristic directions are normal to the ξ for which this determinant vanishes. Thus, any curve of the plane is characteristic and the Cauchy–Kovalevskaya theorem never applies.

Remark 4.2. A consequence of the non-Kovalevskian character of system (2.2) is that when eliminating v_1, v_2 one gets a second (not fourth!) order equation in v_3 (see (2.8) and (2.13)). This property of the lowering of the order by elimination is shared by several systems issuing from differential geometry (cf. Janet [1929]).

Remark 4.3. The characteristic directions of the operator P (see (2.8)) at each point (x_1, x_2) coincide with the asymptotic directions of the surface S (or S_δ) at the corresponding point $(x_1, x_2, \psi(x_1, x_2))$. Thus the surface and the operator are of the same type (elliptic or hyperbolic) at corresponding points.

Remark 4.4. We proved that, under the appropriate hypothesis, the space V is not contained in $\mathcal{D}'(\Omega)^3/\mathcal{R}$ (Theorems 2.1 and 3.1). Consequently, there exist sequences v^n of smooth functions (contained in $\mathcal{V}$) such that

$$\gamma^n_{\alpha\beta} \equiv \gamma_{\alpha\beta}(v^n) \text{ converges in } L^2(\Omega),\ \alpha, \beta = 1, 2, \tag{4.1}$$

but

$$v^n \text{ does not converge in } \mathcal{D}'(\Omega)^3/\mathcal{R}. \tag{4.2}$$

In particular,

$$v^n_3 \text{ does not converge in } \mathcal{D}'(\Omega). \tag{4.3}$$

On the other hand, v^n_3 is the solution of the Dirichlet problem (see (2.13))

$$Pv^n_3 = \partial^2_2\gamma^n_{11} + \partial^2_1\gamma^n_{22} - 2\partial_1\partial_2\gamma^n_{12}, \tag{4.4}$$

$$v^n_3 = 0 \quad \text{on } \partial\Omega, \tag{4.5}$$

where P denotes the second order operator defined in (2.8). It should be noticed that according to (4.1) the right hand side of (4.4) converges in $H^{-2}(\Omega)$, and this does not imply that v^n_3 converges in a space of distributions on Ω.

Indeed, let us consider in general the problem

$$Pu = f, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4.6}$$

where f is given in $H^{-2}(\Omega)$. Using the transposition method, as in J. L. Lions and E. Magenes [1968], one solves first

$$P^*\varphi = \psi, \qquad \varphi = 0 \quad \text{on } \partial\Omega, \tag{4.7}$$

where ψ is in a “suitable” function space and then one “defines” u by

$$\langle u, \psi\rangle = \langle f, \varphi\rangle. \tag{4.8}$$

If $\psi \to \langle f, \varphi \rangle$ is a continuous linear form on the space Ψ of functions ψ, then (4.8) defines u as an element of Ψ', the dual space of Ψ. If f can be an arbitrary element of $H^{-2}(\Omega)$, then in order that the form $\psi \to \langle f, \varphi \rangle$ be continuous, we must have

(4.9) $\psi \to \varphi$ is a continuous mapping from Ψ to $H_0^2(\Omega)$ (= dual space of H^{-2}).

But if we choose $\Psi = \mathcal{D}(\Omega)$ then φ is C^∞ in Ω; it can be $C^\infty(\overline{\Omega})$ if the coefficients of P (or P^*) are C^∞ and if $\Gamma = \partial\Omega$ is C^∞; of course $\varphi = 0$ on Γ; in general $\partial\varphi/\partial n$ is not 0 on Γ.

Other conditions, of a global nature, are needed on $\psi \in \mathcal{D}(\Omega)$ in order to have $\partial\varphi/\partial n = 0$ on Γ. Therefore, except if f satisfies (infinitely many) conditions of a global nature, the solution u of (4.6) is not a distribution on Ω.

The present remark is *independent* of the boundary conditions which can be imposed on v_1 or on v_2.

It gives a new proof of the fact that, in the elliptic case, the space V is not a space of distributions on Ω, *if* one can check that

$$\partial_2^2\gamma_{11} + \partial_1^2\gamma_{22} - 2\partial_1\partial_2\gamma_{12}$$

can be *any* element of $H^{-2}(\Omega)$, or, at least, can be an element of $H^{-2}(\Omega)$ so that (4.6) has no distribution solution. Actually, we need a sequence of "good" functions which approximate (cf. (4.1)) such an element of $H^{-2}(\Omega)$.

REMARK 4.5. We now give an explicit example of a Cauchy sequence v^n of V which does not converge in $\mathcal{D}'(\Omega)/\mathcal{R}$. In fact, we shall prove that v_3^n does not converge in $\mathcal{D}'(\Omega)$ (v_3^n tends to ∞ on compact subsets of Ω), giving an example of the situation considered in Remark 4.4.

Let Ω be the unit circle and let S be the elliptic paraboloid

$$\psi = \tfrac{1}{2}(x_1^2 + x_2^2 - 1) = \tfrac{1}{2}(r^2 - 1) \tag{4.10}$$

where $r^2 = x_1^2 + x_2^2$; for the sake of simplicity, we shall not change the name of the functions when expressing them in cartesian or polar coordinates: $\varphi(x_1, x_2)$ or $\varphi(r)$ are the two expressions in (4.10).

Let $g(r)$ be an increasing function defined on $[0,1]$, vanishing for $r \le 1/2$ and tending to ∞ as $(1-r)^{-\beta}$ with $0 < \beta < 1/2$ for $r \to 1$. Let us also define

$$g^n(r) = \begin{cases} g(r) & \text{for } r < 1 - 1/n, \\ g(1-1/n) & \text{for } r > 1 - 1/n. \end{cases} \tag{4.11}$$

Clearly, $g^n \in L^2(\Omega)$ (and even $H^1(\Omega)$) and

$$g^n \to g \quad \text{in } L^2(\Omega). \tag{4.12}$$

Now, we shall construct (v_1^n, v_2^n, v_3^n) such that the corresponding $\gamma_{\alpha\beta}^n \equiv \gamma_{\alpha\beta}(v^n)$ (see (2.2)) are

$$\gamma_{11}^n = g^n(r), \quad \gamma_{22}^n = g^n(r), \quad \gamma_{12}^n = 0. \tag{4.13}$$

In other words, we wish to construct (v_1^n, v_2^n, v_3^n) satisfying

$$\begin{aligned} &\partial_1 v_1^n - \psi \partial_1^2 v_3^n = g^n(r), \\ &\partial_2 v_2^n - \psi \partial_2^2 v_3^n = g^n(r), \\ &\tfrac{1}{2}(\partial_2 v_1^n + \partial_1 v_2^n) - \psi \partial_1 \partial_2 v_3^n = 0, \end{aligned} \tag{4.14}$$

with the boundary condition

$$v_3^n = 0 \quad \text{on } \partial\Omega. \tag{4.15}$$

In order to solve (4.14), (4.15), we eliminate v_1^n and v_2^n and we obtain (see (2.13) and (2.8) with $a_1 = a_2 = 1, a_3 = 0$, if necessary)

$$-\Delta v_3^n = \Delta g^n, \tag{4.16}$$

so that v_3^n is uniquely defined by (4.15), (4.16). (Note that this is a classical problem as $g^n \in H^1(\Omega)$.)

The solution only depends on r and is

$$v_3^n(r) = -g^n(r) + g^n(1). \tag{4.17}$$

When v_3^n is defined in this way, (4.14) defines v_1^n, v_2^n up to a rigid displacement in the plane (this is a classical property of kinematics of continuous media, which is easily checked as (4.16) is the compatibility condition for (4.13) considered as a system for the unknowns v_1^n, v_2^n). On account of (4.12), the sequence $v^n = (v_1^n, v_2^n, v_3^n)$ constitutes a Cauchy sequence in the space V defined in (2.18). Nevertheless, v_3^n does not converge in $\mathcal{D}'(\Omega)$; indeed,

$$v_3^n = \begin{cases} -g(r) + g(1 - 1/n) & \text{for } r < 1 - 1/n, \\ 0 & \text{for } r > 1 - 1/n, \end{cases} \tag{4.18}$$

tends to ∞ on any compact subset of Ω.

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