Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 8, 1996, 407–412

A SELECTION THEOREM FOR MAPPINGS WITH NONCONVEX NONDECOMPOSABLE VALUES IN L_p-SPACES

Dušan Repovš[1] — Pavel V. Semenov[2]

1. Introduction

Let T be a set equipped with a probability measure μ, B a Banach space and $L_p = L_p(T, B)$ the Banach space of all (equivalence classes of) p-summable mappings from T into B with the usual norm:

$$\|f\| = \left(\int_T |f(t)|_B^p \, d\mu \right)^{1/p}, \quad 1 \leq p < \infty.$$

For every subset $E \subset B$ we define

$$L_p(T, E) = \{ f \in L_p(T, B) \mid f(t) \in E \text{ almost everywhere} \}.$$

If E is a convex subset of B then $L_p(T, E)$ is a convex subset of $L_p(T, B)$. For an arbitrary subset E of B one can, in general, state only the *decomposability* of the set $L_p(T, E)$ in the Banach space $L_p(T, B)$. Recall that by [4] decomposability of $Z \subset L_p(T, B)$ means that for every $f \in Z$, $g \in Z$ and for every μ-measurable subset $A \subset T$, the function which agrees with f over A and with g over $T \setminus A$ is also an element of Z. In [1], [2], and [3] selection theorems were proved for

1991 *Mathematics Subject Classification.* Primary 54C65, 52A30; Secondary 54C60, 26E25.
Key words and phrases. Selections of multivalued maps, Banach spaces, α-paraconvex sets, nondecomposable sets, semi-convex sets, paracompact spaces.
[1]Supported in part by Ministry of Science and Technology of the Republic of Slovenia grant No. J1-7039-0101-95.
[2]Supported in part by the G. Soros International Science Foundation Grant No. NFU000.

decomposable valued lower semicontinuous mappings into spaces $L_1(T,B)$ with nonatomic measure μ and separable B. In other words, decomposability looks like a suitable substitute for convexity in L_1-spaces (cf. [1], [6]).

In the present note we shall consider multivalued mappings whose values are unions of two intersecting sets $L_p(T,E_1)$ and $L_p(T,E_2)$, where E_1 and E_2 are convex. Sets of such type are, in general, nondecomposable and nonconvex. However, we shall prove that a selection theorem for lower semicontinuous mappings holds also in this case under some additional restrictions on E_1 and E_2.

Definition 1.1. A subset $W \subset L_p(T,B)$, $p \geq 1$, is said to be *semiconvex* if

$$W = L_p(T,E_1) \cup L_p(T,E_2)$$

for some nonempty closed convex subsets $E_1 \subset B$, $E_2 \subset B$ with a convex union $E_1 \cup E_2$.

Definition 1.2. A subset $W \subset L_p(T,B)$, $p \geq 1$, is said to be *strongly semiconvex* if

$$W = L_p(T,E_-) \cup L_p(T,E_+)$$

for some nonempty closed convex $E_- \subset B$, $E_+ \subset B$ with a convex union $E_- \cup E_+$ such that $\ell(E_-) \leq c$ and $\ell(E_+) \geq c$, for some $c \in \mathbb{R}$ and for some continuous linear functional $\ell : B \to \mathbb{R}$.

Theorem 1.3. *Every lower semicontinuous mapping from a paracompact space into the space $L_p(T,B)$, $p \geq 1$, with strongly semiconvex values admits a continuous singlevalued selection.*

Theorem 1.3 is a direct corollary of the following theorems:

Theorem 1.4. *Every strongly semiconvex subset of $L_p(T,B)$ is α-paraconvex in $L_p(T,B)$, where $\alpha = (1+2^{-1/p})/2 \in [0,1)$.*

Theorem 1.5. *For every $\alpha \in [0,1)$, every lower semicontinuous α-paraconvex valued mapping from a paracompact space into a Banach space admits a continuous singlevalued selection.*

Theorem 1.5 was proved by Michael [5] where the notion of paraconvexity was also introduced.

Definition 1.6. Let $\alpha \in [0,1)$. A nonempty closed subset P of a normed space E is said to be *α-paraconvex* if for every open ball D with radius r and with $D \cap P \neq \emptyset$, the inequality

$$\operatorname{dist}(q,P) \leq \alpha r$$

holds, for all q from the convex hull $\operatorname{conv}(D \cap P)$.

If f is a Lipschitz function (with some constant k) in n variables and with a convex closed domain, then its graph is an α-paraconvex subset of $\mathbb{R}^{n+1}$, for some $\alpha = \alpha(k,n) < 1$ (see [7]). Another example of a paraconvex subset in the Hilbert space is given by a bouquet of convex sets (see [8]).

We conclude the introduction by two open questions:

QUESTION 1.7. *Let B be a Banach space and $\mathcal{L}$ the family of all of its subsets which admit a representation as the union of two closed convex sets. Is it then true that every lower semicontinuous mapping $F : X \to \mathcal{L}$ from a paracompact space X with equi-LC^0 family $\{F(x)\}_{x\in X}$ of values must always have a selection?*

It is easy to show that in the *Hilbert space* a sufficient condition is that the set of "angles" between two closed convex sets above has a positive lower bound (see [8]).

QUESTION 1.8. *Does there exist a suitable* (*for selection theory*) *notion of paradecomposability, i.e. a controlled version of the weakening of the concept of decomposability?*

As a test one can consider the case of the union $L_p(T, E_1) \cup L_p(T, E_2)$, for nonconvex E_1 and E_2, with E_1 and E_2 separated by a hyperplane.

2. Preliminaries

Given a multivalued mapping $F : X \to Y$ with nonempty values, a *selection* for F is a continuous singlevalued mapping $f : X \to Y$ such that $f(x) \in F(x)$, for each $x \in X$. A multivalued mapping $F : X \to Y$ is said to be *lower semicontinuous* if $\{x \in X \mid F(x) \cap U \neq \emptyset\}$ is open in X whenever U is open in Y.

LEMMA 2.1. *Let P be a closed nonempty subset of a normed space $(E, \|\cdot\|)$, let $x \in P$, $y \in P$, and let*

$$\operatorname{dist}(z_0, P) \leq \alpha r, \quad 0 \leq \alpha < 1,$$

where $2z_0 = x + y$ and $\|x - y\| = 2r$. Then

$$\operatorname{dist}(z, P) \leq \beta r$$

for all $z \in [x, y]$, where $\beta = (1 + \alpha)/2$.

Let W be a strongly semiconvex subset of $L_p(T, B)$ and let

$$W = L_p(T, E_-) \cup L_p(T, E_+)$$

with $E_-, E_+, c \in \mathbb{R}$ and $\ell : B \to \mathbb{R}$ from Definition 1.2. Set $E_0 = E_- \cap E_+$, $W_- = L_p(T, E_-)$, $W_+ = L_p(T, E_+)$ and $W_0 = L_p(T, E_0)$. Clearly, $E_0 = (E_- \cup E_+) \cap \Pi$ where Π is the hyperplane $\{x \in B \mid \ell(x) = c\}$. Observe that $E_- \setminus E_0 = \emptyset$ implies that $E_- \subset E_+$ and $W_- \subset W_+$, i.e. that W is convex. Thus we can assume that $E_- \setminus E_0 \neq \emptyset$.

Lemma 2.2. *Let D be an open ball in $L_p(T,B)$ whose intersection with W is nonconvex. Then the convex hull* $\operatorname{conv}(D \cap W)$ *equals the union of segments:*

$$\bigcup\{[f_-, f_+] \mid f_+ \in W_+ \cap D,\ f_- \in (W_- \setminus W_0) \cap D\}.$$

In order to prove Theorem 1.4 it suffices, by Lemmas 2.1 and 2.2, to show that $\operatorname{dist}(g,W) \le 2^{-1/p}r$ for $2g = f_- + f_+$ with $f_+ \in W_+ \cap D$, $f_- \in (W_- \setminus W_0) \cap D$ and $\|f_- - f_+\| = 2r$.

3. Proofs

Proof of Theorem 1.4. We assume that $f_+ \in W_+ \cap D$ and $f_- \in (W_- \setminus W_0) \cap D$ are mappings from T into $E_- \cup E_+$ with $\|f_- - f_+\| = 2r$ and with $f_+(t) \in E_+$ for almost every $t \in T$ and $f_-(t) \in E_- \setminus E_0$ for almost every $t \in T$, respectively. So, the segment $[f_-(t), f_+(t)]$ intersects the hyperplane Π, for almost every $t \in T$. Because of the convexity of $E_- \cup E_+$ the intersection $\Pi \cap [f_-(t), f_+(t)]$ lies in E_0 and by the assumption $f_-(t) \notin \Pi$, this intersection is a singleton. So, we define a mapping $f_0 : T \to E_0$ by setting $f_0(t) = \Pi \cap [f_-(t), f_+(t)]$, for almost every $t \in T$. Clearly,

$$|g(t) - f_0(t)|_B \le |f_-(t) - f_+(t)|_B/2$$

because $g(t)$ is the middle point of the segment $[f_-(t), f_+(t)]$.

Define mappings $g_+ : T \to E_+$ and $g_- : T \to E_-$ by setting

$$g_+(t) = \begin{cases} g(t) & \text{if } g(t) \in E_+, \\ f_0(t) & \text{if } g(t) \in E_- \setminus E_0, \end{cases}$$

and

$$g_-(t) = \begin{cases} g(t) & \text{if } g(t) \in E_- \setminus E_0, \\ f_0(t) & \text{if } g(t) \in E_+. \end{cases}$$

Assertion 3.1. *The mappings f_0, g_+, g_- are elements of $L_p(T,B)$.*

By Assertion 3.1 we have $g_+ \in W_+ \subset W$ and $g_- \in W_- \subset W$. Thus

$$\operatorname{dist}(g,W) \le \min\{\|g - g_+\|, \|g - g_-\|\}.$$

Let us estimate the right hand side of the inequality above:

$$\begin{aligned}\|g-g_+\|^p+\|g-g_-\|^p &= \int_T |g(t)-g_+(t)|_B^p\,d\mu+\int_T |g(t)-g_-(t)|_B^p\,d\mu\\ &= \int_{\{t|g(t)\in E_-\setminus E_0\}} |g(t)-f_0(t)|_B^p\,d\mu\\ &\quad+\int_{\{t|g(t)\in E_+\}} |g(t)-f_0(t)|_B^p\,d\mu\\ &= \int_T |g(t)-f_0(t)|_B^p\,d\mu\\ &\le \int_T |f_-(t)-f_+(t)|_B^p/2^p\,d\mu=\|f_--f_+\|^p/2^p=r^p.\end{aligned}$$

Hence

$$\min\{\|g-g_+\|,\|g-g_-\|\}\le (r^p/2)^{1/p}=2^{-1/p}r.$$

This completes the proof of Theorem 1.4. □

PROOF OF LEMMA 2.1. If $\|z-z_0\|\le(\beta-\alpha)r$, then

$$\operatorname{dist}(z,P)\le\|z-z_0\|+\operatorname{dist}(z_0,P)\le\beta r$$

by the triangle inequality. If $\|z-z_0\|>(\beta-\alpha)r$, then z is βr-close to $x\in P$ or z is βr-close to $y\in P$. □

PROOF OF LEMMA 2.2. The inclusion

$$\bigcup\{[f_-,f_+]\mid f_+\in W_+\cap D,\ f_-\in(W_-\setminus W_0)\cap D\}\subset\operatorname{conv}(D\cap W)$$

is obvious. To prove the other inclusion, let us fix $f=\sum_{i=1}^n\lambda_i f_i$ with $\lambda_i>0$ and $\sum_{i=1}^n\lambda_i=1$, for some $n\in\mathbb{N}$ and for some $\{f_1,\dots,f_n\}\in D\cap W$. If all $f_1,\dots,f_n$ are elements of $D\cap W_+$ then $f\in D\cap W_+$ and hence $f\in[f_-,f]$, for an arbitrary $f_-\in(W_-\setminus W_0)\cap D$. If $\{f_1,\dots,f_k\}\subset D\cap W_+$, $\{f_{k+1},\dots,f_n\}\subset(W_-\setminus W_0)\cap D$ and $k<n$ then

$$\begin{aligned}f&=(\lambda_1+\dots+\lambda_k)\Big(\frac{\lambda_1}{\lambda_1+\dots+\lambda_k}f_1+\dots+\frac{\lambda_k}{\lambda_1+\dots+\lambda_k}f_k\Big)\\ &\quad+(\lambda_{k+1}+\dots+\lambda_n)\Big(\frac{\lambda_{k+1}}{\lambda_{k+1}+\dots+\lambda_n}f_{k+1}+\dots+\frac{\lambda_n}{\lambda_{k+1}+\dots+\lambda_n}f_n\Big)\\ &=\lambda f_++(1-\lambda)f_-\end{aligned}$$

where $0<\lambda<1$ and $f_+\in D\cap W_+$, $f_-\in D\cap(W_-\setminus W_0)$, by the convexity of D, W_+ and $(W_-\setminus W_0)$. □

PROOF OF ASSERTION 3.1. The mapping $f_0:T\to E_0$ admits an analytic expression

$$f_0(t)=(1-\lambda(t))f_-(t)+\lambda(t)f_+(t)$$

where

$$0<\lambda(t)=\frac{c-\ell(f_-(t))}{\ell(f_+(t)-f_-(t))}\le 1\,.$$

But $f_+, f_- \in L_p(T,B)$ and $\ell : B \to \mathbb{R}$ is continuous. Hence $\lambda \in L_p(T,\mathbb{R})$ and thus $f_0 \in L_p(T,B)$.

Let $T_c = \{t \in T \mid \ell(g(t)) \geq c\}$. Then T_c is a μ-measurable subset of T, since $g \in L_p(T,B)$ and ℓ is continuous. So, the characteristic function κ_c of the set T_c is a simple measurable function. Therefore

$$g_+ = \kappa_c g + (1-\kappa_c) f_0 \in L_p(T,B)$$

and

$$g_- = \kappa_c f_0 + (1-\kappa_c) g \in L_p(T,B).$$

□

References

[1] A. Bressan and G. Colombo, *Extensions and selections of maps with decomposable values*, Studia Math. **40** (1988), 69–86.

[2] A. Fryszkowski, *Continuous selections for a class of non-convex multi-valued maps*, Studia Math. **76** (1983), 163–174.

[3] V. V. Goncharov and A. A. Tolstonogov, *Compatible continuous selectors of multivalued mappings with nonconvex values and their applications*, Mat. Sb. **182** (1991), 946–949. (Russian)

[4] F. Hiai and H. Umegaki, *Integrals, conditional expectations and martingales of multivalued functions*, J. Math. Anal. **7** (1977), 149–182.

[5] E. Michael, *Paraconvex sets*, Math. Scand. **7** (1959), 372–376.

[6] C. Olech, *Decomposability as a substitute for convexity*, Lecture Notes in Math. 1091, Springer-Verlag, Berlin, 1984, pp. 193–205.

[7] D. Repovš and P. V. Semenov, *Paraconvexity of graphs of continuous functions*, Set-Valued Anal. **3** (1995), 23–32.

[8] P. V. Semenov, *On paraconvexity of star-like sets*, Sibirsk. Mat. Zh. (to appear). (Russian)

Manuscript received February 3, 1996

Dušan Repovš
Institute of Mathematics, Physics and Mechanics
University of Ljubljana
P.O.B. 2964
1001 Ljubljana, SLOVENIA

E-mail address: dusan.repovs@uni-lj.si

Pavel V. Semenov
Moscow State Pedagogical University
Ul. M. Pyrogovskaya 1
119882 Moscow, RUSSIA