FIXED POINT THEOREMS AND CHARACTERIZATIONS
OF METRIC COMPLETENESS

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1. Introduction

Let $X$ be a metric space with metric $d$. A mapping $T$ from $X$ into itself is
called contractive if there exists a real number $r \in [0, 1)$ such that $d(Tx, Ty) \leq
rd(x, y)$ for every $x, y \in X$. It is well known that if $X$ is a complete metric space,
then every contractive mapping from $X$ into itself has a unique fixed point in $X$.
However, we exhibit a metric space $X$ such that $X$ is not complete and every
contractive mapping from $X$ into itself has a fixed point in $X$; see Section 4.
On the other hand, in [1], Caristi proved the following theorem: Let $X$ be a
complete metric space and let $\phi : X \to (-\infty, \infty)$ be a lower semicontinuous
function, bounded from below. Let $T : X \to X$ be a mapping satisfying
$$d(x, Tx) \leq \phi(x) - \phi(Tx)$$
for every $x \in X$. Then $T$ has a fixed point in $X$. Later, characterizations of
metric completeness have been discussed by Weston [8], Takahashi [7], Park and
Kang [6] and others. For example, Park and Kang [6] proved the following: Let
$X$ be a metric space. Then $X$ is complete if and only if for every selfmap $T$ of
$X$ with a uniformly continuous function $\phi : X \to [0, \infty)$ such that
$$d(x, Tx) \leq \phi(x) - \phi(Tx)$$

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for every \( x \in X \), \( T \) has a fixed point in \( X \). Recently, Kada, Suzuki and Takahashi [4] introduced the concept of \( w \)-distance on a metric space \( X \) (see Section 2) and improved Caristi’s fixed point theorem [1], Ekeland’s variational principle [3], and the nonconvex minimization theorem according to Takahashi [7].

In this paper, using the concept of \( w \)-distance, we first establish fixed point theorems for set-valued mappings on complete metric spaces which are connected with Nadler’s fixed point theorem [5] and Edelstein’s fixed point theorem [2]. Next, we give characterizations of metric completeness. One of them is as follows: A convex subset \( D \) of a normed linear space is complete if and only if every contractive mapping from \( D \) into itself has a fixed point in \( D \).

2. Preliminaries

Throughout this paper, we denote by \( \mathbb{N} \) the set of positive integers and by \( \mathbb{R} \) the set of real numbers. Let \( X \) be a metric space with metric \( d \). Then a function \( p : X \times X \to [0, \infty) \) is called a \( w \)-distance on \( X \) if the following are satisfied:

1. \( p(x, z) \leq p(x, y) + p(y, z) \) for any \( x, y, z \in X \);
2. for any \( x \in X \), \( p(x, \cdot) : X \to [0, \infty) \) is lower semicontinuous;
3. for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( p(z, x) \leq \delta \) and \( p(z, y) \leq \delta \) imply \( d(x, y) \leq \varepsilon \).

The metric \( d \) is a \( w \)-distance on \( X \). Some other examples of \( w \)-distances are given in [4]. We have the following lemmas regarding \( w \)-distance.

**Lemma 1.** Let \( X \) be a metric space with metric \( d \), let \( p \) be a \( w \)-distance on \( X \), and let \( q \) be a function from \( X \times X \to [0, \infty) \) satisfying (1), (2) in the definition of \( w \)-distance. Suppose that \( q(x, y) \geq p(x, y) \) for every \( x, y \in X \). Then \( q \) is also a \( w \)-distance on \( X \). In particular, if \( q \) satisfies (1), (2) in the definition of \( w \)-distance and \( q(x, y) \geq d(x, y) \) for every \( x, y \in X \), then \( q \) is a \( w \)-distance on \( X \).

**Proof.** We show that \( q \) satisfies (3). Let \( \varepsilon > 0 \). Since \( p \) is a \( w \)-distance, there exists a positive number \( \delta \) such that \( p(z, x) \leq \delta \) and \( p(z, y) \leq \delta \) imply \( d(x, y) \leq \varepsilon \). Then \( q(z, x) \leq \delta \) and \( q(z, y) \leq \delta \) imply \( d(x, y) \leq \varepsilon \). \( \square \)

**Lemma 2.** Let \( F \) be a bounded and closed subset of a metric space \( X \). Assume that \( F \) contains at least two points and \( c \) is a constant with \( c \geq \delta(F) \), where \( \delta(F) \) is the diameter of \( F \). Then the function \( p : X \times X \to [0, \infty) \) defined by

\[
p(x, y) = \begin{cases} d(x, y) & \text{if } x, y \in F, \\ c & \text{if } x \notin F \text{ or } y \notin F, \end{cases}
\]

is a \( w \)-distance on \( X \).
If $x, y, z \in F$, we have
\[ p(x, z) = d(x, z) \leq d(x, y) + d(y, z) = p(x, y) + p(y, z). \]

In the other case, we have
\[ p(x, z) \leq c \leq p(x, y) + p(y, z). \]

Let $x \in X$. If $\alpha \geq c$, we have $\{y \in X : p(x, y) \leq \alpha\} = X$. Let $\alpha < c$. If $x \in F$, then $p(x, y) \leq \alpha$ implies $y \in F$. So, we have
\[ \{y \in X : p(x, y) \leq \alpha\} = \{y \in X : d(x, y) \leq \alpha\} \cap F. \]

If $x \notin F$, we have $\{y \in X : p(x, y) \leq \alpha\} = \emptyset$. In each case, the set $\{y \in X : p(x, y) \leq \alpha\}$ is closed. Therefore $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous.

Let $\varepsilon > 0$. Then there exists $n_0 \in N$ such that $0 < \varepsilon/n_0 < c$. Let $\delta = \varepsilon/(2n_0)$. Then $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $x, y, z \in F$. So, we have
\[ d(x, y) \leq d(x, z) + d(y, z) = p(z, x) + p(z, y) \leq \frac{\varepsilon}{2n_0} + \frac{\varepsilon}{2n_0} = \frac{\varepsilon}{n_0} \leq \varepsilon. \]

Let $\varepsilon \in (0, \infty]$. A metric space $X$ with metric $d$ is called $\varepsilon$-chainable [2] if for every $x, y \in X$ there exists a finite sequence $\{u_0, u_1, \ldots, u_k\}$ in $X$ such that $u_0 = x$, $u_k = y$ and $d(u_i, u_{i+1}) < \varepsilon$ for $i = 0, 1, \ldots, k - 1$. Such a sequence is called an $\varepsilon$-chain in $X$ linking $x$ and $y$.

**Lemma 3.** Let $\varepsilon \in (0, \infty]$ and let $X$ be an $\varepsilon$-chainable metric space with metric $d$. Then the function $p : X \times X \to [0, \infty)$ defined by
\[ p(x, y) = \inf \left\{ \sum_{i=0}^{k-1} d(u_i, u_{i+1}) : \{u_0, u_1, \ldots, u_k\} \text{ is an } \varepsilon \text{-chain linking } x \text{ and } y \right\} \]
is a $w$-distance on $X$.

**Proof.** Note that $p$ is well-defined because $X$ is $\varepsilon$-chainable. Let $x, y, z \in X$ and let $\eta > 0$ be arbitrary. Then there exist $\varepsilon$-chains $\{u_0, u_1, \ldots, u_k\}$ linking $x$ and $y$ and $\{v_0, v_1, \ldots, v_l\}$ linking $y$ and $z$ such that
\[ \sum_{i=0}^{k-1} d(u_i, u_{i+1}) \leq p(x, y) + \eta \quad \text{and} \quad \sum_{i=0}^{l-1} d(v_i, v_{i+1}) \leq p(y, z) + \eta. \]

Since $\{u_0, u_1, \ldots, u_k, v_1, v_2, \ldots, v_l\}$ is an $\varepsilon$-chain linking $x$ and $z$, we have
\[ p(x, z) \leq \sum_{i=0}^{k-1} d(u_i, u_{i+1}) + \sum_{i=0}^{l-1} d(v_i, v_{i+1}) \leq p(x, y) + p(y, z) + 2\eta. \]

Since $\eta > 0$ is arbitrary, we have $p(x, z) \leq p(x, y) + p(y, z)$.

Let us prove (2). Let $x, y \in X$ and let $\{y_n\}$ be a sequence in $X$ with $y_n \to y$. Choose $n_0 \in N$ such that $d(y, y_n) < \varepsilon$ for every $n \geq n_0$. Let $\eta > 0$ be arbitrary.
and let \( n \geq n_0 \). Then there exists an \( \varepsilon \)-chain \( \{u_0, u_1, \ldots, u_k\} \) linking \( x \) and \( y_n \) such that
\[
\sum_{i=0}^{k-1} d(u_i, u_{i+1}) \leq p(x, y_n) + \eta.
\]
Since \( d(y_n) < \varepsilon \), \( \{u_0, u_1, \ldots, u_k, y\} \) is an \( \varepsilon \)-chain linking \( x \) and \( y \). So, we have
\[
p(x, y) \leq \sum_{i=0}^{k-1} d(u_i, u_{i+1}) + d(y_n, y) \leq p(x, y_n) + \eta + d(y_n, y)
\]
and hence
\[
p(x, y) \leq \liminf_{n \to \infty} p(x, y_n) + \eta.
\]
Since \( \eta > 0 \) is arbitrary, we have
\[
p(x, y) \leq \liminf_{n \to \infty} p(x, y_n).
\]
This implies that \( p(x, \cdot) \) is lower semicontinuous. Since \( p(x, y) \geq d(x, y) \) for every \( x, y \in X \), by Lemma 1, \( p \) is a \( w \)-distance. \( \square \)

The following lemma was proved in [4].

**Lemma 4 ([4]).** Let \( X \) be a metric space with metric \( d \) and let \( p \) be a \( w \)-distance on \( X \). Let \( \{x_n\} \) and \( \{y_n\} \) be sequences in \( X \), let \( \{\alpha_n\} \) and \( \{\beta_n\} \) be sequences in \( [0, \infty) \) converging to 0, and let \( x, y, z \in X \). Then the following hold:

1. if \( p(x_n, y) \leq \alpha_n \) and \( p(x_n, z) \leq \beta_n \) for any \( n \in \mathbb{N} \), then \( y = z \); in particular, if \( p(x, y) = 0 \) and \( p(x, z) = 0 \), then \( y = z \);
2. if \( p(x_n, y_n) \leq \alpha_n \) and \( p(x_n, z) \leq \beta_n \) for any \( n \in \mathbb{N} \), then \( \{y_n\} \) converges to \( z \);
3. if \( p(x_n, x_m) \leq \alpha_n \) for any \( n, m \in \mathbb{N} \) with \( m > n \), then \( \{x_n\} \) is a Cauchy sequence;
4. if \( p(y, x_n) \leq \alpha_n \) for any \( n \in \mathbb{N} \), then \( \{x_n\} \) is a Cauchy sequence.

**3. Fixed point theorems**

Let \( X \) be a metric space with metric \( d \). A set-valued mapping \( T \) from \( X \) into itself is called **weakly contractive** or **\( p \)-contractive** if there exist a \( w \)-distance \( p \) on \( X \) and \( r \in [0, 1) \) such that for any \( x_1, x_2 \in X \) and \( y_1 \in Tx_1 \) there is \( y_2 \in Tx_2 \) with \( p(y_1, y_2) \leq rp(x_1, x_2) \).

**Theorem 1.** Let \( X \) be a complete metric space and let \( T \) be a set-valued \( p \)-contractive mapping from \( X \) into itself such that for any \( x \in X \), \( Tx \) is a nonempty closed subset of \( X \). Then there exists \( x_0 \in X \) such that \( x_0 \in Tx_0 \) and \( p(x_0, x_0) = 0 \).
Proof. Let \( p \) be a \( w \)-distance on \( X \) and let \( r \in [0, 1) \) be such that for any \( x_1, x_2 \in X \) and \( y_1 \in Tx_1 \), there exists \( y_2 \in Tx_2 \) with \( p(y_1, y_2) \leq rp(x_1, x_2) \).

Fix \( u_0 \in X \) and \( u_1 \in Tu_0 \). Then there exists \( u_2 \in Tu_1 \) such that \( p(u_1, u_2) \leq rp(u_0, u_1) \). Thus, we have a sequence \( \{u_n\} \) in \( X \) such that \( u_{n+1} \in Tu_n \) and \( p(u_n, u_{n+1}) \leq rp(u_{n-1}, u_n) \) for every \( n \in \mathbb{N} \). For any \( n \in \mathbb{N} \), we have
\[
p(u_n, u_{n+1}) \leq rp(u_{n-1}, u_n) \leq r^2p(u_{n-2}, u_{n-1}) \leq \ldots \leq r^n p(u_0, u_1)
\]
and hence, for any \( n, m \in \mathbb{N} \) with \( m > n \),
\[
p(u_n, u_m) \leq p(u_n, u_{n+1}) + p(u_{n+1}, u_{n+2}) + \ldots + p(u_{m-1}, u_m) \\
\leq r^n p(u_0, u_1) + r^n p(u_0, u_1) + \ldots + p(u_0, u_1) \\
\leq \frac{r^n}{1-r} p(u_0, u_1).
\]

By Lemma 4, \( \{u_n\} \) is a Cauchy sequence. Hence \( \{u_n\} \) converges to a point \( v_0 \in X \). Fix \( n \in \mathbb{N} \). Since \( \{u_m\} \) converges to \( v_0 \) and \( p(u_n, \cdot) \) is lower semicontinuous, we have
\[(*) \quad p(u_n, v_0) \leq \liminf_{m \to \infty} p(u_n, u_m) \leq \frac{r^n}{1-r} p(u_0, u_1).
\]

By hypothesis, we also have \( w_n \in Tv_0 \) such that \( p(u_n, w_n) \leq rp(u_{n-1}, v_0) \). So, for any \( n \in \mathbb{N} \),
\[
p(u_n, w_n) \leq rp(u_{n-1}, v_0) \leq \frac{r^n}{1-r} p(u_0, u_1).
\]

By Lemma 4, \( \{w_n\} \) converges to \( v_0 \). Since \( Tv_0 \) is closed, we have \( v_0 \in Tv_0 \). For such \( v_0 \), there exists \( v_1 \in Tv_0 \) such that \( p(v_0, v_1) \leq rp(v_0, v_0) \). Thus, we also have a sequence \( \{v_n\} \) in \( X \) such that \( v_{n+1} \in Tv_n \) and \( p(v_0, v_{n+1}) \leq rp(v_0, v_n) \) for every \( n \in \mathbb{N} \). So, we have
\[
p(v_0, v_n) \leq rp(v_0, v_{n-1}) \leq \ldots \leq r^n p(v_0, v_0).
\]

By Lemma 4, \( \{v_n\} \) is a Cauchy sequence. Hence \( \{v_n\} \) converges to a point \( x_0 \in X \). Since \( p(v_0, \cdot) \) is lower semicontinuous, \( p(v_0, x_0) \leq \liminf_{n \to \infty} p(v_0, v_n) \leq 0 \) and hence \( p(v_0, x_0) = 0 \). Then, for any \( n \in \mathbb{N} \),
\[
p(v_n, x_0) \leq p(u_n, v_0) + p(v_0, x_0) \leq \frac{r^n}{1-r} p(u_0, u_1).
\]

So, using \((*)\) and Lemma 4, we obtain \( v_0 = x_0 \) and hence \( p(v_0, v_0) = 0 \). \( \square \)

Let \( X \) be a metric space with metric \( d \) and let \( T \) be a mapping from \( X \) into itself. Then \( T \) is called weakly contractive or \( p \)-contractive if there exist a \( w \)-distance \( p \) on \( X \) and \( r \in [0, 1) \) such that \( p(Tx, Ty) \leq rp(x, y) \) for every \( x, y \in X \). In the case of \( p = d \), \( T \) is called contractive.
Theorem 2. Let $X$ be a complete metric space. If a mapping $T$ from $X$ into itself is $p$-contractive, then $T$ has a unique fixed point $x_0 \in X$. Further the $x_0$ satisfies $p(x_0, x_0) = 0$.

Proof. Let $p$ be a $w$-distance and let $r \in [0, 1)$ be such that $p(Tx, Ty) \leq rp(x, y)$ for every $x, y \in X$. Then from Theorem 1, there exists $x_0 \in X$ with $Tx_0 = x_0$ and $p(x_0, x_0) = 0$. If $y_0 = Ty_0$, then

$$p(x_0, y_0) = p(Tx_0, Ty_0) \leq rp(x_0, y_0)$$

and hence $p(x_0, y_0) = 0$. So, by $p(x_0, x_0) = 0$ and Lemma 4, we have $x_0 = y_0$. □

Using Theorem 1, we will prove a fixed point theorem which generalizes Nadler’s fixed point theorem for set-valued mappings and Edelstein’s fixed point theorem on an $\varepsilon$-chainable metric space. Before proving it, we give some definitions and notations. Let $X$ be a metric space with metric $d$. For $x \in X$ and $A \subset X$, set $d(x, A) = \inf\{d(x, y) : y \in A\}$. Denote by $CB(X)$ the class of all nonempty bounded closed subsets of $X$. Let $H$ be the Hausdorff metric with respect to $d$, i.e.,

$$H(A, B) = \max\{\sup_{u \in A} d(u, B), \sup_{v \in B} d(v, A)\}$$

for every $A, B \in CB(X)$. Let $\varepsilon \in (0, \infty)$. A mapping $T$ from $X$ into $CB(X)$ is said to be $(\varepsilon, \sigma)$-uniformly locally contractive [2] if there exists $\sigma \in [0, 1)$ such that $H(Tx, Ty) \leq \sigma d(x, y)$ for every $x, y \in X$ with $d(x, y) < \varepsilon$. In particular, $T$ is said to be contractive when $\varepsilon = \infty$.

Theorem 3. Let $\varepsilon \in (0, \infty]$ and let $X$ be a complete and $\varepsilon$-chainable metric space with metric $d$. Suppose that a mapping $T$ from $X$ into $CB(X)$ is $(\varepsilon, \sigma)$-uniformly locally contractive. Then there exists $x_0 \in X$ with $x_0 \in Tx_0$.

Proof. Define a function $p$ from $X \times X$ into $[0, \infty)$ as follows:

$$p(x, y) = \inf \left\{ \sum_{i=0}^{k-1} d(u_i, u_{i+1}) : \{u_0, u_1, \ldots, u_k\} \text{ is an } \varepsilon\text{-chain linking } x \text{ and } y \right\}.$$ 

From Lemma 3, $p$ is a $w$-distance on $X$. We prove that $T$ is $p$-contractive. Choose a real number $r$ such that $\sigma < r < 1$. Let $x_1, x_2 \in X$, $y_1 \in Tx_1$ and $\eta > 0$. Then there exists an $\varepsilon$-chain $\{u_0, u_1, \ldots, u_k\}$ linking $x_1$ and $x_2$ such that

$$\sum_{i=0}^{k-1} d(u_i, u_{i+1}) \leq p(x_1, x_2) + \eta.$$

Put $v_0 = y_1$. Since $T$ is $(\varepsilon, \sigma)$-uniformly locally contractive, there exists $v_1 \in Tu_1$ such that

$$d(v_0, v_1) \leq rd(u_0, u_1) < r\varepsilon \leq \varepsilon.$$
In a similar way, we define an ε-chain \( \{v_0, v_1, \ldots, v_k\} \) linking \( y_1 \) and \( y_2 \) such that \( v_i \in T u_i \) for every \( i = 0, 1, \ldots, k \) and
\[
d(v_i, v_{i+1}) \leq r d(u_i, u_{i+1}) < \varepsilon
\]
for every \( i = 0, 1, \ldots, k - 1 \). Putting \( y_2 = v_k \), since \( y_2 \in T x_2 \) and \( \{v_0, v_1, \ldots, v_k\} \) is an ε-chain linking \( y_1 \) and \( y_2 \), we have
\[
p(y_1, y_2) \leq \sum_{i=0}^{k-1} d(v_i, v_{i+1}) \leq \sum_{i=0}^{k-1} r d(u_i, u_{i+1}) \leq r p(x_1, x_2) + r \eta < r p(x_1, x_2) + \eta.
\]
Since \( \eta > 0 \) is arbitrary, we have \( p(y_1, y_2) \leq r p(x_1, x_2) \). So, \( T \) is a \( p \)-contractive set-valued mapping from \( X \) into itself. Theorem 1 now gives the desired result. \( \square \)

As direct consequences of Theorem 3, we obtain the following.

**Corollary 1** (Nadler [5]). Let \( X \) be a complete metric space and let \( T \) be a contractive set-valued mapping from \( X \) into \( \text{CB}(X) \). Then there exists \( x_0 \in X \) with \( x_0 \in T x_0 \).

**Proof.** We may assume that there exists \( \sigma \in [0, 1) \) such that \( H(Tx, Ty) \leq \sigma d(x, y) \) for every \( x, y \in X \). Since \( T \) is \((\infty, \sigma)\)-uniformly locally contractive and \( X \) is \( \infty \)-chainable, using Theorem 3, we obtain the desired result. \( \square \)

**Corollary 2** (Edelstein [2]). Let \( \varepsilon \in (0, \infty] \) and let \( X \) be a complete and \( \varepsilon \)-chainable metric space with metric \( d \). Suppose that a mapping \( T \) from \( X \) into itself is \((\varepsilon, \sigma)\)-uniformly locally contractive. Then \( T \) has a unique fixed point.

4. Characterizations of metric completeness

In this section, we discuss characterizations of metric completeness. We first give the following example.

**Example.** Define subsets of \( \mathbb{R}^2 \) as follows:
\[
A_n = \{(t, t/n) : t \in (0, 1]\} \quad \text{for every } n \in \mathbb{N}, \quad S = \bigcup_{n \in \mathbb{N}} A_n \cup \{0\}.
\]
Then \( S \) is not complete and every continuous mapping on \( S \) has a fixed point in \( S \).

**Proof.** It is clear that \( S \) is not complete. Let \( T \) be a continuous mapping from \( S \) into itself. If \( T 0 = 0 \), then 0 is a fixed point of \( T \). Assume that \( T 0 \in A_j \) for some \( j \in \mathbb{N} \) and define a mapping \( U \) on \( A_j \cup \{0\} \) as follows:
\[
U x = \begin{cases} 
Tx & \text{if } Tx \in A_j, \\
0 & \text{if } Tx \notin A_j.
\end{cases}
\]
Then \( U \) is continuous. In fact, let \( \{x_n\} \) be a sequence in \( A_j \cup \{0\} \) which converges to \( x_0 \). Then \( \{Tx_n\} \) converges to \( Tx_0 \). If \( Tx_0 \in A_j \), then \( \{U x_n\} \) also converges
to $Tx_0 = Ux_0$. Otherwise $\{Ux_n\}$ converges to 0 and $Ux_0 = 0$. Hence $U$ is continuous. On the other hand, $A_j \cup \{0\}$ is compact and convex. So, $U$ has a fixed point $z_0$ in $A_j \cup \{0\}$. It is clear that $z_0 \neq 0$ and $z_0$ is a fixed point of $T$. □

Motivated by this example, we obtain the following.

**Theorem 4.** Let $X$ be a metric space. Then $X$ is complete if and only if every weakly contractive mapping from $X$ into itself has a fixed point in $X$.

**Proof.** Since the “only if” part is proved in Theorem 2, we need only prove the “if” part. Assume that $X$ is not complete. Then there exists a sequence $\{x_n\}$ in $X$ which is Cauchy and does not converge. So, we have $\lim_{m \to \infty} d(x_n, x_m) > 0$ for any $n \in \mathbb{N}$ and also $\lim_{n \to \infty} \lim_{m \to \infty} d(x_n, x_m) = 0$. Then, for any $c > 0$, we can choose a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that, for any $i \in \mathbb{N}$,

$$\lim_{m \to \infty} d(x_{n_i}, x_m) > c \lim_{m \to \infty} d(x_{n_i+1}, x_m)$$

and hence

$$\lim_{j \to \infty} d(x_{n_i}, x_{n_j}) > c \lim_{j \to \infty} d(x_{n_i+1}, x_{n_j}).$$

So, we may assume that there exists a sequence $\{x_n\}$ in $X$ satisfying the following conditions:

1. $\{x_n\}$ is Cauchy;
2. $\{x_n\}$ does not converge;
3. $\lim_{n \to \infty} d(x_i, x_n) > 3 \lim_{n \to \infty} d(x_{i+1}, x_n)$ for any $i \in \mathbb{N}$.

Put $F = \{x_n : n \in \mathbb{N}\}$. Then $F$ is bounded and closed. So, the function $p : X \times X \to [0, \infty)$ defined by

$$p(x, y) = \begin{cases} 
    d(x, y) & \text{if } x, y \in F, \\
    2\delta(F) & \text{if } x \notin F \text{ or } y \notin F,
\end{cases}$$

is a $w$-distance on $X$ by Lemma 2. Further, $p(x, y) = p(y, x)$ for any $x, y \in X$.

Define a mapping $T$ from $X$ into itself as follows:

$$Tx = \begin{cases} 
    x_1 & \text{if } x \notin F, \\
    x_{i+1} & \text{if } x = x_i.
\end{cases}$$

Then it is clear that $T$ has no fixed point in $X$. To complete the proof, it is sufficient to show that $T$ is $p$-contractive. If $x \notin F$ or $y \notin F$, then

$$p(Tx, Ty) \leq \delta(F) = \frac{1}{2} \cdot 2\delta(F) = \frac{1}{2} \cdot \frac{1}{2} p(x, y) \leq \frac{2}{3} p(x, y).$$
Let $x, y \in F$. Then, without loss of generality, we may assume that $x = x_i, y = x_j$ and $i < j$. We have

$$\begin{align*}
    d(x_i, x_j) &\geq \lim_{n \to \infty} d(x_i, x_n) - \lim_{n \to \infty} d(x_j, x_n) \\
    &\geq \lim_{n \to \infty} d(x_i, x_n) - \lim_{n \to \infty} d(x_{i+1}, x_n) \\
    &\geq 2 \lim_{n \to \infty} d(x_{i+1}, x_n).
\end{align*}$$

On the other hand,

$$\begin{align*}
    d(x_{i+1}, x_{j+1}) &\leq \lim_{n \to \infty} d(x_{i+1}, x_n) + \lim_{n \to \infty} d(x_{j+1}, x_n) \\
    &\leq \lim_{n \to \infty} d(x_{i+1}, x_n) + \lim_{n \to \infty} d(x_{i+2}, x_n) \\
    &\leq \frac{4}{3} \lim_{n \to \infty} d(x_{i+1}, x_n).
\end{align*}$$

Therefore we have

$$p(Tx, Ty) = p(Tx_i, Tx_j) = d(x_{i+1}, x_{j+1}) \leq \frac{4}{3} \lim_{n \to \infty} d(x_{i+1}, x_n)$$

$$\leq \frac{4}{3} \cdot \frac{1}{2} d(x_i, x_j) = \frac{2}{3} d(x_i, x_j) = \frac{2}{3} p(x_i, x_j) = \frac{2}{3} p(x, y). \quad \square$$

**Theorem 5.** Let $X$ be a normed linear space and let $D$ be a convex subset of $X$. Then $D$ is complete if and only if every contractive mapping from $D$ into itself has a fixed point in $D$.

Before proving Theorem 5, we need two lemmas.

**Lemma 5.** Let $X$ be a normed linear space and let $D$ be a convex subset of $X$ with $0 \in \overline{D}$, where $\overline{D}$ is the closure of $D$. Then for any $x \in D \setminus \{0\}$, there exists $y \in D$ such that $2\|y\| = \|x\|$ and $\|x - y\| \leq 2\|x\| - 2\|y\|$

**Proof.** Let $x \in D \setminus \{0\}$. Then, since $0 \in \overline{D}$, we obtain an element $z \in D$ with $\|z\| \leq \|x\|/3$. So, there exist $y \in D$ and $t \in [0, 1]$ such that $y = tz + (1 - t)x$ and $\|y\| = \|x\|/2$. From

$$\frac{\|x\|}{2} = \|y\| \leq t\|z\| + (1 - t)\|x\| \leq t\frac{\|x\|}{3} + (1 - t)\|x\|,$$

we have $1/2 \leq t/3 + (1 - t)$ and hence $t \leq 3/4$. Then we obtain
\[ \|x - y\| = t\|x - z\| \leq \frac{3}{4}\|x - z\| \leq \frac{3}{4}\|x\| + \frac{3}{4}\|z\| \]
\[ \leq \frac{3}{4}\|x\| + \frac{1}{4}\|x\| = \|x\| = \|x\| + (\|x\| - 2\|y\|) = 2\|x\| - 2\|y\|. \] \(\square\)

**Lemma 6.** Let \(X\) be a normed linear space and let \(D\) be a convex subset of \(X\) with \(0 \in \overline{D} \setminus D\). Then there exist a sequence \(\{v_n\}\) in \(D\) and a mapping \(w\) from \((0, \infty)\) into \(D\) satisfying the following conditions:

1. \(\|v_n\| = \|v_1\|/2^{n-1}\) for every \(n \in \mathbb{N}\);
2. \(w(\|v_n\|) = v_n\) for every \(n \in \mathbb{N}\);
3. \(\|w(s) - w(t)\| \leq 2|s - t|\) for every \(s, t \in (0, \infty)\);
4. \(\|w(t)\| \leq t\) for every \(t \in (0, \infty)\).

**Proof.** Let \(v_1 \in D\). Then from \(v_1 \neq 0\) and Lemma 5 there exists \(v_2 \in D\) such that \(2\|v_2\| = \|v_1\|\) and \(\|v_1 - v_2\| \leq 2\|v_1\| - 2\|v_2\|\). Thus, we can find a sequence \(\{v_n\}\) in \(D\) such that

\[ \|v_n\| = \frac{1}{2^{n-1}}\|v_1\| \] and \(\|v_{n-1} - v_n\| \leq 2\|v_{n-1}\| - 2\|v_n\|\).

Note that \(\|v_n\| \to 0\) and \(\|v_{n+1}\| < \|v_n\|\) for every \(n \in \mathbb{N}\). Define a mapping \(w\) from \((0, \infty)\) into \(D\) as follows:

\[ w(t) = \begin{cases} v_1 & \text{if } \|v_1\| < t, \\ \frac{t - \|v_{n+1}\|}{\|v_n\| - \|v_{n+1}\|} v_n + \frac{\|v_n\| - t}{\|v_n\| - \|v_{n+1}\|} v_{n+1} & \text{if } \|v_{n+1}\| < t \leq \|v_n\| \end{cases} \]

for some \(n \in \mathbb{N}\).

Then it is clear that \(w(\|v_n\|) = v_n\) for every \(n \in \mathbb{N}\). We shall show (3). In fact, if \(\|v_1\| \leq s \leq t\), it is obvious that \(\|w(t) - w(s)\| \leq 2(t - s)\) and if \(\|v_{n+1}\| \leq s \leq t \leq \|v_n\|\) for some \(n \in \mathbb{N}\), we have

\[ \|w(s) - w(t)\| = \frac{t - s}{\|v_n\| - \|v_{n+1}\|} \|v_n - v_{n+1}\| \leq 2(t - s). \]

Further, if \(\|v_{m+1}\| < s \leq \|v_m\| \leq \|v_n\| \leq t < \|v_{n-1}\|\) for some \(m, n \in \mathbb{N}\) with \(m \geq n \geq 1\), where \(\|v_0\| = \infty\), we have

\[ \|w(s) - w(t)\| \leq \|w(s) - w(\|v_m\|)\| \]
\[ + \sum_{i=n}^{m-1} \|w(\|v_{i+1}\|) - w(\|v_i\|)\| + \|w(\|v_n\|) - w(t)\| \]
\[ \leq 2\|v_n\| - s + \sum_{i=n}^{m-1} 2(\|v_i\| - \|v_{i+1}\|) + 2(t - \|v_n\|) = 2(t - s). \]
We shall show (4). In fact, if \( \|v_1\| < t \), it is obvious that \( \|w(t)\| = \|v_1\| \leq t \). And if \( \|v_{n+1}\| < t \leq \|v_n\| \) for some \( n \in \mathbb{N} \), we have
\[
\|w(t)\| \leq \frac{t - \|v_{n+1}\|}{\|v_n\| - \|v_{n+1}\|} \|v_n\| + \frac{\|v_n\| - t}{\|v_n\| - \|v_{n+1}\|} \|v_{n+1}\| = t.
\]
\[\square\]

**Proof of Theorem 5.** Since the “only if” part is well known, we need only prove the “if” part. Suppose that \( D \) is not complete. We denote the completion of \( X \) by \( \hat{X} \) and the closure of \( D \) in \( \hat{X} \) by \( \hat{D} \). Since \( D \) is not complete, we obtain \( z_0 \in \hat{D} \setminus D \). Since \( D - z_0 \) is convex in \( \hat{X} \) and the closure of \( D - z_0 \) in \( \hat{X} \) includes 0, there exists a mapping \( w \) from \((0, \infty)\) into \( D - z_0 \) satisfying (3) and (4) of Lemma 6. Now, define a mapping \( T \) from \( D \) into itself as follows:
\[
T(x) = w\left(\frac{\|x - z_0\|}{4}\right) + z_0 \quad \text{for every} \quad x \in D.
\]
Then we have, for any \( x, y \in D \),
\[
\|Tx - Ty\| = \left\| w\left(\frac{\|x - z_0\|}{4}\right) - w\left(\frac{\|y - z_0\|}{4}\right) \right\|
\leq 2 \left| \frac{\|x - z_0\|}{4} - \frac{\|y - z_0\|}{4} \right| \leq \frac{1}{2} \|x - y\|.
\]
Further, we have, for every \( x \in D \),
\[
\|Tx - z_0\| = \left\| w\left(\frac{\|x - z_0\|}{4}\right) \right\| \leq \frac{\|x - z_0\|}{4} < \|x - z_0\|.
\]
So, \( T \) has no fixed point in \( D \). \[\square\]

As a direct consequence of Theorem 5, we obtain the following.

**Corollary 3.** Let \( X \) be a normed linear space. Then \( X \) is a Banach space if and only if every contractive mapping from \( X \) into itself has a fixed point in \( X \).

**References**


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