1. Introduction

Takahashi [10] introduced a notion of convexity in metric spaces and studied some fixed point theorems for nonexpansive mappings in such spaces. Let $X$ be a metric space and $I = [0, 1]$. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on $X$ if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

$X$ together with a convex structure $W$ is called a convex metric space.

Recently, Shimizu and Takahashi [9] proved the following result:

Let $X$ be a bounded convex metric space and let $T$ be a multivalued nonexpansive mapping of $X$ into itself such that $T(x)$ is a nonempty compact set for each $x \in X$. Then $T$ has the almost fixed point property in $X$, i.e.,

$$\inf_{x \in X} d(x, Tx) = 0.$$

metric spaces. They showed that a hyperbolic metric is, in some sense, uniformly convex, and showed fixed point theorems for single-valued nonexpansive mappings.

In this paper, we introduce a notion of uniform convexity in convex metric spaces and prove a fixed point theorem for multivalued nonexpansive mappings in such spaces by applying ultrafilters, without using the notion of regular sequences. This is a generalization of Lim's result [5] and the proof is simpler than that of [5].

2. Preliminaries

Let $X$ be a nonempty set. A nonempty family $\mathcal{F}$ of subsets of $X$ is called a filter on $X$ if it has the following properties: (1) $\emptyset \not\in \mathcal{F}$; (2) if $A \subseteq B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$; (3) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are filters on $X$ with $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then we say that $\mathcal{F}_2$ is finer than $\mathcal{F}_1$. A filter $\mathcal{U}$ on $X$ is called an ultrafilter if there is no filter on $X$ which is strictly finer than $\mathcal{U}$. A nonempty class $\mathcal{B}$ of subsets of $X$ is called a filterbase on $X$ if it has the following properties: (1) $\emptyset \not\in \mathcal{B}$; (2) for any $A_1$ and $A_2$ in $\mathcal{B}$, there exists $A_3$ in $\mathcal{B}$ such that $A_3 \subset A_1 \cap A_2$. If $\mathcal{B}$ is a filterbase on $X$, then

$$\mathcal{F} = \{A \subset X : B \subset A, \; B \in \mathcal{B}\}$$

is a filter on $X$. In this case, $\mathcal{B}$ is said to be a base of $\mathcal{F}$ or to generate $\mathcal{F}$. Let $X$ be a topological space and let $\mathcal{B}$ be a filterbase on $X$. Then $\mathcal{B}$ is said to converge to a point $x$ in $X$ or to have a limit $x$ in $X$ if for any neighbourhood $V$ of $x$, there is a set $A$ in $\mathcal{B}$ such that $A \subset V$. If $\mathcal{U}$ is an ultrafilter on a compact set $X$, then $\mathcal{U}$ has a limit in $X$. Let $\mathcal{U}$ be an ultrafilter on a set $X$ and $P$ be a mapping of $X$ into a set $D$. Then $P(\mathcal{U})$ is a filterbase on $D$ and it generates an ultrafilter on $D$. In fact, it is obvious that since $\mathcal{U}$ is an ultrafilter on $X$, then $P(\mathcal{U})$ is a filterbase on $D$. Let

$$\mathcal{B} = \{B \subset D : P(A) \subset B \text{ for some } A \in \mathcal{U}\}$$

and let $\mathcal{K}$ be a filter on $D$ with $\mathcal{K} \supset \mathcal{B}$. If $K \in \mathcal{K}$, then $P^{-1}K \in \mathcal{U}$ or $P^{-1}K^c \in \mathcal{U}$, where $K^c$ is the complement of $K$. Suppose $A = P^{-1}K^c \in \mathcal{U}$. Then $P(A) = P(P^{-1}K^c) \subset K^c$ and hence $K^c \in \mathcal{B}$. This is a contradiction. So, $P^{-1}K \in \mathcal{U}$. Since $P(P^{-1}K) \subset K$, we have $K \in \mathcal{B}$ and hence $\mathcal{K} = \mathcal{B}$. This implies that $\mathcal{B}$ is an ultrafilter on $D$; for details, see [1, 8].

Let $X$ be a convex metric space. A nonempty subset $K \subset X$ is convex if $W(x, y, \lambda) \in K$ whenever $(x, y, \lambda) \in K \times K \times I$. Takahashi [10] has shown that open spheres $B(x, r) = \{y \in X : d(x, y) < r\}$ and closed spheres $B[x, r] = \{y \in X : d(x, y) \leq r\}$ are convex. Also, if $\{K_\alpha : \alpha \in A\}$ is a family of convex subsets of $X$, then $\bigcap\{K_\alpha : \alpha \in A\}$ is convex. For $A \subset X$, we denote by $\overline{A}$ the intersection
of all closed convex sets containing $A$ and by $\delta(A)$ the diameter of $A$. A convex metric space $X$ is said to have the property (C) if every decreasing sequence of nonempty bounded closed convex subsets of $X$ has nonempty intersection.

Let $X$ be a convex metric space and let $\mathcal{B}$ be a filterbase on $X$ which contains at least one bounded subset of $X$. Then we define

$$r(x, \mathcal{B}) = \inf_{A \in \mathcal{B}} \sup_{y \in A} d(x, y) = \lim_{A \in \mathcal{B}} \sup_{y \in A} d(x, y)$$

for every $x \in X$. Since for every $x, y \in X$, $|r(x, \mathcal{B}) - r(y, \mathcal{B})| \leq d(x, y)$, the real-valued function $r(\cdot, \mathcal{B})$ on $X$ is continuous. Further, for any real number $\alpha$, the set

$$C = \{z \in X : r(z, \mathcal{B}) \leq \alpha\}$$

is convex. In fact, let $z_1, z_2 \in C$ and $\lambda \in [0, 1]$. Then

$$r(W(z_1, z_2, \lambda), \mathcal{B}) = \inf_{A \in \mathcal{B}} \sup_{y \in A} d(W(z_1, z_2, \lambda), y) = \lambda r(z_1, \mathcal{B}) + (1 - \lambda) r(z_2, \mathcal{B})$$

$$\leq \lambda \alpha + (1 - \lambda) \alpha = \alpha$$

and hence $W(z_1, z_2, \lambda) \in C$.

3. Uniformly convex metric spaces

A convex metric space $X$ is said to be uniformly convex if for any $\varepsilon > 0$, there exists $\alpha = \alpha(\varepsilon)$ such that, for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r$, $d(z, y) \leq r$ and $d(x, y) \geq r\varepsilon$,

$$d(z, W(x, y, 1/2)) \leq r(1 - \alpha) < r.$$

Example 1. Uniformly convex Banach spaces are uniformly convex metric spaces.

Example 2. Let $H$ be a Hilbert space and let $X$ be a nonempty closed subset of $\{x \in H : \|x\| = 1\}$ such that if $x, y \in X$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$, then $(\alpha x + \beta y)/\|\alpha x + \beta y\| \in X$ and $\delta(X) \leq \sqrt{2}/2$; see [7]. Let $d(x, y) = \cos^{-1}\{(x, y)\}$ for every $x, y \in X$, where $(\cdot, \cdot)$ is the inner product of $H$. When we define a convex structure $W$ for $(X, d)$ properly, it is easily seen that $(X, d)$ becomes a complete and uniformly convex metric space.

Remark. The module of convexity of Banach spaces and Goebel, Sękowski and Stachura’s $\delta$ in Theorem 1 of [4] are continuous functions, but we only assume the existence of a positive number $\alpha$ such that $\alpha$ is a function of $\varepsilon$. Goebel, Sękowski and Stachura’s $\delta$ depends on $\gamma$ and $\varepsilon$, but our $\alpha$ only depends on $\varepsilon$. 
Theorem 1. Let $X$ be a complete and uniformly convex metric space. Then $X$ has the property $(C)$.

Proof. Let $\{K_n\}$ be a decreasing sequence of nonempty bounded closed convex subsets of $X$. If $\delta(K_n) > 0$ for every positive integer $n$, then there exist $x, y \in K_n$ such that $d(x, y) \geq \delta(K_n)/2$. Since $d(z, x) \leq \delta(K_n)$, $d(z, y) \leq \delta(K_n)$ for all $z \in K_n$ and the space is uniformly convex, there exists $\alpha > 0$ such that
\[
d(z, W(x, y, 1/2)) \leq \delta(K_n)(1 - \alpha) < \delta(K_n)
\]
for all $z \in K_n$ and hence we obtain $u_n^1 \in K_n$ such that
\[
d(z, u_n^1) \leq \delta(K_n)(1 - \alpha)
\]
for all $z \in K_n$. Let
\[
K_n^1 = \{u_n^1, u_{n+1}^1, u_{n+2}^1, \ldots\}.
\]
Then it is obvious that $K_n^1 \neq \emptyset$ and $K_n^1 \supset K_{n+1}^1$ for every $n$. Suppose $\delta(K_n^1) > 0$ for every $n$. Then there exist $x, y \in K_n^1$ such that $d(x, y) \geq \delta(K_n^1)/2$. Put
\[
B_n^1 = \bigcap_{k=0}^{\infty} B[u_{n+k}^1, \delta(K_n^1)].
\]
Then $B_n^1 \supset \overline{\text{co}}(K_n^1)$ and $d(z, x) \leq \delta(K_n^1)$, $d(z, y) \leq \delta(K_n^1)$ for every $z \in \overline{\text{co}} K_n^1$. Since $X$ is uniformly convex, there exists $u_n^2 \in \overline{\text{co}} K_n^1 \subset K_n$ such that
\[
d(z, u_n^2) \leq \delta(K_n^1)(1 - \alpha)^2
\]
for all $z \in \overline{\text{co}} K_n^1$. By the same method, we obtain $\overline{\text{co}} K_n^2, \overline{\text{co}} K_n^3, \ldots$ and $u_n^3, u_n^4, \ldots$. It is obvious that
\[
K_n \supset \overline{\text{co}} K_n^1 \supset \overline{\text{co}} K_n^2 \supset \ldots \quad \text{and} \quad \delta(\overline{\text{co}} K_n^m) \to 0
\]
as $m \to \infty$. Since $X$ is complete, there exists $u_n \in X$ such that
\[
\bigcap_{m=1}^{\infty} \overline{\text{co}} K_n^m = \{u_n\}
\]
for every $n$. From
\[
\bigcap_{m=1}^{\infty} \overline{\text{co}} K_n^m \supset \bigcap_{m=1}^{\infty} \overline{\text{co}} K_{n+1}^m,
\]
we obtain $u_1 = u_2 = u_3 = \ldots$. Therefore, there exists $u$ with $u \in K_n$ for all $n$ and hence $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$. 
**Lemma.** Let $X$ be a complete and uniformly convex metric space. Let $K$ be a nonempty closed convex subset of $X$. If $\mathcal{F}$ is a filter on $X$ which contains at least a bounded subset of $X$, then there exists a unique point $u_0 \in K$ such that

$$r(u_0, \mathcal{F}) = \inf_{x \in K} r(x, \mathcal{F}).$$

**Proof.** Let $r = \inf_{x \in K} r(x, \mathcal{F})$ and define

$$K_n = \{ z \in K : r(z, \mathcal{F}) \leq r + 1/n \}$$

for every positive integer $n$. Then it is obvious that $K_n$ is nonempty, closed and convex. Further, $K_n$ is bounded. In fact, let $u, v \in K_n$. Then there exists $A \in \mathcal{F}$ such that

$$\sup_{y \in A} d(u, y) < r + 2/n \quad \text{and} \quad \sup_{y \in A} d(v, y) < r + 2/n.$$ 

So, we have

$$d(u, v) \leq \sup_{y \in A} d(u, y) + \sup_{y \in A} d(v, y) < 2(r + 2/n).$$

Since $\{K_n\}$ is a bounded decreasing sequence of nonempty closed convex subsets of $K$, we have

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

Further, we prove that $\bigcap_{n=1}^{\infty} K_n$ consists of one point. Let $x, y \in \bigcap_{n=1}^{\infty} K_n$. If $r = 0$, then $d(x, y) < 4/n$ for every positive integer $n$. Hence $x = y$. In the case of $r > 0$, suppose $x \neq y$. Then, for a fixed positive number $b$, there exists a positive number $\varepsilon$ such that

$$d(x, y) \geq (r + a)\varepsilon$$

for every $a \in [0, b]$. We can also choose $a_0 \in (0, b)$ such that

$$(r + a_0)(1 - \alpha(\varepsilon)) < r.$$ 

Then there exists $A \in \mathcal{F}$ such that

$$\sup_{z \in A} d(x, z) < r + a_0 \quad \text{and} \quad \sup_{z \in A} d(y, z) < r + a_0.$$ 

Since $X$ is uniformly convex, we have

$$d(z, W(x, y, 1/2)) \leq (r + a_0)(1 - \alpha(\varepsilon)) < r$$

for every $z \in A$. This implies

$$\sup_{z \in A} d(z, W(x, y, 1/2)) \leq (r + a_0)(1 - \alpha(\varepsilon)) < r$$

and hence $r(W(x, y, 1/2), \mathcal{F}) < r$. This is a contradiction, because $W(x, y, 1/2) \in K$. Therefore we have $x = y$. 


4. Fixed point theorem

Let \( X \) be a metric space. Then, for \( x \in X \) and \( A \subset X \), we define \( d(x, A) = \inf \{ d(x, y) : y \in A \} \). Let \( BC(X) \) be the family of all nonempty bounded closed subsets of \( X \). Then a mapping \( T \) of \( X \) into \( BC(X) \) is said to be nonexpansive if

\[
H(Tx, Ty) \leq d(x, y) \quad \text{for every } x, y \in X,
\]

where \( H \) is the Hausdorff metric with respect to \( d \), i.e.,

\[
H(A, B) = \max \{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \}
\]

for every \( A, B \in BC(X) \). Now, we can prove a fixed point theorem for multi-valued nonexpansive mappings in uniformly convex metric spaces.

**Theorem 2.** Let \( X \) be a bounded, complete and uniformly convex metric space. If \( T \) is a multi-valued nonexpansive mapping which assigns to each point of \( X \) a nonempty compact subset of \( X \), then \( T \) has a fixed point in \( X \).

**Proof.** By Theorem 1 of [9], there exists a sequence \( \{x_n\} \) in \( X \) such that \( d(x_n, Tx_n) \to 0 \) as \( n \to \infty \). For every positive integer \( n \), define

\[
A_n = \{x_n, x_{n+1}, \ldots\}.
\]

Then \( \{A_n\} \) is a filterbase on \( X \) and generates a filter \( \mathcal{F} \) on \( X \). From [1, 8], we know that there is an ultrafilter \( \mathcal{U} \) finer than \( \mathcal{F} \). Clearly we have

\[
\inf_{A \in \mathcal{U}} \sup_{x \in A} d(x, Tx) = 0.
\]

By the Lemma, there exists a unique element \( u_0 \in X \) such that

\[
r(u_0, \mathcal{U}) = \inf_{x \in X} r(x, \mathcal{U}).
\]

Since for each \( x \in X \), \( Tx \) is nonempty and compact, we obtain elements \( Sx \in Tx \) and \( Px \in Tu_0 \) such that

\[
d(x, Sx) = d(x, Tx) \quad \text{and} \quad d(Sx, Px) = d(Sx, Tu_0).
\]

Thus, we have got a mapping \( P : X \to Tu_0 \). We know that \( P(\mathcal{U}) \) is a filterbase on \( Tu_0 \) and the filter generated by \( P(\mathcal{U}) \) is an ultrafilter on \( Tu_0 \). Since
$Tu_0$ is compact, $P(U)$ has a limit $p_0$ in $Tu_0$. So, we have

$$r(p_0, U) = \inf_{A \in U} \sup_{x \in A} d(p_0, x) \leq \inf_{A \in U} \sup_{x \in A} \{d(p_0, Px) + d(Px, Sx) + d(Sx, x)\}$$

$$= \inf_{A \in U} \sup_{x \in A} \{d(p_0, Px) + d(Sx, Tu_0) + d(x, Tx)\}$$

$$\leq \inf_{A \in U} \sup_{x \in A} \{d(p_0, Px) + H(Tx, Tu_0) + d(x, Tx)\}$$

$$\leq \inf_{A \in U} \sup_{x \in A} \{d(p_0, Px) + d(x, u_0) + d(x, Tx)\}$$

$$= \inf_{A \in U} \sup_{x \in A} d(x, u_0) = r(u_0, U).$$

By the Lemma, we have $u_0 = p_0 \in Tu_0$. This completes the proof.

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Tomoo Shimizu and Wataru Takahashi
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology
Oh-Okayama, Meguro-ku, Tokyo 152, JAPAN

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