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SINGULAR NONLINEAR DIFFERENTIAL EQUATIONS ON THE HALF LINE

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1. Introduction

This paper presents existence results to second order problems on the semi-infinite interval of the form

$$\text{(1.1)} \qquad \begin{cases} -y'' = q(t)f(t,y,y'), \quad 0 < t < \infty, \\ y(0) = 0, \\ y \text{ bounded on } [0,\infty), \end{cases}$$

where f may be singular at $y = 0$. The boundary condition at infinity will also be discussed. Problems of the above form occur in many applications. For example in power law fluids [4, 5, 15] we encounter the equation

$$\text{(1.2)} \qquad y'' + \phi(t)y^{-\lambda} = 0, \quad 0 < t < \infty, \quad y(0) = \alpha \geq 0,$$

with $\lambda > 0$ and ϕ nonnegative and continuous. Also in nonlinear mechanics in the study of unsteady flow of gas through a semi-infinite porous medium [1, 3, 9] the problem

$$\text{(1.3)} \qquad \begin{cases} -y'' = 2ty^{-1/2}y', \quad 0 < t < \infty, \\ y(0) = 0, \quad \lim_{t\to\infty} y(t) = 1, \end{cases}$$

occurs. In the literature (1.2) was examined by Taliaferro [15] for the nonsingular problem (i.e. $\alpha > 0$). The goal in this paper is to tackle the more general problem

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(1.1). This automatically produces a result for (1.2) in the singular case ($\alpha = 0$). Our theory also includes the problem (1.3).

The discussion of the boundary value problem on the half line will be in three stages. We first establish the existence of solutions to

$$\begin{cases} -y'' = q(t)f(t,y,y'), \quad 0 < t < n, \\ y(0) = 0, \quad y(n) = b > 0, \quad n \in \mathbb{N}^+ = \{1,2,\dots\}. \end{cases}$$

This together with the Arzelà–Ascoli theorem and a diagonalization argument [3, 8, 14] will establish the existence of a global solution to (1.1). Finally, the limit condition at infinity will be discussed.

To conclude this section we state a well known existence principle [7, 10].

Theorem 1.1. *Suppose $g : [c,d] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous with $q \in C(c,d)$, $q > 0$ on (c,d) and $q \in L^1[c,d]$. In addition assume there is a constant M, independent of λ, with*

$$\max\{\sup_{[c,d]} |y(t)|, \sup_{[c,d]} |y'(t)|\} \le M \tag{1.4}$$

for each solution y to

$$\begin{cases} y'' = \lambda q(t)g(t,y,y'), \quad c < t < b, \\ y(c) = e_1, \quad y(d) = e_2, \end{cases} \tag*{$(1.5)_\lambda$}$$

for each $\lambda \in (0,1)$. Then $(1.5)_1$ has at least one solution $y \in C^1[c,d] \cap C^2(c,d)$.

2. The equation $y'' + f(t,y) = 0$

Our goal in this section will be to examine

$$\begin{cases} -y'' = q(t)f(t,y), \quad 0 < t < \infty, \\ y(0) = 0, \\ y \text{ bounded on } [0,\infty). \end{cases} \tag{2.1}$$

The boundary condition at infinity will also be discussed. Now let $n \in \mathbb{N}^+$ be fixed. We first discuss a boundary value problem on the finite interval, namely

$$\begin{cases} -y'' = q(t)f(t,y), \quad 0 < t < n, \\ y(0) = 1/m, \\ y(n) = b > 0, \end{cases} \tag*{$(2.2)^m$}$$

where $m \in \{N, N+1, \dots .\}$ is fixed; here $N \in \mathbb{N}^+$ and $b > 1/N$. The idea is to show that $(2.2)^m$ has a solution for each $m \in \{N, N+1, \dots\}$ and then this will be used to establish the existence of a solution to

$$\begin{cases} -y'' = q(t)f(t,y), \quad 0 < t < n, \\ y(0) = 0, \quad y(n) = b > 0. \end{cases} \tag{2.3}$$

THEOREM 2.1. *Suppose that*

(2.4) $q \in C(0,\infty)$ *with* $q > 0$ *on* $(0,\infty)$,

(2.5) $\int_0^\infty xq(x)\,dx < \infty$ *and* $\int_0^\infty q(x)\,dx < \infty$,

(2.6) $0 \le f(t,y) \le g(y) + h(y)$ *on* $(0,\infty)\times(0,\infty)$ *with* f *continuous on* $[0,\infty)\times(0,\infty)$, $g > 0$ *continuous and nonincreasing on* $(0,\infty)$ *and* $h \ge 0$ *continuous on* $[0,\infty)$,

(2.7) h/g *is nondecreasing on* $(0,\infty)$ *and there exists a constant* $M_0 > 0$ *such that for* $z > 0$,

$$\int_0^z \frac{dx}{g(x)} \le \left(1 + \frac{h(z)}{g(z)}\right)\int_0^\infty xq(x)\,dx + \int_0^1 \frac{dx}{g(x)}$$

implies $z \le M_0$,

(2.8) *for each constant* $H > 0$ *there exists a function* ψ_H *continuous on* $[0,\infty)$ *and positive on* $(0,\infty)$ *such that* $f(t,y) \ge \psi_H(t)$ *on* $(0,\infty)\times(0,H]$; *in addition,* $\int_0^\infty xq(x)\psi_H(x)\,dx < \infty$,

and

(2.9) *for any* $R > 0$, $1/g$ *is differentiable on* $(0,R]$ *with* $g'/g^2 < 0$ *integrable on* $[0,R]$; *in addition,* $\int_0^\infty (|g'(t)|^{1/2}/g(t))\,dt = \infty$.

Then (2.3) *has a solution* $y \in C[0,n] \cap C^2(0,n]$.

PROOF. Consider the modified problem

$$(2.10)_\lambda^m \qquad \begin{cases} -y'' = \lambda q(t) f^\star(t,y), & 0 < t < n, \\ y(0) = 1/m, \\ y(n) = b > 0, \end{cases}$$

where $0 < \lambda < 1$ with $m \in \{N, N+1, \dots\}$, $N \in \mathbb{N}^+$ and $b > 1/N$. Here $f^\star \ge 0$ is any continuous extension of f from $y \ge 1/m$. Let $y \in C^1[0,n] \cap C^2(0,n]$ be a solution to $(2.10)_\lambda^m$. Now $y(0) = 1/m$, $y(n) = b > 0$ together with $y'' \le 0$ on $(0,n)$ implies $y(t) \ge 1/m > 0$ for $t \in [0,n]$. (To see this suppose there exists a $\delta \in (0,n)$ with $y(\delta) < 1/m$. Now since there exists a $\nu \in (\delta,n)$ with $y'(\nu)(n-\delta) = y(n) - y(\delta) > 0$ we have $y'(\nu) > 0$ and this together with $y'' \le 0$ on $(0,n)$ implies $y' > 0$ on $(0,\delta)$. In particular $y(\delta) > y(0) = 1/m$, a contradiction.)

Now there are two cases to consider: either $y' \ge 0$ on $(0,n)$ or there exists a $t_0 \in (0,n)$ with $y' \ge 0$ on $(0,t_0)$ and $y' \le 0$ on (t_0,n).

REMARK. It is of interest to note that $y'(0) > 0$ for if $y'(0) \le 0$ then $y' \le 0$ on $(0,n)$ and so $y(n) \le y(0)$, a contradiction.

CASE (i): $y' \ge 0$ *on* $(0,n)$. Then we have

$$(2.11) \qquad 1/m \le y(t) \le b \quad \text{for } t \in [0,n].$$

Case (ii): *$y' \geq 0$ on $(0, t_0)$ and $y' \leq 0$ on (t_0, n); $t_0 \in (0, n)$.* Now for $x \in (0, n)$ we have

$$-y''(x) \leq q(x) g(y(x)) \left\{ 1 + \frac{h(y(x))}{g(y(x))} \right\} \tag{2.12}$$

and so integration from t ($t < t_0$) to t_0 yields

$$y'(t) \leq g(y(t)) \left\{ 1 + \frac{h(y(t_0))}{g(y(t_0))} \right\} \int_t^{t_0} q(x)\, dx.$$

Consequently, for $t \in (0, t_0)$ we have

$$\frac{y'(t)}{g(y(t))} \leq \left\{ 1 + \frac{h(y(t_0))}{g(y(t_0))} \right\} \int_t^{t_0} q(x)\, dx$$

and integration from 0 to t_0 yields

$$\int_{1/m}^{y(t_0)} \frac{du}{g(u)} \leq \left\{ 1 + \frac{h(y(t_0))}{g(y(t_0))} \right\} \int_0^{t_0} \int_s^{t_0} q(x)\, dx\, ds.$$

Hence

$$\int_0^{y(t_0)} \frac{du}{g(u)} \leq \left\{ 1 + \frac{h(y(t_0))}{g(y(t_0))} \right\} \int_0^{\infty} x q(x)\, dx + \int_0^1 \frac{du}{g(u)}$$

and so (2.7) implies that there exists a constant M_0 with $y(t_0) \leq M_0$. Thus

$$1/m \leq y(t) \leq M_0 \quad \text{for } t \in [0, n]. \tag{2.13}$$

Remark. Notice that M_0 is independent of m, n and λ.

Combining both cases yields

$$1/m \leq y(t) \leq \max\{b, M_0\} \equiv M \quad \text{for } t \in [0, n]. \tag{2.14}$$

In addition (2.12) implies for $x \in (0, n)$ that

$$-y''(x) \leq g\left(\frac{1}{m}\right) \left\{ 1 + \frac{h(M)}{g(M)} \right\} q(x). \tag{2.15}$$

Now since $y(0) = 1/m$, $y(n) = b$ there exists $\nu \in (0, n)$ with $y'(\nu) = (b - 1/m)/n$.

Case (i): *$y' \geq 0$ on $(0, n)$.* Integrating (2.15) from 0 to ν yields

$$y'(0) \leq b + g\left(\frac{1}{m}\right) \left\{ 1 + \frac{h(M)}{g(M)} \right\} \int_0^n q(x)\, dx \equiv K_0$$

and so since $y'' \leq 0$ on $(0, n)$ we have

$$0 \leq y'(t) \leq y'(0) \leq K_0 \quad \text{for } t \in [0, n]. \tag{2.16}$$

Case (ii): *$y' \geq 0$ on $(0, t_0)$ and $y' \leq 0$ on (t_0, n); $t_0 \in (0, n)$.* Integrating (2.15) from t ($t < t_0$) to t_0 yields

$$0 \leq y'(t) \leq g\left(\frac{1}{m}\right) \left\{ 1 + \frac{h(M)}{g(M)} \right\} \int_0^n q(x)\, dx \equiv K_1 \quad \text{for } t \in [0, t_0]. \tag{2.17}$$

Similarly integrating (2.15) from t_0 to t $(t > t_0)$ yields

$$0 \leq -y'(t) \leq K_1 \quad \text{for } t \in [t_0, n]. \tag{2.18}$$

Combining (2.17) and (2.18) yields

$$|y'(t)| \leq K_1 \quad \text{for } t \in [0, n]. \tag{2.19}$$

Thus in both cases

$$|y'(t)| \leq K_0 \quad \text{for } t \in [0, n]. \tag{2.20}$$

Now (2.14), (2.20) and Theorem 1.1 imply that $(2.10)_1^m$ has a solution $y_m \in C^1[0,n] \cap C^2(0,n]$ with $1/m \leq y_m(t) \leq M$ for $t \in [0,n]$. Also since $y_m \geq 1/m$ on $[0,n]$ it follows that y_m is a solution of $(2.2)^m$.

Additional estimates must be derived before we can show that (2.3) has a solution. We first claim that there is a constant N independent of m and n with

$$\int_0^n \left\{ \frac{-g'(y_m(s))}{g^2(y_m(s))} \right\} [y_m'(s)]^2 \, ds \leq N,$$

where y_m is the solution to $(2.2)^m$ constructed above.

Case (i): *$y_m' \geq 0$ on $(0,n)$.* Now there exists $\nu \in (0,n)$ with $y_m'(\nu) = (b - 1/m)/n \leq b$. Also we have

$$\frac{-y_m''(x)}{g(y_m(x))} \leq q(x) \left\{ 1 + \frac{h(M)}{g(M)} \right\} \quad \text{for } x \in (0,n). \tag{2.21}$$

First since $y_m'' \leq 0$ on $(0,n)$ we have

$$0 \leq y_m'(t) \leq b \quad \text{for } t \in [\nu, n]. \tag{2.22}$$

Now integrating (2.21) from 0 to n yields

$$\begin{aligned} \frac{-y_m'(n)}{g(b)} + \frac{y_m'(0)}{g(1/m)} + \int_0^n \left\{ \frac{-g'(y_m(s))}{g^2(y_m(s))} \right\} [y_m'(s)]^2 \, ds \\ \leq \left\{ 1 + \frac{h(M)}{g(M)} \right\} \int_0^\infty q(x) \, dx \end{aligned} \tag{2.23}$$

and so using (2.22) we obtain

$$\begin{aligned} \int_0^n \left\{ \frac{-g'(y_m(s))}{g^2(y_m(s))} \right\} [y_m'(s)]^2 \, ds \\ \leq \left\{ 1 + \frac{h(M)}{g(M)} \right\} \int_0^\infty q(x) \, dx + \frac{b}{g(b)} \equiv N. \end{aligned} \tag{2.24}$$

Case (ii): *$y_m' \geq 0$ on $(0, t_0)$ and $y_m' \leq 0$ on (t_0, n); $t_0 \in (0,n)$.* Integrate (2.21) from 0 to n to obtain (2.23) and so

$$\int_0^n \left\{ \frac{-g'(y_m(s))}{g^2(y_m(s))} \right\} [y_m'(s)]^2 \, ds \leq \left\{ 1 + \frac{h(M)}{g(M)} \right\} \int_0^\infty q(x) \, dx. \tag{2.25}$$

Combining (2.24) and (2.25) yields

$$\int_0^n \left\{ \frac{-g'(y_m(s))}{g^2(y_m(s))} \right\} [y_m'(s)]^2 \, ds \le N, \tag{2.26}$$

where N is defined in (2.24).

Remark. Notice that N is independent of m and n.

Next assumption (2.8) implies that there exists a function $\psi_M(t)$ (which may depend on M) continuous on $[0,\infty)$ and positive on $(0,\infty)$ with $f(t,y) \ge \psi_M(t)$ for $(t,y) \in (0,\infty) \times (0,M]$; here M is as defined in (2.14). Notice y_m satisfies

$$\begin{aligned} y_m(t) = \frac{1}{m} + \frac{b}{n} t + \left(1 - \frac{t}{n}\right) \int_0^t x q(x) f(x, y_m(x)) \, dx \\ + t \int_t^n \left(1 - \frac{x}{n}\right) q(x) f(x, y_m(x)) \, dx \end{aligned}$$

and so

$$y_m(t) \ge \frac{b}{n} t + \left(1 - \frac{t}{n}\right) \int_0^t x q(x) \psi_M(x) \, dx. \tag{2.27}$$

Choose a constant k_0 with

$$k_0 \le \min\left\{ 1, \frac{b}{\int_0^\infty x q(x) \psi_M(x) \, dx} \right\}. \tag{2.28}$$

Then $b \ge k_0 \int_0^\infty x q(x) \psi_M(x) \, dx$ and putting this into (2.27) yields for $t \in [0,n]$ that

$$\begin{aligned} y_m(t) &\ge \frac{k_0 t}{n} \int_0^\infty x q(x) \psi_M(x) \, dx + k_0 \int_0^t x q(x) \psi_M(x) \, dx \\ &\quad + \left(1 - k_0 - \frac{t}{n}\right) \int_0^t x q(x) \psi_M(x) \, dx \\ &= k_0 \int_0^t x q(x) \psi_M(x) \, dx + (1 - k_0) \left(1 - \frac{t}{n}\right) \int_0^t x q(x) \psi_M(x) \, dx \\ &\quad + \frac{k_0 t}{n} \int_t^\infty x q(x) \psi_M(x) \, dx. \end{aligned}$$

Consequently, for $t \in [0,n]$ we have

$$y_m(t) \ge k_0 \int_0^t x q(x) \psi_M(x) \, dx \equiv \Phi_M(t) \tag{2.29}$$

where k_0 is chosen as in (2.28).

Remark. Notice that $\Phi_M(t)$ is independent of m and n.

For later purposes it will be of benefit to note that

$$y_m(x) \ge \Phi_M(1) x \quad \text{for } x \in [0,1]. \tag{2.30}$$

To see this let

$$r(x) = y_m(x) - \left\{ \frac{1}{m}(1-x) + y_m(1)x \right\}.$$

Then $r''(x) = y_m''(x) \le 0$ on $(0,n)$ and $r(0) = r(1) = 0$. Consequently, $r \ge 0$ on $[0,1]$ so

$$y_m(x) \ge \frac{1}{m}(1-x) + y_m(1)x \ge y_m(1)x \ge \Phi_M(1)x \quad \text{for } x \in [0,1].$$

Hence (2.30) is true and combining with (2.29) yields

$$y_m(t) \ge \Omega_M(t) \quad \text{for } t \in [0,n], \tag{2.31}$$

where

$$\Omega_M(t) = \begin{cases} \Phi_M(1)t, & 0 \le t \le 1, \\ \Phi_M(t), & t \ge 1. \end{cases}$$

Thus

$$\Omega_M(t) \le y_m(t) \le M \quad \text{for } t \in [0,n] \tag{2.32}$$

and

$$\int_0^n \left\{ \frac{-g'(y_m(s))}{g^2(y_m(s))} \right\} [y_m'(s)]^2 \, ds \le N, \tag{2.33}$$

where M and N are as in (2.14) and (2.24) respectively. Consider

$$I(z) = \int_0^z \frac{\{-g'(u)\}^{1/2}}{g(u)} \, du.$$

Now I is an increasing map from $[0,\infty)$ onto $[0,\infty)$ with I continuous on $[0,K]$ for any $K > 0$. First notice that $\{I(y_m)\}_{m=N}^{\infty}$ is uniformly bounded since $|y_m(t)| \le M$ for $t \in [0,n]$ and I is continuous; in particular, there exists a constant $Q > 0$ with

$$|I(y_m(t))| \le Q \quad \text{for } t \in [0,n]. \tag{2.34}$$

Also $\{I(y_m)\}$ is equicontinuous on $[0,n]$. To see this take $t, s \in [0,n]$; then Hölder's inequality yields

$$\begin{aligned} |I(y_m(t)) - I(y_m(s))| &= \left| \int_{y_m(s)}^{y_m(t)} \frac{\{-g'(u)\}^{1/2}}{g(u)} \, du \right| \\ &= \left| \int_s^t \frac{\{-g'(y_m(z))\}^{1/2}}{g(y_m(z))} y_m'(z) \, dz \right| \\ &\le |t-s|^{1/2} \left(\int_0^n \left\{ \frac{-g'(y_m(z))}{g^2(y_m(z))} \right\} [y_m'(z)]^2 \, dz \right)^{1/2} \\ &\le N^{1/2} |t-s|^{1/2}. \end{aligned}$$

The Arzelà–Ascoli theorem guarantees the existence of a subsequence $\{I(y_{m'})\}$ converging uniformly on $[0,n]$ to some $I(y) \in C[0,n]$. This together with the fact that I^{-1} is a continuous increasing map and $\{I(y_{m'})\}$ is uniformly bounded implies that the subsequence $y_{m'}$ converges uniformly on $[0,n]$ to $y \in C[0,n]$. Also $y(0)=0$, $y(n)=b$ and

$$\Omega_M(t) \le y(t) \le M \quad \text{for } t \in [0,n]. \tag{2.35}$$

Remark. Note that $y(t) > 0$ for $t \in (0,n]$.

Now $y_{m'}$ satisfies the integral equation

$$y_{m'}(t) = y_{m'}\left(\frac{1}{2}\right) + y'_{m'}\left(\frac{1}{2}\right)\left(t-\frac{1}{2}\right) + \int_{1/2}^{t} (s-t)q(s)f(s,y_{m'}(s))\,ds. \tag{2.36}$$

Remark. It is of interest to note that (2.36) (take $t=3/4$ say) implies that $\{y'_{m'}(1/2)\}$ is a bounded sequence (bound independent of m') since $\Omega_M(t) \le y(t) \le M$ for $t \in [0,n]$. Thus $\{y'_{m'}(1/2)\}$ has a convergent subsequence; for convenience let $\{y'_{m'}(1/2)\}$ denote this subsequence also and let r be its limit. (In fact, the original sequence $\{y'_{m'}(1/2)\}$ is Cauchy from (2.36).)

Fix $t \in (0,n]$. Since f is uniformly continuous on compact subsets of $[\min(1/2,t),\max(1/2,t)] \times (0,M]$, let $m' \to \infty$ in (2.36) to obtain ($r \in \mathbb{R}$)

$$y(t) = y\left(\frac{1}{2}\right) + r\left(t-\frac{1}{2}\right) + \int_{1/2}^{t} (s-t)q(s)f(s,y(s))\,ds. \tag{2.37}$$

From (2.37) we have $y \in C^2(0,n]$ and $-y''(t) = q(t)f(t,y(t))$ for $t \in (0,n)$. Also $y'' \le 0$ on $(0,n)$ since $y > 0$ on $(0,n]$. In addition, essentially the same reasoning as that used to derive (2.26) (in this case we integrate (2.21) from ε to n; here $\varepsilon > 0$ is small) yields

$$\int_0^n \left\{\frac{-g'(y(s))}{g^2(y(s))}\right\}[y'(s)]^2\,ds \le N,$$

where N is as defined in (2.24). □

If (2.9) is replaced by

$$\int_0^1 q(t)g(t)\,dt < \infty \tag{2.38}$$

then the solution to (2.3) will have added smoothness, as the next result shows.

Theorem 2.2. *Suppose* (2.2)–(2.8) *and* (2.38) *are satisfied. Then* (2.3) *has a solution* $y \in C^1[0,n] \cap C^2(0,n]$.

Proof. As in Theorem 2.1 we find that $(2.2)^m$ has a solution $y_m \in C^1[0,n] \cap C^2(0,n]$ with

$$\Omega_M(t) \le y_m(t) \le M \quad \text{for } t \in [0,n]. \tag{2.39}$$

We now claim that there exists a constant Q_0 (independent of m and n) with

$$|y_m'(t)| \le Q_0 \quad \text{for } t \in [0,n]. \tag{2.40}$$

Case (i): $y_m' \ge 0$ *on* $(0,n)$. Now there exists $\xi \in (0,1)$ with $y_m'(\xi) = y_m(1) - y_m(0) \le b - 1/m \le b$ so

$$0 \le y_m'(t) \le b \quad \text{for } t \in [\xi, n]. \tag{2.41}$$

On the other hand, since

$$-y_m''(t) \le \left(1 + \frac{h(M)}{g(M)}\right) q(t) g(\Omega_M(t)) \tag{2.42}$$

we see on integrating from t ($t < \xi$) to ξ that

$$0 \le y_m'(t) \le b + \left(1 + \frac{h(M)}{g(M)}\right) \int_0^1 q(x) g(\Omega_M(x))\, dx. \tag{2.43}$$

Thus (2.40) is true in this case.

Remark. Note that

$$\int_0^1 q(x) g(\Omega_M(x))\, dx = \int_0^1 q(x) g(\Phi_M(1) x)\, dx < \infty.$$

Case (ii): $y_m' \ge 0$ *on* $(0, t_0)$ *and* $y_m' \le 0$ *on* (t_0, n); $t_0 \in (0,n)$. Integrate (2.42) from t ($t < t_0$) to t_0 to obtain

$$0 \le y_m'(t) \le \left(1 + \frac{h(M)}{g(M)}\right) \int_0^\infty q(x) g(\Omega_M(x))\, dx \quad \text{for } t \in [0, t_0]. \tag{2.44}$$

Remark. Note that

$$\int_1^\infty q(x) g(\Omega_M(x))\, dx = \int_1^\infty q(x) g(\Phi_M(x))\, dx \le g(\Phi_M(1)) \int_1^\infty q(x)\, dx < \infty.$$

On the other hand, for $t > t_0$ we have

$$0 \le -y_m'(t) \le \left(1 + \frac{h(M)}{g(M)}\right) \int_0^\infty q(x) g(\Omega_M(x))\, dx \quad \text{for } t \in [t_0, n]. \tag{2.45}$$

Once again (2.40) is true.

Now (2.38), (2.39), (2.40) and (2.42) imply that $\{y_m^{(j)}\}$, $j = 0,1$, is uniformly bounded and equicontinuous on $[0,n]$ so the Arzelà–Ascoli theorem guarantees the existence of a subsequence $\{y_{m'}\}$ and a function $y \in C^1[0,n]$ with $y_{m'}^{(j)}$ converging uniformly on $[0,n]$ to $y^{(j)}$, $j = 0,1$. Also $y(0) = 0$, $y(n) = b$ and

$$\Omega_M(t) \le y(t) \le M, \quad |y'(t)| \le Q_0 \quad \text{for } t \in [0,n].$$

It is easy to check as in Theorem 2.1 that y is a solution of (2.3). $\square$

We now discuss the semi-infinite problems

$$\begin{cases} -y'' = q(t)f(t,y), \quad 0 < t < \infty, \\ y(0) = 0, \\ y \text{ bounded on } [0,\infty), \end{cases} \tag{2.46}$$

and

$$\begin{cases} -y'' = q(t)f(t,y), \quad 0 < t < \infty, \\ y(0) = 0, \\ \lim_{t\to\infty} y(t) \text{ exists}. \end{cases} \tag{2.47}$$

Theorem 2.3. *Suppose* (2.4)–(2.9) *are satisfied. Then* (2.46) *and* (2.47) *have a solution* $y \in C[0,\infty) \cap C^2(0,\infty)$.

Proof. Now Theorem 2.1 implies that for each $n \in \mathbb{N}^+$ there exists a solution $y_n \in C[0,n] \cap C^2(0,n]$ with

$$\Omega_M(t) \le y_n(t) \le M \quad \text{for } t \in [0,n] \tag{2.48}$$

and

$$\int_0^n \left\{ \frac{-g'(y_n(s))}{g^2(y_n(s))} \right\} [y_n'(s)]^2 \, ds \le N. \tag{2.49}$$

Let

$$I(z) = \int_0^z \frac{\{-g'(u)\}^{1/2}}{g(u)} \, du$$

and as in Theorem 2.1 there exists a constant Q (see (2.34)) with

$$|I(y_n(t))| \le Q \quad \text{for } t \in [0,n] \tag{2.50}$$

and

$$|I(y_n(t)) - I(y_n(s))| \le N|t-s|^{1/2} \quad \text{for } t,s \in [0,n]. \tag{2.51}$$

Define

$$u_n(x) = \begin{cases} y_n(x), & x \in [0,n], \\ b, & x \in (n,\infty). \end{cases}$$

Then u_n is continuous on $[0,\infty)$ and $u_n(t) \ge \Omega_M(t)$ for $t \in [0,n]$. Now for $t,s \in [0,\infty)$ it is easy to check that

$$|I(u_n(t))| \le Q \tag{2.52}$$

and

$$|I(u_n(t)) - I(u_n(s))| \le N|t-s|^{1/2}. \tag{2.53}$$

Let $S = \{I(u_n)\}_{n=1}^{\infty}$. By the Arzelà–Ascoli theorem there is a subsequence N_1 of $\mathbb{N}^+$ and a continuous function $I(z_1)$ on $[0,1]$ such that $I(u_n) \to I(z_1)$ uniformly on $[0,1]$ as $n \to \infty$ through N_1. Since I^{-1} is a continuous increasing map and

(2.52) holds, we have $u_n(x) \to z_1(x)$ uniformly on $[0,1]$ as $n \to \infty$ through N_1. Also $\Omega_M(t) \leq z_1(t) \leq M$ for $t \in [0,1]$ and $z_1(0) = 0$. Again the Arzelà–Ascoli theorem implies there is a subsequence N_2 of N_1 and a function $I(z_2) \in C[0,2]$ such that $u_n(x) \to z_2(x)$ uniformly on $[0,2]$ as $n \to \infty$ through N_2. Note $z_2 = z_1$ on $[0,1]$ since $N_2 \subseteq N_1$. Also $\Omega_M(t) \leq z_2(t) \leq M$ for $t \in [0,2]$ and $z_2(0) = 0$. Proceed inductively to obtain for $k = 1, 2, \ldots$ a subsequence $N_k \subseteq \mathbb{N}^+$ with $N_k \subseteq N_{k-1}$ and a function $z_k \in C[0,k]$ such that $u_n(x) \to z_k(x)$ uniformly on $[0,k]$ as $n \to \infty$ through N_k. Also $z_k = z_{k-1}$ on $[0,k-1]$ with $\Omega_M(t) \leq z_k(t) \leq M$ for $t \in [0,k]$ and $z_k(0) = 0$.

Define a function y as follows. Fix $x \in [0,\infty)$ and let $k \in \mathbb{N}^+$ with $x \leq k$. Define $y(x) = z_k(x)$. Now y is well defined with $y \in C[0,\infty)$, $y(0) = 0$ and $\Omega_M(t) \leq y(t) \leq M$ for $t \in [0,\infty)$. Again fix $x > 0$ and choose $k > x$, $k \in \mathbb{N}^+$. Then for $t \in (0,k)$ we have

$$u_n(t) = u_n\left(\frac{1}{2}\right) + u_n'\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) + \int_{1/2}^{t} (s-t) q(s) f(s, u_n(s))\, ds.$$

Let $n \to \infty$ through N_k so for $t \in (0,k]$ we obtain ($r \in \mathbb{R}$),

$$z_k(t) = z_k\left(\frac{1}{2}\right) + r\left(t - \frac{1}{2}\right) + \int_{1/2}^{t} (s-t) q(s) f(s, z_k(s))\, ds$$

since f is uniformly continuous on compact subsets of $[\min(1/2,t), \max(1/2,t)] \times (0,M]$ and $z_k(t) \geq \Omega_M(t) > 0$ for $t \in (0,k]$. Thus for $t \in (0,k]$ we have

$$y(t) = y\left(\frac{1}{2}\right) + r\left(t - \frac{1}{2}\right) + \int_{1/2}^{t} (s-t) q(s) f(s, y(s))\, ds$$

and so $-y''(x) = q(x) f(x, y(x))$. Thus for $0 < s < \infty$ we have $-y''(s) = q(s) f(s, y(s))$ and $y \in C^2(0,\infty)$. Thus we have shown that there exists a solution $y \in C[0,\infty) \cap C^2(0,\infty)$ to (2.46) with $\Omega_M(t) \leq y(t) \leq M$ for $t \in [0,\infty)$. Now since $y(t) \geq \Omega_M(t) > 0$ for $t > 0$ and $-y''(t) = q(t) f(t, y(t))$, $t > 0$, it follows that $y''(t) \leq 0$ for $t \in (0,\infty)$. Thus y' is nonincreasing on $(0,\infty)$ so y will be eventually monotonic (either y' is of fixed sign on $(0,\infty)$ or there exists $t_0 \in (0,\infty)$ with $y'(t_0) = 0$) i.e. there exists $\mu \in (0,\infty)$ with y monotonic for $t \geq \mu$. This together with $0 \leq y(t) \leq M$ for $t \in [0,\infty)$ implies that $\lim_{t\to\infty} y(t)$ exists. Consequently, y is a solution to (2.47). In fact, $\lim_{t\to\infty} y(t) \in [0,M]$. $\square$

Example. The boundary value problem

$$\begin{cases} -y'' = q(t) y^{-\alpha}, \ 0 < t < \infty, \\ y(0) = 0, \\ y \text{ bounded on } [0,\infty), \end{cases}$$

with $\alpha > 0$ and (2.4), (2.5) holding, has a solution $y \in C[0,\infty) \cap C^2(0,\infty)$.

To see this we apply Theorem 2.3. Clearly (2.6) with $g(y) = y^{-\alpha}$ and $h(y) = 0$, (2.7), (2.8) with $\psi_H(t) = H^{-\alpha}$, and (2.9) are satisfied. The result now follows from Theorem 2.3.

THEOREM 2.4. *Suppose* (2.4)–(2.8) *and* (2.38) *are satisfied. Then* (2.46) *and* (2.47) *have a solution* $y \in C^1[0,\infty) \cap C^2(0,\infty)$.

PROOF. Essentially the same reasoning as in Theorem 2.3 (except that we use Theorem 2.2) establishes the result. For more details see the proof of Theorem 3.2. □

Finally, we examine the boundary condition at infinity. In particular, we discuss

$$\begin{cases} -y'' = q(t)f(t,y), \quad 0 < t < \infty, \\ y(0) = 0, \\ \lim_{t\to\infty} y'(t) = 0. \end{cases} \tag{2.54}$$

THEOREM 2.5. *Suppose* (2.4)–(2.8) *and* (2.38) *are satisfied. Then* (2.54) *has a solution* $y \in C[0,\infty) \cap C^2(0,\infty)$ (*in fact, in* $C^1[0,\infty) \cap C^2(0,\infty)$).

PROOF. Theorem 2.3 implies that (2.46) has a solution y with $\Omega_M(t) \le y(t) \le M$ for $t \in [0,\infty)$. Note that since y is a solution to $-y'' = qf(t,y)$, $0 < t < \infty$, with $y(0) = 0$ we have

$$y(t) = At + \int_0^t xq(x)f(x,y(x))\,dx - t\int_0^t q(x)f(x,y(x))\,dx,$$

where A is a constant. However, since $y(t) \le M$ for $t \in [0,\infty)$ we get

$$y(t) = t\int_t^\infty q(x)f(x,y(x))\,dx + \int_0^t xq(x)f(x,y(x))\,dx. \tag{2.55}$$

REMARK. (i) Notice that $\int_0^\infty xq(x)f(x,y(x))\,dx < \infty$ since

$$\int_0^1 xq(x)f(x,y(x))\,dx \le \left(1 + \frac{h(M)}{g(M)}\right)\int_0^1 q(x)g(\Phi_M(1)x)\,dx < \infty$$

and

$$\int_1^\infty xq(x)f(x,y(x))\,dx \le \left(1 + \frac{h(M)}{g(M)}\right)g(\Phi_M(1))\int_1^\infty xq(x)\,dx < \infty.$$

(ii) Also $\int_0^\infty q(x)f(x,y(x))\,dx < \infty$ and $\lim_{t\to\infty} t\int_t^\infty q(x)f(x,y(x))\,dx = 0$ since for $t > 0$ we have

$$0 \le t\int_t^\infty q(x)f(x,y(x))\,dx \le \int_t^\infty xq(x)f(x,y(x))\,dx.$$

Now (2.55) implies

$$y'(t) = \int_t^\infty q(x) f(x, y(x))\, dx \quad \text{for } t > 0$$

so $y' \geq 0$ on $(0,\infty)$ with $\lim_{t\to\infty} y'(t) = 0$. Thus y is a solution to (2.54). □

3. The equation $y'' + f(t, y, y') = 0$

In this section we discuss

$$\text{(3.1)} \qquad \begin{cases} -y'' = q(t) f(t, y, y'), \quad 0 < t < \infty, \\ y(0) = 0, \\ y \text{ bounded on } [0,\infty). \end{cases}$$

Two cases will be examined, the case when q is nonincreasing on $(0,\infty)$ and the case when q is nondecreasing on $(0,\infty)$.

Existence theory I. In this subsection we examine the situation when q is nonincreasing on $(0,\infty)$. The strategy will be to examine for $n \in \mathbb{N}^+$ the boundary value problem

$$\text{(3.2)}^m \qquad \begin{cases} -y'' = q(t) f(t, y, y'), \quad 0 < t < n, \\ y(0) = 1/m, \\ y(n) = b > 0, \end{cases}$$

where $m \in \{N, N+1, \dots\}$ with $N \in \mathbb{N}^+$ and $b > 1/N$. Existence of a solution to $(3.2)^m$ will then be used to establish existence

$$\text{(3.3)} \qquad \begin{cases} -y'' = q(t) f(t, y, y'), \quad 0 < t < n, \\ y(0) = 0, \quad y(n) = b > 0. \end{cases}$$

Theorem 3.1. *Suppose* (2.4), (2.5) *hold and in addition*

(3.4) *q is nonincreasing on $(0,\infty)$ and bounded on $[0,1]$,*

(3.5) $0 \leq f(t, y, p) \leq [g(y) + h(y)][A|p| + B]$ *on* $(0,\infty) \times (0,\infty) \times (-\infty,\infty)$ *with f continuous on $[0,\infty) \times (0,\infty) \times (-\infty,\infty)$, $g > 0$ continuous and nonincreasing on $(0,\infty)$, $h \geq 0$ continuous on $[0,\infty)$ and $A > 0$, $B > 0$ constants,*

(3.6) *h/g is nondecreasing on $(0,\infty)$ and there exists a constant $M_0 > 0$ such that for $z > 0$,*

$$\int_0^z \frac{dx}{g(x)} \leq \left(1 + \frac{h(z)}{g(z)}\right)\left\{Az \int_0^\infty q(x)\, dx + B \int_0^\infty x q(x)\, dx\right\} + \int_0^1 \frac{dx}{g(x)}$$

implies $z \leq M_0$,

$$\text{(3.7)} \qquad \int_0^1 g(x)\, dx < \infty,$$

and

(3.8) *for constants $H > 0$, $K > 0$ there exists a function $\psi_{H,K}$ continuous on $[0,\infty)$ and positive on $(0,\infty)$ such that $f(t,y,p) \ge \psi_{H,K}(t)$ on $(0,\infty) \times (0,H] \times [-K,K]$; in addition $\int_0^\infty xq(x)\psi_{H,K}(x)\,dx < \infty$.*

Then (3.3) *has a solution $y \in C^1[0,n] \cap C^2(0,n]$.*

Proof. Consider the modified problem

$$(3.9)_\lambda^m \qquad \begin{cases} -y'' = \lambda q(t) f^\star(t,y,y'), \quad 0 < t < n, \\ y(0) = 1/m, \\ y(n) = b > 0, \end{cases}$$

where $0 < \lambda < 1$ and $f^\star \ge 0$ is any continuous extension of f from $y \ge 1/m$. Let $y \in C^1[0,n] \cap C^2(0,n]$ be a solution to $(3.9)_\lambda^m$. Now $y'' \le 0$ on $(0,n)$ and $y \ge 1/m$ on $(0,n)$.

Case (i): *$y' \ge 0$ on $(0,n)$.* Then

$$(3.10) \qquad 1/m \le y(t) \le b \quad \text{for } t \in [0,n].$$

Case (ii): *$y' \ge 0$ on $(0,t_0)$ and $y' \le 0$ on (t_0,n); $t_0 \in (0,n)$.* Now for $x \in (0,t_0)$ we have

$$-y''(x) \le q(x)g(y(x))\left\{1 + \frac{h(y(x))}{g(y(x))}\right\}\Big(Ay'(x) + B\Big)$$

so integration from t ($t < t_0$) to t_0 yields

$$y'(t) \le g(y(t))\left\{1 + \frac{h(y(t_0))}{g(y(t_0))}\right\}\left(Aq(t)\int_t^{t_0} y'(x)\,dx + B\int_t^{t_0} q(x)\,dx\right).$$

Thus

$$\frac{y'(t)}{g(y(t))} \le \left\{1 + \frac{h(y(t_0))}{g(y(t_0))}\right\}\left(Aq(t)y(t_0) + B\int_t^{t_0} q(x)\,dx\right)$$

and integration from 0 to t_0 yields

$$\int_0^{y(t_0)} \frac{du}{g(u)} \le \left\{1+\frac{h(y(t_0))}{g(y(t_0))}\right\}\left(Ay(t_0)\int_0^\infty q(x)\,dx + B\int_0^\infty xq(x)\,dx\right) + \int_0^1 \frac{du}{g(u)}.$$

Now (3.6) implies that there exists a constant M_0 with $y(t_0) \le M_0$ and so

$$(3.11) \qquad 1/m \le y(t) \le M_0 \quad \text{for } t \in [0,n].$$

Combining both cases yields

$$(3.12) \qquad 1/m \le y(t) \le \max\{b, M_0\} \equiv M \quad \text{for } t \in [0,n].$$

Next we show there is a constant V, independent of λ and n, with

$$(3.13) \qquad |y'(t)| \le V \quad \text{for } t \in [0,n].$$

Case (i): $y' \geq 0$ *on* $(0,n)$. There exists $\xi \in (0,1)$ with $y'(\xi) = y(1) - y(0) \leq b - 1/m \leq b$ and so

$$0 \leq y'(t) \leq b \quad \text{for } t \in [\xi, n]. \tag{3.14}$$

Also

$$\frac{-y'y''}{Ay' + B} \leq q(t)[g(y) + h(y)]y'$$

so integration from t $(t < \xi)$ to ξ yields

$$\int_0^{y'(t)} \frac{u}{Au + B}\, du \leq \int_0^b \frac{u}{Au + B}\, du + (\sup_{[0,1]} q(t)) \int_0^b [g(u) + h(u)]\, du. \tag{3.15}$$

Let

$$J(z) = \int_0^z \frac{u}{Au + B}\, du \tag{3.16}$$

and note $J(\infty) = \infty$ with $J : [0,\infty) \to [0,\infty)$ an increasing function.

Remark. We can of course calculate J explicitly.

Now (3.15) implies

$$\begin{aligned} y'(t) &\leq J^{-1}\left(\int_0^b \frac{u}{Au + B}\, du + (\sup_{[0,1]} q(t)) \int_0^b [g(u) + h(u)]\, du \right) \\ &\equiv R_0 \quad \text{for } t \in [0, \xi]. \end{aligned} \tag{3.17}$$

Combining (3.14) and (3.17) yields

$$0 \leq y'(t) \leq \max\{b, R_0\} \equiv R_1 \quad \text{for } t \in [0, n]. \tag{3.18}$$

Case (ii): $y' \geq 0$ *on* $(0, t_0)$ *and* $y' \leq 0$ *on* (t_0, n); $t_0 \in (0, n)$.

Now for $t \in (0, t_0)$ we have

$$\frac{-y'y''}{Ay' + B} \leq q(t)[g(y) + h(y)]y'$$

so integration from t $(t < t_0)$ to t_0 yields

$$0 \leq y'(t) \leq J^{-1}\left(\sup_{[0,1]} q(t) \int_0^{M_0} [g(u) + h(u)]\, du \right) \equiv R_2 \quad \text{for } t \in [0, t_0]. \tag{3.19}$$

On the other hand, for $t \in (t_0, n)$ we have

$$\frac{y'y''}{A(-y') + B} \leq q(t)[g(y) + h(y)](-y')$$

and integration from t_0 to t $(t > t_0)$ yields

$$0 \leq -y'(t) \leq R_2 \quad \text{for } t \in [t_0, n]. \tag{3.20}$$

Now (3.19) and (3.20) yield

$$|y'(t)| \leq R_2 \quad \text{for } t \in [0, n]. \tag{3.21}$$

With $V = \max\{R_1, R_2\}$ we obtain (3.13). Now (3.12), (3.13) and Theorem 1.1 imply that $(3.9)_1^m$ has a solution $y_m \in C^1[0,n] \cap C^2(0,n]$ with

$$1/m \le y_m(t) \le M \quad \text{and} \quad |y_m'(t)| \le V \quad \text{for } t \in [0,n]. \tag{3.22}$$

In addition, since $y_m \ge 1/m$ on $[0,n]$, it follows that y_m is a solution of $(3.2)^m$.

Next assumption (3.8) implies that there exists a function $\psi_{M,V}(t)$ continuous on $[0,\infty)$ and positive on $(0,\infty)$ with $f(t,y,p) \ge \psi_{M,V}(t)$ for $(t,y,p) \in (0,\infty) \times (0,M] \times [-V,V]$. Now with

$$k_0 \le \min\left\{1, \frac{b}{\int_0^\infty xq(x)\psi_{M,V}(x)\,dx}\right\}$$

we deduce by essentially the same argument as in Theorem 2.1 that

$$y_m(t) \ge k_0 \int_0^t xq(x)\psi_{M,V}(x)\,dx \equiv \Phi_{M,V}(t) \quad \text{for } t \in [0,n]. \tag{3.23}$$

Let

$$\Omega_{M,V}(t) = \begin{cases} \Phi_{M,V}(1)t, & 0 \le t \le 1, \\ \Phi_{M,V}(t), & t \ge 1. \end{cases}$$

and as in Theorem 2.1,

$$y_m(t) \ge \Omega_{M,V}(t) \quad \text{for } t \in [0,n]. \tag{3.24}$$

Also for $x \in (0,n)$,

$$-y_m''(x) \le q(x)g(y_m(x))\left(1 + \frac{h(y_m(x))}{g(y_m(x))}\right)[A|y_m'(x)| + B]$$

so

$$-y_m''(x) \le \left(1 + \frac{h(M)}{g(M)}\right)[AV + B]q(x)g(\Omega_{M,V}(x)) \quad \text{for } x \in (0,n). \tag{3.25}$$

Now (3.22) and (3.25) imply that $\{y_m^{(j)}\}$, $j = 0,1$, is uniformly bounded and equicontinuous on $[0,n]$ so the Arzelà–Ascoli theorem guarantees the existence of a subsequence $\{y_{m'}\}$ and a function $y \in C^1[0,n]$ with $y_{m'}^{(j)}$ converging uniformly on $[0,n]$ to $y^{(j)}$, $j = 0,1$. Also $y(0) = 0$, $y(n) = b$ with

$$\Omega_{M,V}(t) \le y(t) \le M \quad \text{and} \quad |y'(t)| \le V \quad \text{for } t \in [0,n]. \tag{3.26}$$

Essentially the same reasoning as in Theorem 2.1 now implies

$$y(t) = y\left(\frac{1}{2}\right) + y'\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) + \int_{1/2}^t (s-t)q(s)f(s,y(s),y'(s))\,ds.$$

Consequently, $y \in C^2(0,n]$ and $-y''(t) = q(t)f(t,y(t),y'(t))$ for $t \in (0,n)$. Also $y'' \le 0$ on $(0,n)$ and

$$-y''(x) \le \left(1 + \frac{h(M)}{g(M)}\right)[AV + B]q(x)g(\Omega_{M,V}(x)) \quad \text{for } x \in (0,n). \qquad \square \tag{3.27}$$

We next discuss the semi-infinite problems

$$
\begin{cases} -y'' = q(t)f(t,y,y'), \quad 0 < t < \infty, \\ y(0) = 0, \\ y \text{ bounded on } [0,\infty), \end{cases} \tag{3.28}
$$

and

$$
\begin{cases} -y'' = q(t)f(t,y,y'), \quad 0 < t < \infty, \\ y(0) = 0, \\ \lim_{t\to\infty} y(t) \text{ exists.} \end{cases} \tag{3.29}
$$

Theorem 3.2. *Suppose* (2.4), (2.5) *and* (3.4)–(3.8) *are satisfied. Then* (3.28) *and* (3.29) *have a solution* $y \in C^1[0,\infty) \cap C^2(0,\infty)$.

Proof. Theorem 3.1 guarantees for each $n \in \mathbb{N}^+$ a solution $y_n \in C^1[0,n] \cap C^2(0,n]$ with

$$
\Omega_{M,V}(t) \le y_n(t) \le M \quad \text{and} \quad |y_n'(t)| \le V \quad \text{for } t \in [0,n] \tag{3.30}
$$

and

$$
-y_n''(x) \le \left(1 + \frac{h(M)}{g(M)}\right)[AV + B]q(x)g(\Omega_{M,V}(x)) \quad \text{for } x \in (0,n). \tag{3.31}
$$

Notice that for $t, s \in [0,n]$ we have

$$
|y_n(t) - y_n(s)| \le V|t-s| \tag{3.32}
$$

and

$$
|y_n'(t) - y_n'(s)| \le \left(1 + \frac{h(M)}{g(M)}\right)[AV + B]\left|\int_s^t q(x)g(\Omega_{M,V}(x))\,dx\right|. \tag{3.33}
$$

Define

$$
u_n(x) = \begin{cases} y_n(x), & x \in [0,n], \\ b, & x \in (n,\infty). \end{cases}
$$

Each u_n is continuous on $[0,\infty)$ and is twice continuously on $(0,\infty)$ except possibly at $x = n$. Let $S = \{u_n\}_{n=1}^{\infty}$. By the Arzelà–Ascoli theorem there is a subsequence $N_1^\star$ of $\mathbb{N}^+$ and a function $z_1 \in C^1[0,1]$ with $u_n^{(j)}(x) \to z_1^{(j)}(x)$, $j = 0,1$, uniformly on $[0,1]$ as $n \to \infty$ through $N_1^\star$. Let $N_1 = N_1^\star/\{1\}$. Then there is a subsequence $N_2^\star$ of N_1 and a function $z_2 \in C^1[0,2]$ with $u_n^{(j)}(x) \to z_2^{(j)}(x)$, $j = 0,1$, uniformly on $[0,2]$ as $n \to \infty$ through $N_2^\star$. Note that $z_2 = z_1$ on $[0,1]$. Let $N_2 = N_2^\star/\{2\}$ and proceed inductively to obtain for $k = 1,2,\ldots$ a subsequence $N_k^\star \subseteq N_{k-1}$ and a function $z_k \in C^1[0,k]$ with $u_n^{(j)}(x) \to z_k^{(j)}(x)$, $j = 0,1$, uniformly on $[0,k]$ as $n \to \infty$ through $N_k^\star$.

Define the function y as follows. Fix $x \in [0,\infty)$ and let $k \in \mathbb{N}^+$ with $x \le k$. Then define $y(x) = z_k(x)$ so $y \in C^1[0,\infty)$, $y(0) = 0$ with $\Omega_{M,V}(t) \le y(t) \le M$,

$t \in [0,\infty)$ and $|y'(t)| \leq V$, $t \in [0,\infty)$. Essentially the same argument as in Theorem 2.3 shows that y is a solution to (3.28) and (3.29). $\square$

Next we discuss

$$\begin{cases} -y'' = q(t)f(t,y,y'), & 0 < t < \infty, \\ y(0) = 0, \\ \lim_{t\to\infty} y'(t) = 0. \end{cases} \tag{3.34}$$

Theorem 3.3. *Suppose* (2.4), (2.5) *and* (3.4)–(3.8) *are satisfied. Then* (3.34) *has a solution* $y \in C^1[0,\infty) \cap C^2(0,\infty)$.

Proof. Theorem 3.2 implies that (3.28) has a solution y with $\Omega_{M,V}(t) \leq y(t) \leq M$ for $t \in [0,\infty)$ and $|y'(t)| \leq V$ for $t \in [0,\infty)$. Consequently,

$$y(t) = t\int_t^\infty q(x)f(x,y(x),y'(x))\,dx + \int_0^t xq(x)f(x,y(x),y'(x))\,dx. \tag{3.35}$$

Remark. Notice that

$$\int_0^\infty xq(x)f(x,y(x),y'(x))\,dx < \infty \quad \text{and} \quad \int_0^\infty q(x)f(x,y(x),y'(x))\,dx < \infty$$

since

$$\int_0^1 xq(x)f(x,y(x),y'(x))\,dx \leq \left(1 + \frac{h(M)}{g(M)}\right)[AV + B]\sup_{[0,1]} q(t)\int_0^1 g(\Phi_{M,V}(1)x)\,dx < \infty.$$

Now (3.35) implies

$$y'(t) = \int_t^\infty q(x)f(x,y(x),y'(x))\,dx \quad \text{for } t > 0$$

so $y' \geq 0$ on $(0,\infty)$ with $\lim_{t\to\infty} y'(t) = 0$. $\square$

Existence theory II. In this subsection we will assume q is nondecreasing on $(0,\infty)$. Our existence theory is motivated by (1.3); see [1, 3]. We first discuss the boundary value problem (3.3).

Theorem 3.4. *Suppose* (2.4) *holds and in addition assume that*

(3.36) f *is continuous on* $[0,\infty) \times (0,\infty) \times (-\infty,\infty)$,

(3.37) $pf(t,y,p) \geq 0$ *on* $[0,\infty) \times (0,\infty) \times (-\infty,\infty)$,

(3.38) $f(t,y,p) \leq [g(y) + h(y)]\mu(|p|)$ *on* $(0,\infty) \times (0,\infty) \times (-\infty,\infty)$ *with* $g > 0$ *continuous and nonincreasing on* $(0,\infty)$, $h \geq 0$ *continuous on* $[0,\infty)$ *and* h/g *nondecreasing on* $(0,\infty)$,

(3.39) q *is nondecreasing on* $(0,\infty)$ *and bounded on* $[0,1]$, (3.40) $\int_0^1 g(x)\,dx < \infty$,

(3.41) *$\mu > 0$ is continuous on $(-\infty,\infty)$ with*

$$\int_0^\infty \frac{x}{\mu(x)}\,dx > \sup_{[0,1]} q(t)\left(1 + \frac{h(b)}{g(b)}\right)\int_0^b g(x)\,dx + \int_0^b \frac{x}{\mu(x)}\,dx$$

and

(3.42) *for constants $H > 0$, $K > 0$ there exists a function $\psi_{H,K}$ continuous on $[0,\infty)$, positive and nondecreasing on $(0,\infty)$ and constants $A > 0$, $B \ge 0$, $1 \le r < 2$ such that $f(t,y,p) \ge \psi_{H,K}(t)[Ap^r + B]$ on $(0,\infty)\times(0,H]\times[0,K]$.*

Then (3.3) *has a solution $y \in C^1[0,n]\cap C^2(0,n]$.*

Proof. Let y be a solution to $(3.9)^m_\lambda$ with

$$f^\star(t,u,p) = \begin{cases} f(t,1/m,p), & u \le 1/m,\\ f(t,u,p), & u \ge 1/m.\end{cases}$$

Remark. Note that $pf^\star(t,u,p) \ge 0$ on $[0,\infty)\times(-\infty,\infty)\times(-\infty,\infty)$.

We claim that

$$y' \ge 0 \quad \text{on } [0,n]. \tag{3.43}$$

To see this notice that for any $\tau \in [0,n)$ and $t > \tau$ we have

$$-[y'(t)]^2 + [y'(\tau)]^2 = 2\lambda\int_\tau^t q(x)y'(x)f^\star(x,y(x),y'(x))\,dx \ge 0$$

so

$$[y'(t)]^2 \le [y'(\tau)]^2 \quad \text{for } t \ge \tau. \tag{3.44}$$

If $y'(0) = 0$ then (3.44) with $\tau = 0$ implies $y'(t) = 0$ for $t \in [0,n]$, a contradiction since $b > 1/m$. Thus $y'(0) \ne 0$. Then either $y' \ne 0$ on $[0,n]$ or there exists $\delta \le n$ with $y'(t) \ne 0$ for $t \in [0,\delta)$ and $y'(\delta) = 0$.

Case (i): *$y' \ne 0$ on $[0,n]$.* If $y' < 0$ on $[0,n]$ then $1/m = y(0) > y(n) = b$, a contradiction. Thus $y' > 0$ on $[0,n]$.

Case (ii): *There exists $\delta \le n$ with $y'(t) \ne 0$ for $t \in [0,\delta)$ and $y'(\delta) = 0$.* Now (3.44) with $\tau = \delta$ implies $y' = 0$ on $[\delta,n]$ so $y(t) = b$ for $t \in [\delta,n]$. If $y' < 0$ on $[0,\delta)$ then $y(\delta) < 1/m$, a contradiction. Hence $y' > 0$ on $[0,\delta)$.

Consequently, (3.43) is true. Now the definition of $f^\star$ together with (3.37) implies $y'' \le 0$ on $(0,n)$. Also

$$1/m \le y(t) \le b \quad \text{for } t \in [0,n]. \tag{3.45}$$

Now there exists $\xi \in (0,1)$ with $y'(\xi) = y(1) - y(0) \le b$ and so

$$0 \le y'(t) \le b \quad \text{for } t \in [\xi, n]. \tag{3.46}$$

Also since

$$\frac{-y'y''}{\mu(y')} \le q(t)g(y(t))\left(1 + \frac{h(y(t))}{g(y(t))}\right)y'$$

we deduce on integration from t $(t < \xi)$ to ξ that

$$\int_0^{y'(t)} \frac{u}{\mu(u)}\,du \le \sup_{[0,1]} q(t)\left(1 + \frac{h(b)}{g(b)}\right)\int_0^b g(u)\,du + \int_0^b \frac{u}{\mu(u)}\,du.$$

Define

$$K(z) = \int_0^z \frac{u}{\mu(u)}\,du$$

so

$$\begin{aligned} y'(t) &\le K^{-1}\left(\sup_{[0,1]} q(t)\left(1 + \frac{h(b)}{g(b)}\right)\int_0^b g(u)\,du + \int_0^b \frac{u}{\mu(u)}\,du\right) \\ &\equiv N_0 \quad \text{for } t \in [0, \xi]. \end{aligned} \tag{3.47}$$

Combining (3.46) and (3.47) yields

$$0 \le y'(t) \le \max\{N_0, b\} \equiv V \quad \text{for } t \in [0, n]. \tag{3.48}$$

Theorem 1.1 implies that $(3.9)_1^m$ (and consequently $(3.2)^m$) has a solution y_m with

$$1/m \le y_m(t) \le b \quad \text{and} \quad 0 \le y_m'(t) \le V \quad \text{for } t \in [0, n]. \tag{3.49}$$

Next assumption (3.42) implies there is a function $\psi_{b,V}(t)$ with $f(t,y,p) \ge \psi_{b,V}(t)$ for $(t,y,p) \in (0,\infty) \times (0,b] \times [0,V]$. Thus

$$-y_m''(t) \ge q(t)\psi_{b,V}(t)[A(y_m'(t))^r + B] \quad \text{for } t \in (0, n). \tag{3.50}$$

Case (i): $r = 1$. Integrating (3.50), with $r = 1$, from t to n yields

$$y_m'(t) \ge Aq(t)\psi_{b,V}(t)\int_t^n y_m'(x)\,dx$$

so

$$y_m'(t) \ge Aq(t)\psi_{b,V}(t)[b - y_m(t)].$$

Integration from 0 to t now yields

$$y_m(t) \ge b - b\exp\left(-A\int_0^t q(x)\psi_{b,V}(x)\,dx\right) \equiv \Phi_1(t), \quad t \in [0, n]. \tag{3.51}$$

Remark. Note that $\Phi_1(t) \to b$ as $t \to \infty$.

Case (ii): $1 < r < 2$. We know either $y_m' > 0$ on $[0, n]$ or there exists $\delta \le n$ with $y_m' > 0$ on $[0, \delta)$ and $y_m' = 0$ on $[\delta, n]$. Multiply (3.50) by $(y_m')^{1-r}$ and integrate from t to n if $y_m' > 0$ on $[0, n]$ whereas integrate from t $(t < \delta)$ to δ if

$y_m' > 0$ on $[0,\delta)$ and $y_m'(\delta) = 0$, to obtain (using the fact that q and $\psi_{b,V}$ are nondecreasing on $(0,\infty)$)

$$y_m'(t) \geq (A(2-r)q(t)\psi_{b,V}(t)[b - y_m(t)])^{1/(2-r)}.$$

Hence

$$[b - y_m(t)]^{-1/(2-r)} y_m'(t) \geq (A(2-r)q(t)\psi_{b,V}(t))^{1/(2-r)}.$$

Integrate from 0 to t to obtain

$$[b - y_m(t)]^{(1-r)/(2-r)} \geq b^{(1-r)/(2-r)} + \left(\frac{2-r}{1-r}\right)\int_0^t (A(2-r)q(x)\psi_{b,V}(x))^{1/(2-r)}\,dx.$$

Thus on $[0,n]$ we have

$$\begin{aligned} &y_m(t) \\ &\geq b - \frac{b}{(1 + b^{(r-1)/(2-r)}(\frac{2-r}{r-1})\int_0^t (A(2-r)q(x)\psi_{b,V}(x))^{1/(2-r)}\,dx)^{(r-1)/(2-r)}} \\ &\equiv \Phi_r(t). \end{aligned} \tag{3.52}$$

Remark. Note that $\Phi_r(t) \to b$ as $t \to \infty$.

Let

$$\Omega_r(t) = \begin{cases} \Phi_r(1)t, & 0 \leq t \leq 1, \\ \Phi_r(t), & t \geq 1. \end{cases}$$

As in Theorem 2.1 we have

$$\Omega_r(t) \leq y_m(t) \leq b \quad \text{and} \quad 0 \leq y_m'(t) \leq V \quad \text{for } t \in [0,n]. \tag{3.53}$$

Remark. Note that $\Omega_r(t) \to b$ as $t \to \infty$.

Finally, for $t \in (0,n)$ we have

$$-y_m''(t) \leq q(t)g(\Omega_r(t))\left(1 + \frac{h(b)}{g(b)}\right)\max_{[0,V]}\mu(p).$$

Essentially the same reasoning as in Theorem 2.1 now implies that (3.3) has a solution. □

The argument in Theorem 3.2 immediately implies the following result.

Theorem 3.5. *Suppose* (2.4) *and* (3.36)–(3.42) *are satisfied. Then* (3.28) *and* (3.29) *have a solution* $y \in C^1[0,\infty) \cap C^2(0,\infty)$.

Consider the boundary value problem

$$\begin{cases} -y'' = q(t)f(t,y,y'), & 0 < t < \infty, \\ y(0) = 0, \\ \lim_{t\to\infty} y(t) = b > 0. \end{cases} \tag{3.54}$$

Theorem 3.6. *Suppose* (2.4) *and* (3.36)–(3.42) *are satisfied. Then* (3.54) *has a solution* $y \in C^1[0,\infty) \cap C^2(0,\infty)$.

Proof. Theorem 3.5 guarantees a solution $y \in C^1[0,\infty) \cap C^2(0,\infty)$ to (3.28) with $\Omega_r(t) \leq y(t) \leq b$ for $t \in [0,\infty)$. Now since $\Omega_r(t) \to b$ as $t \to \infty$ the result follows. □

Example. The boundary value problem

$$\begin{cases} -y'' = 2ty'y^{-1/2}, & 0 < t < \infty, \\ y(0) = 0, \\ \lim_{t\to\infty} y(t) = 1, \end{cases} \tag{3.55}$$

has a solution.

Let $q(t) = 2t$, $f(t,y,p) = py^{-1/2}$, $g(y) = y^{-1/2}$, $h(y) = 0$, $\mu(p) = |p| + 1$ and $b = 1$. Notice that (2.4), (3.36)–(3.40), (3.41) since

$$\int_0^\infty \frac{x}{\mu(x)}\,dx = \int_0^\infty \frac{x}{x+1}\,dx = \infty,$$

and (3.42), with $\psi_{H,K} = H^{-1/2}$, $A = 1$, $r = 1$, $B = 0$, are satisfied. Existence of a solution to (3.55) is guaranteed by Theorem 3.6.

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