

ON THE PERIODIC SOLUTIONS PROBLEM FOR PARABOLIC INCLUSIONS WITH A LARGE PARAMETER

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0. Introduction

In this paper we study the dependence on a parameter of the periodic solutions of a system of parabolic inclusions of the form

$$(1) \quad y'_i(t) \in A_i y_i + f_i(t, y_1, \dots, y_n), \quad i = 1, \dots, n,$$

where X_1, \dots, X_n are separable Banach spaces, A_i are the generators of analytic semigroups $e^{A_i t}$ in X_i , $i = 1, \dots, n$, and

$$f_i : \mathbb{R} \times X_1 \times \dots \times X_n \multimap X_i, \quad i = 1, \dots, n,$$

are nonlinear multivalued maps which are T -periodic with respect to the first variable. We consider here the case when the analytic semigroups $e^{A_i t}$ are not compact.

We consider two different types of dependence of the right hand side of (1) on a large parameter μ . In both cases the right hand side does not have a limit as $\mu \rightarrow \infty$. But we can construct a formal limit of inclusions in such a way that the vector fields whose fixed points represent the periodic solutions of the inclusions depending on the parameter and the vector fields of the limit inclusions are homotopic.

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The first case is the following:

$$(2) \quad y'_i(t) \in A_i y_i + f_i(\mu t, y_1, \dots, y_n), \quad i = 1, \dots, n.$$

This inclusion describes the influence on the system of exterior forces of high frequency. We will provide an averaging principle for system (2). For this, we propose a method which permits formulating the averaging principle in terms of the topological index for multivalued mappings. This formulation allows us to obtain, in a topological setting, an analogue of the classical second theorem of N. N. Bogolyubov for o.d.e.'s (see e.g. [2]). Since the involved semigroups are not compact we cannot apply the topological degree theory for compact multivalued operators; indeed, we employ the degree theory for condensing operators (see [1], [3]). For parabolic equations with compact semigroups we refer to [7], [8]. With respect to the results contained in [11] and [12] we give new and more general conditions in order to guarantee that the integral operator corresponding to the periodic problem is condensing. The measure of noncompactness we use here is the one proposed in [9] for abstract parabolic differential equations with noncompact semigroups. This measure of noncompactness is not semi-additive, and so we cannot use a linear homotopy to the limit operator. We overcome this difficulty by using a linear homotopy to a compact multivalued operator which belongs to the same class of homotopy of the limit condensing operator. For recent investigations on parabolic inclusions in Banach spaces based on the degree theory for multivalued maps we refer to [10]–[12], [14] and [15].

In the second case the parameter μ can be interpreted as a large coefficient of a diffusion problem and the relative model is of the form

$$(3) \quad y'_i(t) \in \mu A_i y_i + f_i(t, y_1, \dots, y_n), \quad i = 1, \dots, n.$$

We consider here the case when zero is a point of the spectrum of the natural operator generated by the operators A_i in the product space X . Under our conditions zero is an eigenvalue whose geometric and algebraic multiplicities coincide. In this case the limit inclusion is an ordinary differential inclusion in the finite-dimensional eigenspace corresponding to the eigenvalue zero.

1. Definitions and preliminary results

Let X and Y be topological spaces; a multivalued map M from X to Y will be denoted by the symbol $M : X \multimap Y$.

DEFINITION 1.1. A multivalued map $M : X \multimap Y$ is said to be *upper semi-continuous* (briefly u.s.c.) if the set

$$M^{-1}(V) = \{x \in X : M(x) \subseteq V\}$$

is open in X for every open set $V \subseteq Y$.

A multivalued map $M : X \multimap Y$ is said to be *closed* if its graph

$$\text{Gr } M = \{(x, y) \in X \times Y : y \in M(x)\}$$

is a closed subset of $X \times Y$.

Let us recall the following concepts (see, e.g. [1], [3]).

DEFINITION 1.2. Let X be a Banach space. A function ψ defined on the collection of all subsets of X with values in some partially ordered (\leq) set is called a *measure of noncompactness* in X if

$$\psi(\overline{\text{co}} \Omega) = \psi(\Omega)$$

for every $\Omega \in X$. A measure of noncompactness ψ is called

- (i) *monotone* if $\Omega_0, \Omega_1 \in 2^X$ and $\Omega_0 \subseteq \Omega_1$ imply $\psi(\Omega_0) \leq \psi(\Omega_1)$;
- (ii) *compactly invariant* if $\psi(\Omega \cup K) = \psi(\Omega)$ for every compact set $K \subset X$ and $\Omega \in 2^X$;
- (iii) *regular* if $\psi(\Omega) = 0$ is equivalent to the relative compactness of Ω ;
- (iv) *semi-additive* if $\psi(\Omega_0 \cup \Omega_1) = \max\{\psi(\Omega_0), \psi(\Omega_1)\}$ for every $\Omega_0, \Omega_1 \in 2^X$.

A well known example of a measure of noncompactness with properties (i)–(iv) is the Hausdorff measure of noncompactness

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

We now give another example of a measure of noncompactness which we will use in the sequel.

EXAMPLE 1.3. Let $X = X_1 \times \dots \times X_n$ and let $C_T(X)$ be the space of all T -periodic continuous functions with values in X . Consider the measure of noncompactness φ in $C_T(X)$ defined by the formula

$$\varphi(\Omega) = (\chi_1(\Omega_1(t)), \dots, \chi_n(\Omega_n(t)), \limsup_{\delta \rightarrow 0} \max_{x \in \Omega} \max_{0 \leq \tau \leq \delta} \|x(t) - x(t + \tau)\|)$$

where Ω_i is the projection of Ω into $C_T(X_i)$ and $\Omega_i(t) = \{y(t) : y \in \Omega_i\}$. The values of this measure of noncompactness belong to the space $M_T(\mathbb{R}^n) \times \mathbb{R}$ where $M_T(\mathbb{R}^n)$ is the space of T -periodic, measurable functions with values in \mathbb{R}^n . The measure of noncompactness φ has properties (i)–(iii), but it does not fulfill (iv). Observe that the ordering related to property (i) is defined as follows: the space \mathbb{R}^n is ordered by the cone \mathbb{R}_+^n of positive coordinates while the space $M_T(\mathbb{R}^n)$ is ordered by the cone

$$\mathbb{K} = \{y : y \in M_T(\mathbb{R}^n), y(t) \in \mathbb{R}_+^n \text{ a.e.}\},$$

and the space $M_T(\mathbb{R}^n) \times \mathbb{R}$ is ordered by $\mathbb{K} \times [0, \infty)$.

Let $L(E)$ be the space of all bounded linear operators defined on the Banach space E . The χ -norm (see [1]) of the operator $B \in L(E)$ is defined as

$$\|B\|^{(\chi)} := \chi(BS),$$

where S is the unit sphere in E . We will use the following properties of the χ -norm.

LEMMA 1.4 (cf. [1]). *If $B \in L(E)$, then*

$$\chi(B\Omega) \leq \|B\|^{(\chi)} \chi(\Omega)$$

for every nonempty bounded set $\Omega \subseteq E$.

LEMMA 1.5 (cf. [10]). *Let E be separable, $B \in L(E)$, $\|B\|^{(\chi)} < 1$ and the operator $I - B$ be invertible. Then*

$$\|(I - B)^{-1}\|^{(\chi)} \leq (1 - \|B\|^{(\chi)})^{-1}.$$

DEFINITION 1.6 (cf. [3]). Let E be a Banach space, and let $U \subset E$ be a bounded, open set. Let $F : \bar{U} \multimap E$ be an u.s.c., multivalued mapping with nonempty, compact, convex values, and let ψ be a measure of noncompactness in E . The operator F is said to be ψ -condensing if $\Omega \subseteq \bar{U}$ and $\psi(F(\Omega)) \geq \psi(\Omega)$ imply that Ω is relatively compact.

DEFINITION 1.7 (cf. [3]). An u.s.c. family of multivalued maps

$$(4) \quad \mathcal{F} : [0, 1] \times \bar{U} \multimap E$$

with nonempty, compact, convex values is called ψ -condensing if $\Omega \subset \bar{U}$ and $\psi(\mathcal{F}([0, 1] \times \Omega)) \geq \psi(\Omega)$ imply that Ω is relatively compact.

PROPOSITION 1.8 (cf. [1]). *Let ψ be a monotone, invariant measure of noncompactness. Let F be an u.s.c., ψ -condensing operator on \bar{U} and \widehat{F} be a closed, compact operator on \bar{U} . Then the family $\mathcal{F}(\lambda, x) = (1 - \lambda)F(x) + \lambda\widehat{F}(x)$ is an u.s.c. ψ -condensing family.*

DEFINITION 1.9 (cf. [3]). Two ψ -condensing multivalued maps

$$F_0, F_1 : \bar{U} \rightarrow 2^E$$

are said to be *homotopic* if there exists a ψ -condensing family (4) such that $F_0(\cdot) = \mathcal{F}(0, \cdot)$, $F_1(\cdot) = \mathcal{F}(1, \cdot)$ and $\mathcal{F}(\lambda, \cdot)$ has no fixed points on ∂U for $\lambda \in [0, 1]$. The family \mathcal{F} is called a *homotopy*. If the measure of noncompactness ψ is monotone and compactly invariant and F has no fixed points on ∂U , one can define (see [1], [3]) an integer-valued characteristic $\deg(I - F, U)$, which is called *degree* and has the following properties:

1°. Homotopic ψ -condensing operators have equal degrees. In the class of ψ -condensing operators homotopic to F there exists a compact operator \widehat{F} such that $\mathcal{F}(\lambda, x) = (1 - \lambda)F(x) + \lambda\widehat{F}(x)$ is a homotopy.

2°. Let $U_i, i \in \mathbb{N}$, be a family of pairwise disjoint open subsets of U and suppose F has no fixed points on $\overline{U} \setminus \bigcup_{i=1}^{\infty} U_i$. Then the degrees $\deg(I - F, U_i)$ are defined for all $i \in \mathbb{N}$; only a finite number of them are different from zero, and

$$\deg(I - F, U) = \sum_{i=1}^{\infty} \deg(I - F, U_i).$$

3°. If $F(x) \equiv x_0$, then $\deg(I - F, U) = \begin{cases} 1 & \text{if } x_0 \in U, \\ 0 & \text{if } x_0 \notin U. \end{cases}$

4°. If $\deg(I - F, U) \neq 0$ then F has at least one fixed point in U .

THEOREM 1.10 (cf. [11]). *Let L be a closed subspace of E and let $F : \overline{U} \multimap L$ be an u.s.c., ψ -condensing multivalued mapping with nonempty, compact, convex values which has no fixed points on ∂U . Then*

$$\deg_E(I - F, U) = \deg_L(I - F, U).$$

A multivalued map $G : [a, b] \multimap E$ with nonempty, compact, convex values is said to be *measurable* if it satisfies any of the following two equivalent conditions:

- (i) the set $G^{-1}(V) = \{t \in [a, b] : G(t) \subseteq V\}$ is measurable for every open set $V \subseteq E$;
- (ii) there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of measurable functions $g_n : [a, b] \rightarrow E$ such that $G(t) = \overline{\{g_n(t)\}_{n=1}^{\infty}}$ for all $t \in [a, b]$ (see e.g. [4]).

We denote by S_G^1 the set of all Bochner integrable selectors of the multivalued map $G : [a, b] \multimap E$, i.e.

$$S_G^1 = \{g \in L^1([a, b], E) : g(t) \in G(t) \text{ a.e.}\}.$$

If $S_G^1 \neq \emptyset$, then G is called *integrable* and

$$\int_{\mathcal{I}} G(s) ds := \left\{ \int_{\mathcal{I}} g(s) ds : g \in S_G^1 \right\}$$

for every measurable set $\mathcal{I} \subseteq [a, b]$.

Clearly, if G is measurable and *integrably bounded* (i.e. there exists $\alpha \in L^1_+([a, b])$ such that $\|G(t)\| := \max\{\|y\| : y \in G(t)\} \leq \alpha(t)$ a.e.), then G is integrable.

We also need the following property:

LEMMA 1.11 (cf. [11]). *Let $G : [a, b] \multimap E$ be a multivalued map with nonempty bounded values. Assume G is integrable, integrably bounded and*

$$\chi(G(t)) \leq \gamma(t) \quad \text{a.e. on } [a, b],$$

where $\gamma \in L_+^1([a, b])$. Then

$$\chi\left(\int_{\mathcal{I}} G(s) ds\right) \leq \int_{\mathcal{I}} \gamma(s) ds$$

for every measurable set $\mathcal{I} \subseteq [a, b]$. In particular, if $\chi(G(\cdot)) \in L_+^1([a, b])$, then

$$\chi\left(\int_{\mathcal{I}} G(s) ds\right) \leq \int_{\mathcal{I}} \chi(G(s)) ds.$$

We also recall some results from the theory of semigroups which we will use in the sequel (see e.g. [13]).

THEOREM 1.12. *A closed operator A is the infinitesimal generator of the analytic semigroup e^{At} if and only if the resolvent set of A contains a half-plane $\operatorname{Re} \lambda \leq \sigma_0$ and the resolvent satisfies there the inequality*

$$(5) \quad \|(\lambda I - A)^{-1}\| \leq C(1 + |\lambda|)^{-1} \quad \text{for some } C > 0.$$

If A is the infinitesimal generator of the analytic semigroup e^{At} , then

$$e^{At} = -\frac{1}{2\pi i} \int_{\Pi(\beta, \sigma)} e^{\lambda t} (\lambda I - A)^{-1} d\lambda, \quad t > 0,$$

where $\Pi(\beta, \sigma)$ consists of the two rays

$$\lambda = \sigma + \rho e^{-i\beta} \quad \text{and} \quad \lambda = \sigma + \rho e^{i\beta}, \quad \sigma \leq \sigma_0, \quad \arcsin \frac{1}{C} < \beta < \frac{\pi}{2}.$$

If $\sigma_0 < 0$, then the negative fractional powers of A are defined by the formula

$$A^{-\alpha} = -\frac{1}{2\pi i} \int_{\Pi(\beta, \sigma)} \lambda^{-\alpha} (\lambda I - A)^{-1} d\lambda, \quad 0 < \alpha < 1.$$

The operator

$$A^\alpha e^{At} = -\frac{1}{2\pi i} \int_{\Pi(\beta, \sigma)} \lambda^\alpha e^{\lambda t} (\lambda I - A)^{-1} d\lambda, \quad t > 0,$$

satisfies the estimate

$$(6) \quad \|A^\alpha e^{At}\| \leq Ct^{-\alpha}.$$

From this inequality one obtains

$$(7) \quad \|A^{-\alpha}(e^{At} - I)\| \leq Ct^\alpha.$$

In what follows, when no confusion can arise, the constants will be indicated by the same letter C .

2. The integral operator

In this section we want to investigate the existence of periodic solutions of (1) when f_i , $i = 1, \dots, n$, have nonempty convex values. We study first the properties of the integral operator

$$\Gamma x = \left\{ y : y \in C_T(X), y(t) = e^{At}[I - e^{AT}]^{-1} \int_0^T e^{A(T-s)}g(s) ds + \int_0^t e^{A(t-s)}g(s) ds, g \in S_{\Phi(\cdot, x(\cdot))}^1, g \text{ is } T\text{-periodic} \right\}$$

where

$$(8) \quad e^{A\tau}g = (e^{A_1\tau}g_1, \dots, e^{A_n\tau}g_n),$$

$$(9) \quad \Phi(s, y) = (f_1(s, y_1, \dots, y_n), \dots, f_n(s, y_1, \dots, y_n)).$$

For one parabolic inclusion this operator was considered in [11], [12].

We assume the following conditions on the operators f_i and A_i , $i = 1, \dots, n$:

(F1) f_i is T -periodic in the first variable, i.e.

$$f_i(t + T, \cdot) \equiv f_i(t, \cdot), \quad t \in \mathbb{R};$$

(F2) for every $x \in X$ the multivalued map $f_i(\cdot, x) : [0, T] \rightarrow \text{Kc}(X_i)$ admits a measurable selector. Here $\text{Kc}(X_i)$ denotes the collection of all nonempty, compact, convex subsets of X_i ;

(F3) for almost all $t \in [0, T]$, the multivalued map $f_i(t, \cdot) : X \rightarrow \text{Kc}(X_i)$ is u.s.c.;

(F4) there exists an $(n \times n)$ -matrix M with nonnegative components (m_{ij}) such that

$$\chi_i(f_i([0, T] \times D_1 \times \dots \times D_n)) \leq \sum_{j=1}^n m_{ij}\chi_j(D_j)$$

for every bounded $D_1 \subset X_1, \dots, D_n \subset X_n$. Here χ_i denotes the Hausdorff measure of noncompactness in the space X_i , $i = 1, \dots, n$;

(A) for any $i = 1, \dots, n$, the closed linear operator A_i defined on the separable Banach space X_i generates the analytic semigroup $e^{A_i t}$ satisfying

$$(10) \quad \|e^{A_i t}\|(x) \leq e^{-\gamma_i(t-s)},$$

where $\gamma_i > 0$ and the zero solution of the system of ordinary differential equations

$$(11) \quad z'_i = -\gamma_i z_i + \sum_{j=1}^n m_{ij}z_j, \quad i = 1, \dots, n,$$

is exponentially stable.

Without loss of generality we can assume that the semigroup

$$e^{At}x = (e^{A_1 t}x_1, \dots, e^{A_n t}x_n)$$

in the space X satisfies the inequality

$$(12) \quad \|e^{At}\| \leq Ce^{-dt},$$

where $d > 0$ and $C \geq 1$. In order to obtain this estimate, it is sufficient to subtract and add to the right hand side of every inclusion (1) the term Ly_i with the constant L sufficiently large and to consider as new A_i the operator $A_i - LI$ and as new f_i the multivalued map $f_i(t, y_1, \dots, y_n) + Ly_i$. It is easy to see that for the new A_i and f_i conditions (A), (F1)–(F4) and the estimate (12) are still satisfied. Observe that (12) implies $1 \notin \sigma(e^{AT})$ and so $0 \notin \sigma(A)$.

From condition (F4) it follows that f_i is bounded on every bounded set and so every measurable selector from (F2) is integrable. As in [12] one can prove the following result.

PROPOSITION 2.1. *For every sequence $\{x_m\}$ of continuous functions, where $x_m \in C([0, 1], X)$, uniformly convergent to x_0 , the sequence of selectors $g_m(s) \in f_i(s, x_m(s))$ is weakly compact in $L^1([0, T], X_i)$ and its limit points are selectors of $f_i(\cdot, x_0(\cdot))$.*

Following the lines of [12] it is easy to see that the integral operator Γ is u.s.c. The following theorem shows that it is also condensing with respect to the measure of noncompactness φ given in Example 1.3.

THEOREM 2.2. *Let conditions (F1)–(F4) and (A) be satisfied. Then Γ is a φ -condensing operator.*

To prove the theorem we need the following lemma.

LEMMA 2.3. *Let $v \in C_T(\mathbb{R}^n)$. Assume that its components $v_i(t)$, $i = 1, \dots, n$, satisfy the following integral inequalities:*

$$\begin{aligned} v_i(t) &\leq e^{-\gamma_i t} (I - e^{-\gamma_i T})^{-1} \int_0^T e^{-\gamma_i (T-s)} \sum_{j=1}^n m_{ij} v_j(s) ds \\ &+ \int_0^t e^{-\gamma_i (t-s)} \sum_{j=1}^n m_{ij} v_j(s) ds, \quad i = 1, \dots, n, \end{aligned}$$

where γ_i and m_{ij} are the constants of condition (A). Then $v(t) \equiv 0$.

PROOF. Let $u(t) = (u_1(t), \dots, u_n(t))$ be the function in $C_T(\mathbb{R}^n)$ defined by

$$\begin{aligned} u_i(t) &= e^{-\gamma_i t} (I - e^{-\gamma_i T})^{-1} \int_0^T e^{-\gamma_i(T-s)} \sum_{j=1}^n m_{ij} v_j(s) ds \\ &\quad + \int_0^t e^{-\gamma_i(t-s)} \sum_{j=1}^n m_{ij} v_j(s) ds, \quad i = 1, \dots, n. \end{aligned}$$

Then for $i = 1, \dots, n$, we have $v_i(t) \leq u_i(t)$. The functions $u_i(t)$ are continuously differentiable and

$$u_i'(t) = -\gamma_i u_i(t) + \sum_{j=1}^n m_{ij} v_j(t) \leq -\gamma_i u_i(t) + \sum_{j=1}^n m_{ij} u_j(t).$$

From the comparison theorem it follows that $u_i(t) \leq z_i(t)$, where $z(t) = (z_1(t), \dots, z_n(t))$ is the solution of system (11) with the initial condition

$$z_i(0) = (I - e^{-\gamma_i T})^{-1} \int_0^T e^{-\gamma_i(T-s)} \sum_{j=1}^n m_{ij} v_j(s) ds, \quad i = 1, \dots, n.$$

On the other hand, all solutions of (11) tend to zero as $t \rightarrow \infty$. Hence by the T -periodicity of v we obtain the assertion.

PROOF OF THEOREM 2.2. Let Ω be a bounded subset of $C_T(X)$ such that

$$(14) \quad \varphi(\Omega) \leq \varphi(\Gamma \Omega),$$

where the ordering (\leq) is the one defined in Example 1.3. Note that the set $\Gamma \Omega$ is bounded and equicontinuous (see [11]). Then Ω is equicontinuous and therefore the functions $\chi_i(\Omega_i(\cdot))$, $i = 1, \dots, n$, are continuous. From (11), (14) and Lemmas 1.5 and 1.11, we have

$$\begin{aligned} \chi_i(\Omega_i(t)) &\leq \chi_i \left(\left\{ e^{A_i t} [I - e^{A_i T}]^{-1} \int_0^T e^{A_i(T-s)} g_i(s) ds \right. \right. \\ &\quad \left. \left. + \int_0^t e^{A_i(t-s)} g_i(s) ds, \quad g_i \in S_{f_i(\cdot, \Omega(\cdot))}^1, \quad g_i \text{ is } T\text{-periodic} \right\} \right) \\ &\leq e^{-\gamma_i t} (I - e^{-\gamma_i T})^{-1} \int_0^T e^{-\gamma_i(T-s)} \chi_i(f_i(s, \Omega_1(s), \dots, \Omega_n(s))) ds \\ &\quad + \int_0^t e^{-\gamma_i(t-s)} \chi_i(f_i(s, \Omega_1(s), \dots, \Omega_n(s))) ds \\ &\leq e^{-\gamma_i t} (I - e^{-\gamma_i T})^{-1} \int_0^T e^{-\gamma_i(T-s)} \sum_{j=1}^n m_{ij} \chi_j(\Omega_j(s)) ds \\ &\quad + \int_0^t e^{-\gamma_i(t-s)} \sum_{j=1}^n m_{ij} \chi_j(\Omega_j(s)) ds, \quad i = 1, \dots, n. \end{aligned}$$

Since the functions $v_i(t) = \chi_i(\Omega_i(t))$ satisfy the conditions of Lemma 2.3, we have $\chi_i(\Omega_i(t)) \equiv 0$. So by the Arzelà criterion we have the relative compactness of Ω .

3. Averaging principle

In this section we consider the system (2) where the operators f_i and A_i satisfy conditions (F1)–(F4) and (A).

If we consider the change of variables $\tau = \mu t$, $x(\tau) = y(\mu^{-1}\tau)$, then the problem of finding (T/μ) -periodic solutions of (2) is equivalent to the problem of finding T -periodic solutions of the system

$$(15) \quad x'_i(t) \in \varepsilon A_i x_i + \varepsilon f_i(t, x_1, \dots, x_n), \quad i = 1, \dots, n,$$

where $\varepsilon = 1/\mu$.

Denote by f_i^0 the multioperator defined by the formula

$$(16) \quad f_i^0(v) = \frac{1}{T} \int_0^T f_i(s, v) ds.$$

Clearly, $f_i^0 : X \rightarrow \text{Kc}(X_i)$. In fact, a convex combination of selectors is also a selector. Moreover, if $y_m \in f_i^0(v)$ and $y_m \rightarrow y_0$, then the sequence of selectors g_m such that

$$y_m = \frac{1}{T} \int_0^T g_m(s) ds$$

is relatively weakly compact by Proposition 2.1. Therefore if a subsequence $\{g_{m_k}\}$ weakly converges to $g \in S_{f(\cdot, v)}^1$, then

$$\frac{1}{T} \int_0^T g_{m_k}(s) ds \xrightarrow{w} \frac{1}{T} \int_0^T g(s) ds$$

and so $y_0 = \frac{1}{T} \int_0^T g(s) ds$.

LEMMA 3.1. *The operator f_i^0 defined in (16) is u.s.c. on X .*

PROOF. Assume by contradiction that f_i^0 is not u.s.c. at some point v_0 . Then there exist $v_m \rightarrow v_0$, a $2d$ -neighbourhood V_{2d} of $f_i^0(v_0)$ and $\{y_m\}$ such that $y_m \in f_i^0(v_m)$ and $y_m \notin V_{2d}$ for any $m \in \mathbb{N}$. Because the sequence $\{y_m\}$ is relatively compact and $f_i^0(v_0)$ is closed and convex, without loss of generality, we can assume that there exists a functional ℓ such that

$$(17) \quad \langle \ell, y_m \rangle \geq d \quad \text{and} \quad \langle \ell, f_i^0(v_0) \rangle \leq 0.$$

Consider a sequence $\{g_m(s)\}$ of selectors of $f_i(s, v_m)$ such that

$$y_m = \frac{1}{T} \int_0^T g_m(s) ds$$

and $\{g_{m_k}\}$ is a subsequence weakly convergent to g . Then

$$\frac{1}{T} \int_0^T g(s) ds = y \in f_i^0(v_0)$$

and

$$\frac{1}{T} \int_0^T \langle \ell, g_{m_k}(s) - g(s) \rangle ds \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

On the other hand, from (16) we have

$$\frac{1}{T} \int_0^T \langle \ell, g_m(s) - g(s) \rangle ds = \langle \ell, y_m \rangle - \langle \ell, y \rangle \geq d > 0,$$

which is a contradiction. This concludes the proof.

Observe that by Lemma 1.11,

$$\chi_i(f_i^0(D_1 \times \cdots \times D_n)) \leq \sum_{j=1}^n m_{ij} \chi_j(D_j).$$

Thus the operators f_i^0 , $i = 1, \dots, n$, satisfy assumptions (F1)–(F4) for an arbitrary T . Observe that any constant function with value in $f_i^0(v)$ is a selector of $f_i^0(v)$.

Denote by Φ_0 the operator defined by

$$\Phi_0(y) = (f_1^0(y_1, \dots, y_n), \dots, f_n^0(y_1, \dots, y_n)).$$

Let us consider the operator $A^{-1}\Phi_0$. If we take as selectors the constant functions as described above, then it is easy to see that

$$A^{-1}\Phi_0(v) = \left\{ y : y \in C_T(X), y(t) = e^{At}[I - e^{AT}]^{-1} \int_0^T e^{A(T-s)} g ds + \int_0^t e^{A(t-s)} g ds, g \in S_{\Phi_0(v)}^1, g \text{ is constant} \right\}.$$

So, as in Theorem 2.2, we deduce that $A^{-1}\Phi_0$ is ψ -condensing in X with respect to the measure of noncompactness ψ defined by

$$\psi(\Omega) = (\chi_1(\Omega_1), \dots, \chi_n(\Omega_n)),$$

where Ω_i is the projection of Ω to X_i . Here the ordering in \mathbb{R}^n is given by the cone \mathbb{R}_+^n (see Example 1.3).

THEOREM 3.2. *Let conditions (F1)–(F4) and (A) be satisfied. Suppose that the inclusion $v \in A^{-1}\Phi_0(v)$ has a solution v^* and $\deg_X(I - A^{-1}\Phi_0, B_X(v^*, r)) \neq 0$ for some r . Then, for ε sufficiently small, the inclusion (15) has a T -periodic solution x^ε such that*

$$\max_t \|x^\varepsilon(t) - v^*\| \leq r.$$

In order to prove this theorem, we need the following lemma.

LEMMA 3.3. *Assume that the infinitesimal generator A of the analytic semi-group e^{At} satisfies (10). Let $\varepsilon_m \rightarrow 0$, let $\{g_m\}$, $g_m \in L^1([0, T], X)$, be a sequence weakly convergent to g and suppose the set*

$$(18) \quad \{g_m(s) : s \in [0, T], m = 1, 2, \dots\}$$

is bounded. Then

$$\varepsilon_m(I - e^{\varepsilon_m AT})^{-1} \int_0^T e^{\varepsilon_m A(T-s)} g_m(s) ds \xrightarrow{w} A^{-1} \frac{1}{T} \int_0^T g(s) ds \quad \text{as } m \rightarrow \infty.$$

PROOF. We have to prove that for every functional $\ell \in X^*$ we have

$$\begin{aligned} \left\langle \ell, \varepsilon_m(I - e^{\varepsilon_m AT})^{-1} \int_0^T e^{\varepsilon_m A(T-s)} g_m(s) ds \right\rangle \\ \rightarrow \left\langle \ell, A^{-1} \frac{1}{T} \int_0^T g(s) ds \right\rangle \quad \text{as } m \rightarrow \infty. \end{aligned}$$

For this, for a given $\nu > 0$ we choose $d > 0$ such that

$$\left| \left\langle \ell, \varepsilon_m(I - e^{\varepsilon_m AT})^{-1} \int_{T-d}^T e^{\varepsilon_m A(T-s)} g_m(s) ds \right\rangle \right| < \frac{\nu}{3}, \quad m = 1, 2, \dots,$$

and

$$\left| \left\langle \ell, A^{-1} \frac{1}{T} \int_{T-d}^T g(s) ds \right\rangle \right| < \frac{\nu}{3}.$$

This is possible since the set in (18) is bounded, $\|e^{\varepsilon_m A(T-s)}\| \leq C$ for some $C > 0$ and

$$\|\varepsilon_m(I - e^{\varepsilon_m AT})^{-1}\| \leq \varepsilon_m \left(1 + C \sum_{k=1}^{\infty} e^{-\varepsilon_m k d T}\right) \leq C \varepsilon_m \sum_{k=0}^{\infty} e^{-\varepsilon_m k d T} \leq 2C/(dT).$$

Moreover, for any $\alpha \in (0, 1)$,

$$(19) \quad \varepsilon_m(I - e^{\varepsilon_m AT})^{-1} A^\alpha e^{\varepsilon_m A(T-s)} \rightarrow \frac{1}{T} A^{-1+\alpha}, \quad s \in [0, T-d],$$

where the convergence is in the $L(X)$ -norm. In fact,

$$\begin{aligned} & \varepsilon_m(I - e^{\varepsilon_m AT})^{-1} A^\alpha e^{\varepsilon_m A(T-s)} \\ &= -\frac{1}{2\pi i} \int_{\Pi(\beta, \sigma)} \varepsilon_m (1 - e^{\varepsilon_m \lambda T})^{-1} \lambda^\alpha e^{\varepsilon_m \lambda(T-s)} (\lambda I - A)^{-1} d\lambda \\ &= \frac{1}{T} \left(-\frac{1}{2\pi i} \int_{\Pi(\beta, \sigma)} \varepsilon_m \lambda T (1 - e^{\varepsilon_m \lambda T})^{-1} e^{\varepsilon_m \lambda(T-s)} \lambda^{-1+\alpha} (\lambda I - A)^{-1} d\lambda \right), \end{aligned}$$

and the last integral converges uniformly, since the function

$$\varepsilon_m \lambda T (1 - e^{\varepsilon_m \lambda T})^{-1} e^{\varepsilon_m \lambda(T-s)}$$

is uniformly bounded for $s \in [0, T-d]$ and the resolvent $(\lambda I - A)^{-1}$ satisfies (5). Therefore, one can pass to the limit under the integral obtaining (19). Furthermore, by (19) and the boundedness of the set defined by (18) we have

$$\left\langle (A^{-\alpha})^* \ell, \varepsilon_m (I - e^{\varepsilon_m A T})^{-1} \int_0^{T-d} A^\alpha e^{\varepsilon_m A(T-s)} g_m(s) ds - A^{-1+\alpha} \frac{1}{T} \int_0^{T-d} g_m(s) ds \right\rangle \rightarrow 0,$$

and so we can choose m_1 in such a way that the absolute value of this term is less than $\nu/6$ for $m > m_1$. Furthermore, since $g_m \xrightarrow{w} g$ we obtain

$$\left\langle (A^{-\alpha})^* \ell, A^{-1+\alpha} \frac{1}{T} \int_0^{T-d} g_m(s) ds - A^{-1+\alpha} \frac{1}{T} \int_0^{T-d} g(s) ds \right\rangle \rightarrow 0,$$

and so we can choose $m_2 > m_1$ such that for $m > m_2$ the inequality

$$\left| \left\langle (A^{-\alpha})^* \ell, A^{-1+\alpha} \frac{1}{T} \int_0^{T-d} g_m(s) ds - A^{-1+\alpha} \frac{1}{T} \int_0^{T-d} g(s) ds \right\rangle \right| \leq \frac{\nu}{6}$$

is satisfied.

Finally, for $m > m_2$ we have

$$\begin{aligned} & \left| \left\langle \ell, \varepsilon_m (I - e^{\varepsilon_m A T})^{-1} \int_0^T e^{\varepsilon_m A(T-s)} g_m(s) ds - A^{-1} \frac{1}{T} \int_0^T g(s) ds \right\rangle \right| \\ & \leq \left| \left\langle \ell, \varepsilon_m (I - e^{\varepsilon_m A T})^{-1} \int_{T-d}^T e^{\varepsilon_m A(T-s)} g_m(s) ds \right\rangle \right| + \left| \left\langle \ell, A^{-1} \frac{1}{T} \int_{T-d}^T g(s) ds \right\rangle \right| \\ & + \left| \left\langle (A^{-\alpha})^* \ell, \varepsilon_m (I - e^{\varepsilon_m A T})^{-1} \int_0^{T-d} A^\alpha e^{\varepsilon_m A(T-s)} g_m(s) ds - A^{-1+\alpha} \frac{1}{T} \int_0^{T-d} g(s) ds \right\rangle \right| \\ & + \left| \left\langle (A^{-\alpha})^* \ell, A^{-1+\alpha} \frac{1}{T} \int_0^{T-d} g_m(s) ds - A^{-1+\alpha} \frac{1}{T} \int_0^{T-d} g(s) ds \right\rangle \right| \leq \nu. \end{aligned}$$

PROOF OF THEOREM 3.2. Since $A^{-1}\Phi_0$ is a ψ -condensing operator, there exists (see property 1° of Definition 1.9) a compact operator $\widehat{A^{-1}\Phi_0}$ such that the following condition is satisfied:

(LH) the linear homotopy $\lambda A^{-1}\Phi_0(v) + (1-\lambda)\widehat{A^{-1}\Phi_0}(v)$ has no fixed points on $\partial B_X(v^*, r) = S_X(v^*, r)$.

Consider in $C_T(X)$ the operator which associates with every $x \in C_T(X)$ the set of constant functions $\widehat{A^{-1}\Phi_0}(x(0))$. This operator is compact. We now prove that the integral operator Γ_ε , associated with (15), is linearly homotopic to $\widehat{A^{-1}\Phi_0}(x(0))$ on $S_{C_T}(v^*, r)$ for ε sufficiently small. Assume the contrary. Then

there are sequences $\varepsilon_m \rightarrow 0$, $\lambda_m \in [0, 1]$, $\lambda_m \rightarrow \lambda_0$ and $x_m \in S_{C_T}(v^*, r)$ such that

$$x_m \in \lambda_m \Gamma_{\varepsilon_m} x_m + (1 - \lambda_m) \widehat{A^{-1}\Phi_0}(x_m(0)).$$

Or equivalently,

$$\begin{aligned} x_m(t) &= \lambda_m \varepsilon_m e^{\varepsilon_m A t} (I - e^{\varepsilon_m A T})^{-1} \int_0^T e^{\varepsilon_m A(T-s)} g_m(s) ds \\ &\quad + \lambda_m \varepsilon_m \int_0^t e^{\varepsilon_m A(t-s)} g_m(s) ds + (1 - \lambda_m) w_m, \end{aligned}$$

where $g_m \in S_{\Phi(\cdot, x_m(\cdot))}^1$, g_m is T -periodic and $w_m \in \widehat{A^{-1}\Phi_0}(x_m(0))$. The sequence $\{w_m\}$ is relatively compact.

We want to prove that the sequence $\{x_m\}$ is relatively compact. For this, we first prove that

$$(20) \quad \|x_m(t) - x_m(0)\| \rightarrow 0 \quad \text{as } m \rightarrow \infty, \text{ whenever } t \in [0, T].$$

Since the functions x_m are T -periodic it is sufficient to estimate the difference $x_m(t) - x_m(0)$ only for $t \in [0, T]$. We have

$$\begin{aligned} x_m(t) - x_m(0) &= \lambda_m \varepsilon_m (e^{\varepsilon_m A t} - I)(I - e^{\varepsilon_m A T})^{-1} \int_0^T e^{\varepsilon_m A(T-s)} g_m(s) ds \\ &\quad + \lambda_m \varepsilon_m \int_0^t e^{\varepsilon_m A(t-s)} g_m(s) ds, \quad t \in [0, T]. \end{aligned}$$

For the second term on the right hand side we have

$$\left\| \lambda_m \varepsilon_m \int_0^t e^{\varepsilon_m A(t-s)} g_m(s) ds \right\| \leq CT \varepsilon_m \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

For the first term, if we put $N = [1/\varepsilon_m]$ and $M = \sup_m \sup_s \|g_m(s)\|$ we obtain

$$\begin{aligned} &\left\| \varepsilon_m (e^{\varepsilon_m A t} - I)(I - e^{\varepsilon_m A T})^{-1} \int_0^T e^{\varepsilon_m A(T-s)} g_m(s) ds \right\| \\ &= \left\| A^{-\alpha} (e^{\varepsilon_m A t} - I) \varepsilon_m \sum_{k=0}^{\infty} e^{k\varepsilon_m A T} \int_0^T A^\alpha e^{\varepsilon_m A(T-s)} g_m(s) ds \right\| \\ &\leq (\varepsilon_m t)^\alpha \left(\left\| \varepsilon_m \sum_{k=0}^N \int_0^T A^\alpha e^{\varepsilon_m A(T(k+1)-s)} g_m(s) ds \right\| \right. \\ &\quad \left. + \left\| \varepsilon_m \sum_{k=0}^{\infty} e^{k\varepsilon_m A T} \int_0^T A^\alpha e^{\varepsilon_m A(T(N+2)-s)} g_m(s) ds \right\| \right) \\ &\leq (\varepsilon_m T)^\alpha \left(CM \varepsilon_m \sum_{k=0}^N \int_0^T \frac{ds}{\varepsilon_m^\alpha (T(k+1) - s)^\alpha} \right) \end{aligned}$$

$$\begin{aligned}
 & + \frac{C\varepsilon_m}{1 - e^{-\varepsilon_m dT}} \int_0^T \frac{ds}{(\varepsilon_m T(N+1))^\alpha} \\
 & \leq (\varepsilon_m T)^\alpha CM \left(\frac{\varepsilon_m^{1-\alpha}}{1-\alpha} \sum_{k=0}^N ((T(k+1))^{1-\alpha} - (Tk)^{1-\alpha}) \right. \\
 & \quad \left. + \frac{2}{d(\varepsilon_m T(N+1))^\alpha} \right) \\
 & \leq (\varepsilon_m T)^\alpha C(\alpha) M((\varepsilon_m(N+1))^{1-\alpha} + (\varepsilon_m(N+1))^{-\alpha}) \\
 & \leq 3C(\alpha) M(\varepsilon_m T)^\alpha \rightarrow 0 \quad \text{as } m \rightarrow \infty.
 \end{aligned}$$

The relation (20) implies that

$$(21) \quad \chi_i(\{(x_m(t))_i\}) \equiv \chi_i(\{(x_m(0))_i\}), \quad i = 1, \dots, n.$$

As the functions x_m are T -periodic,

$$x_m(\lceil 1/\varepsilon_m \rceil T) = x_m(0).$$

Therefore,

$$\begin{aligned}
 & \lambda_m \varepsilon_m e^{\varepsilon_m A(\lceil 1/\varepsilon_m \rceil T)} (I - e^{\varepsilon_m AT})^{-1} \int_0^T e^{\varepsilon_m A(T-s)} g_m(s) ds \\
 & + \lambda_m \varepsilon_m \int_0^{\lceil 1/\varepsilon_m \rceil T} e^{\varepsilon_m A(\lceil 1/\varepsilon_m \rceil T - s)} g_m(s) ds + (1 - \lambda_m) w_m \\
 & = \lambda_m \varepsilon_m (I - e^{\varepsilon_m AT})^{-1} \int_0^T e^{\varepsilon_m A(T-s)} g_m(s) ds + (1 - \lambda_m) w_m,
 \end{aligned}$$

since

$$\begin{aligned}
 & (I - e^{\varepsilon_m AT})^{-1} \int_0^T e^{\varepsilon_m A(T-s)} g_m(s) ds \\
 & = (I - e^{\varepsilon_m A(\lceil 1/\varepsilon_m \rceil T)})^{-1} \int_0^{\lceil 1/\varepsilon_m \rceil T} e^{\varepsilon_m A(\lceil 1/\varepsilon_m \rceil T - s)} g_m(s) ds.
 \end{aligned}$$

Finally,

$$\begin{aligned}
 x_m(t) & = \lambda_m \varepsilon_m e^{\varepsilon_m AT} (I - e^{\varepsilon_m A(\lceil 1/\varepsilon_m \rceil T)})^{-1} \\
 & \quad \cdot \int_0^{\lceil 1/\varepsilon_m \rceil T} e^{\varepsilon_m A(\lceil 1/\varepsilon_m \rceil T - s)} g_m(s) ds \\
 & \quad + \lambda_m \varepsilon_m \int_0^t e^{\varepsilon_m A(T-s)} g_m(s) ds + (1 - \lambda_m) w_m.
 \end{aligned}$$

Making the change of variables $x_m(\tau/\varepsilon_m) = v_m(\tau)$, $\varepsilon_m s = \xi$, we obtain

$$\begin{aligned} v_m(\tau) &= \lambda_m e^{A\tau} (I - e^{\varepsilon_m A([1/\varepsilon_m]+1)T})^{-1} \\ &\quad \cdot \int_0^{\varepsilon_m([1/\varepsilon_m]+1)T} e^{A((1/\varepsilon_m)+1)T-\xi} \tilde{g}_m(\xi) d\xi \\ &\quad + \lambda_m \int_0^\tau e^{A(\tau-\xi)} \tilde{g}_m(\xi) d\xi + (1 - \lambda_m)v_m, \end{aligned}$$

where $\tilde{g}_m(\xi) \in \Phi(\xi/\varepsilon_m, v_m(\xi))$.

From (20) we have

$$\|v_m(\tau) - v_m(0)\| = \|x_m(\tau/\varepsilon_m) - x_m(0)\| \text{ to } 0 \quad \text{as } m \rightarrow \infty.$$

And so from (21),

$$(22) \quad \chi_i(\{(v_m(\tau))_i\}) \equiv \chi_i(\{(v_m(0))_i\}) = \chi_i(\{(x_m(0))_i\}), \quad i = 1, \dots, n.$$

In order to estimate $\chi_i(\{(v_m(\tau))_i\})$, we now prove that the sequence $\{B_m w\}$ defined by

$$B_m w = (I - e^{\varepsilon_m A([1/\varepsilon_m]+1)T})^{-1} \int_0^{\varepsilon_m([1/\varepsilon_m]+1)T} e^{A((1/\varepsilon_m)+1)T-\xi} w(\xi) d\xi$$

converges in norm to

$$Bw = (I - e^{AT})^{-1} \int_0^T e^{A(T-\xi)} w(\xi) d\xi.$$

In fact, let $\varepsilon_m([1/\varepsilon_m] + 1) = 1 + \varrho \varepsilon_m$, where $\varrho \in [0, 1]$, and consider

$$\begin{aligned} (23) \quad &\|B_m w - Bw\| \\ &\leq \|(I - e^{AT(1+\varrho\varepsilon_m)})^{-1}\| \cdot \|(e^{AT(1+\varrho\varepsilon_m)} - e^{AT})\| \\ &\quad \cdot \|(I - e^{AT})^{-1}\| \left\| \int_0^{T(1+\varrho\varepsilon_m)} e^{A(T(1+\varrho\varepsilon_m)-\xi)} w(\xi) d\xi \right\| \\ &\quad + \|(I - e^{AT})^{-1}\| \left\| \int_T^{T(1+\varrho\varepsilon_m)} e^{A(T(1+\varrho\varepsilon_m)-\xi)} w(\xi) d\xi \right\| \\ &\quad + \|(I - e^{AT})^{-1}\| \\ &\quad \cdot \int_0^T \|A^\alpha e^{A(T-\xi)}\| \cdot \|A^{-\alpha}(e^{A\varrho\varepsilon_m} - I)\| \cdot \|w(\xi)\| d\xi. \end{aligned}$$

All the terms on the right hand side of (23) tend to zero since

$$\|e^{AT(1+\varrho\varepsilon_m)} - e^{AT}\| \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

and

$$\left\| \int_T^{T(1+\varrho\varepsilon_m)} e^{A(T+(1+\varrho\varepsilon_m)-\xi)} w(\xi) d\xi \right\| \leq C \varrho \varepsilon_m \|w\|$$

and from (6), (7) we have

$$\int_0^T \|A^\alpha e^{A(T-\xi)}\| \cdot \|A^{-\alpha}(e^{A\rho\varepsilon_m} - I)\| \cdot \|w(\xi)\| d\xi \leq CT^\alpha (C\rho\varepsilon_m)^\alpha \|w\|.$$

Let us note now that

$$\begin{aligned} \chi_i(\{(\lambda_m \tilde{g}_m(\xi))_i\}) &\leq \chi_i(\{(\lambda_m \Phi(\xi/\varepsilon_m, v_m(\xi)))_i\}) \\ &\leq \chi_i(\overline{\text{co}} f_i([0, T] \times \{v_m(\xi)\})) = \chi_i(f_i([0, T] \times \{v_m(\xi)\})) \\ &\leq \sum_{j=1}^n m_{ij} \chi_j(\{(v_m(s))_j\}). \end{aligned}$$

On the other hand (see (22)), $\chi_i(\{(v_m(\tau))_i\})$ satisfies the following inequalities

$$\begin{aligned} \chi_i(\{(v_m(\tau))_i\}) &\leq \chi_i\left(\left\{e^{A_i\tau}(I - e^{\varepsilon_m A_i([1/\varepsilon_m]+1)T})^{-1}\right.\right. \\ &\quad \cdot \left.\int_0^{\varepsilon_m([1/\varepsilon_m]+1)T} e^{A_i((1/\varepsilon_m)+1)T-\xi} \lambda_m(\tilde{g}_m(\xi))_i d\xi\right. \\ &\quad \left.+\int_0^\tau e^{\varepsilon_m A(\tau-\xi)} \lambda_m(\tilde{g}_m(\xi))_i d\xi + (1 - \lambda_m)w_m\right\}) \\ &= \chi_i\left(\left\{e^{A_i\tau}(I - e^{\varepsilon_m A_i T})^{-1} \int_0^T e^{A_i(T-\xi)} \lambda_m(\tilde{g}_m(\xi))_i d\xi\right.\right. \\ &\quad \left.\left.+\int_0^\tau e^{A(\tau-\xi)} \lambda_m(\tilde{g}_m(\xi))_i d\xi\right\}\right) \\ &\leq e^{-\gamma_i t}(I - e^{-\gamma_i T})^{-1} \int_0^T e^{-\gamma_i(T-s)} \sum_{j=1}^n m_{ij} \chi_j(\{(v_m(s))_j\}) ds \\ &\quad + \int_0^t e^{-\gamma_i(t-s)} \sum_{j=1}^n m_{ij} \chi_j(\{(v_m(s))_j\}) ds, \quad i = 1, \dots, n. \end{aligned}$$

Since the functions $\chi_i(\{(v_m(\tau))_i\})$, $i = 1, \dots, n$, satisfy the conditions of Lemma 2.3, we have $\chi_i(\{(v_m(\tau))_i\}) \equiv 0$, $i = 1, \dots, n$. Therefore, from (22), $\chi_i(\{(x_m(0))_i\}) = 0$, $i = 1, \dots, n$, and in virtue of (20) the sequence $\{x_m\}$ is relatively compact. Without loss of generality, we may suppose that

$$x_m \rightarrow x_0 \in S_{C_T}(v^*, r) \quad \text{and} \quad w_m \rightarrow w_0 \quad \text{as } m \rightarrow \infty.$$

Then from (20), x_0 is a constant function and $x_0 \in S_X(v^*, r)$. As already noticed in the proof of Proposition 2.1, the sequence $\{g_m\}$ is weakly compact. Assume that $\{g_m\}$ weakly converges to some g_0 ; then $g_0 \in S_{\Phi(\cdot, x_0)}^1$ (see Section 2). Since the operator $\widehat{A^{-1}\Phi_0}$ is closed, we have $w_0 \in \widehat{A^{-1}\Phi_0}(x_0)$. By Lemma 3.3 we can pass to the limit in the equality

$$x_m(0) = \lambda_m \varepsilon_m (I - e^{\varepsilon_m A T})^{-1} \int_0^T e^{\varepsilon_m A(T-s)} g_m(s) ds + (1 - \lambda_m) w_m,$$

obtaining

$$x_0(0) = \lambda_0 A^{-1} \frac{1}{T} \int_0^T g_0(s) ds + (1 - \lambda_0) w_0.$$

Hence, $x_0 \in S_X(v^*, r)$ and

$$x_0 \in \lambda_0 A^{-1} \Phi_0(x_0) + (1 - \lambda_0) \widehat{A^{-1} \Phi_0}(x_0),$$

which contradicts (LH). Therefore, by Proposition 1.8 and property 1° of Definition 1.9, for $\varepsilon > 0$ sufficiently small, we obtain

$$\deg_{C_T}(I - \Gamma_\varepsilon, B_{C_T}(v^*, r)) = \deg_{C_T}(I - A^{-1} \Phi_0, B_{C_T}(v^*, r))$$

and by Theorem 1.10,

$$\deg_{C_T}(I - A^{-1} \Phi_0, B_{C_T}(v^*, r)) = \deg_X(I - A^{-1} \Phi_0, B_X(v^*, r)) \neq 0.$$

Finally, by property 4° of Definition 1.9, there exists a fixed point of Γ_ε in $B_{C_T}(v^*, r)$ and so a T -periodic solution x^ε of the inclusion (15). This concludes the proof.

4. Inclusions with large coefficient of diffusion

This section deals with the problem of finding T -periodic solutions of inclusion (3), which we rewrite in the form

$$(24) \quad y'(t) \in \mu Ay + \Phi(t, y),$$

where the operators A and Φ are defined in (8) and (9).

In addition to conditions (F1)–(F4) and (A) we assume:

(A⁺) $0 \in \sigma(A)$ and it is a simple pole of the resolvent $(\lambda I - A)^{-1}$, that is,

$$(\lambda I - A)^{-1} = \lambda^{-1} P + P_0 + \lambda P_1 + \dots,$$

where P is the Riesz projector. Furthermore,

$$(25) \quad \|e^{At}(I - P)\| \leq C e^{-dt}$$

for some $d > 0$.

From inequality (10) it follows that the projector P has a finite-dimensional range and zero is an eigenvalue of A . The expansion of the resolvent means simply that the geometric and the algebraic multiplicities of the zero eigenvalue coincide.

Consider now the finite-dimensional inclusion

$$(26) \quad x'(t) \in P\Phi(t, x(t)),$$

where $x(t) \in PX$, $t \in [0, T]$, and Φ is defined by (9). A T -periodic solution of (26) is a solution of the integral inclusion $x \in \Gamma_0 x$, where

$$\Gamma_0 x = \left\{ y : y \in C_T(X), y(t) = e^{-t}[I - e^{-T}]^{-1} \int_0^T e^{-(T-s)} g(s) ds + \int_0^t e^{-(t-s)} g(s) ds, g \in S_{P\Phi(\cdot, x(\cdot)) + Px(\cdot)}^1, g \text{ is } T\text{-periodic} \right\}.$$

Note that Γ_0 is a compact operator since the projector P is finite-dimensional.

THEOREM 4.1. *Let conditions (F1)–(F4), (A), (A⁺) be satisfied and assume that the inclusion (26) has a T -periodic solution x^0 such that $\deg(I - \Gamma_0, B(x^0, r)) \neq 0$ for some $r > 0$. Then for μ sufficiently large, the inclusion (3) has a T -periodic solution x^μ such that*

$$\max_t \|x^\mu(t) - x^0(t)\| \leq r.$$

PROOF. Rewrite the inclusion (24) in the form

$$(27) \quad y'(t) \in (\mu A - P)y + \Phi(t, y) + Py.$$

The operators $\Phi + P$ and $\mu A - P$ satisfy conditions (F1)–(F4) and (A), respectively. Since $1 \notin \sigma(e^{(\mu A - P)T})$, we can define the integral operator Γ for (27) as in Section 2, and we denote it by Γ_μ . By Proposition 1.9 the family

$$\mathcal{F}(\lambda, x) = \lambda \Gamma_\mu x + (1 - \lambda) \Gamma_0 x$$

is φ -condensing. We want to prove that for μ sufficiently large, $\mathcal{F}(\lambda, x)$ is a homotopy on $S_{C_T}(x^0, r)$. Assume the contrary; then there exist sequences $\mu_m \rightarrow \infty$, $\lambda_m \in [0, 1]$, $\lambda_m \rightarrow \lambda_0$ and $x_m \in S_{C_T}(x^0, r)$ such that

$$x_m \in \lambda_m \Gamma_{\mu_m} x_m + (1 - \lambda_m) \Gamma_0 x_m.$$

Or equivalently,

$$(28) \quad \begin{aligned} x_m(t) &= \lambda_m e^{(\mu_m A - P)t} (I - e^{(\mu_m A - P)T})^{-1} \\ &\quad \cdot \int_0^T e^{(\mu_m A - P)(T-s)} g_m(s) ds \\ &\quad + \lambda_m \int_0^t e^{(\mu_m A - P)(t-s)} g_m(s) ds \\ &\quad + (1 - \lambda_m) e^{-t} (I - e^{-T})^{-1} \int_0^T e^{-(T-s)} \widehat{g}_m(s) ds \\ &\quad + (1 - \lambda_m) \int_0^t e^{-(t-s)} \widehat{g}_m(s) ds, \end{aligned}$$

where $g_m \in S_{\Phi(\cdot, x_m(\cdot)) + Px_m(\cdot)}^1$, $\widehat{g}_m \in S_{P\Phi(\cdot, x_m(\cdot)) + Px_m(\cdot)}^1$ and g_m, \widehat{g}_m are T -periodic.

Now the sequence of operators B_m defined by the formulas

$$B_m y(t) = e^{(\mu_m A - P)t} (I - e^{(\mu_m A - P)T})^{-1} \int_0^T e^{(\mu_m A - P)(T-s)} y(s) ds \\ + \int_0^t e^{(\mu_m A - P)(t-s)} y(s) ds$$

converges to the compact operator

$$B_0 y(t) = e^{-t} (I - e^{-T})^{-1} \int_0^T e^{-(T-s)} P y(s) ds + \int_0^t e^{-(t-s)} P y(s) ds.$$

in the operator norm. In fact, $P B_m y = B_0 y$, and

$$\|(I - P) B_m y\| \\ \leq C \left(\int_0^{T-\delta} \|e^{(\mu_m A - P)(T-s)} (I - P)\| ds + \int_{T-\delta}^T \|e^{(\mu_m A - P)(T-s)}\| ds \right) \|y\| \\ + C \left(\int_0^{t-\delta} \|e^{(\mu_m A - P)(t-s)} (I - P)\| ds + \int_{t-\delta}^t \|e^{(\mu_m A - P)(t-s)}\| ds \right) \|y\|.$$

Using (25) we obtain $\|(I - P) B_m y\| \leq C(1/(d\mu_m) + \delta)$, and from this the assertion.

Since the sequence $\{x_m\}$ is relatively compact, by the results of Section 2 the sequences $\{g_m\}$ and $\{\widehat{g}_m\}$ are weakly compact. Without loss of generality, we assume that

$$x_m \longrightarrow x_0 \in S_{C_T}(x^0, r), \\ g_m \xrightarrow{w} g_0 \in S_{\Phi(\cdot, x_0(\cdot)) + P x_0(\cdot)}^1, \\ \widehat{g}_m \xrightarrow{w} \widehat{g}_0 \in S_{P\Phi(\cdot, x_0(\cdot)) + P x_0(\cdot)}^1.$$

Therefore we can pass to the limit in equation (28) to obtain

$$x_0(t) = e^{-t} (I - e^{-T})^{-1} \int_0^T e^{-(T-s)} (\lambda_0 P g_0(s) (1 - \lambda_0) \widehat{g}_0(s)) ds \\ + \int_0^t e^{-(t-s)} (\lambda_0 P g_0(s) (1 - \lambda_0) \widehat{g}_0(s)) ds.$$

On the other hand, $P g_0 \in S_{P\Phi(\cdot, x_0(\cdot)) + P x_0(\cdot)}^1$ and so $x_0 \in \Gamma_0 x_0$, which is a contradiction. Thus, from properties 1^o, 4^o of Definition 1.9 and Theorem 1.10 we obtain the assertion.

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