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ON THE OBLIQUE DERIVATIVE PROBLEM IN AN INFINITE ANGLE

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Dedicated to Louis Nirenberg on the occasion of his 70th birthday

1. Introduction

Let $d_\vartheta \subset \mathbb{R}^2$ be the infinite angle of opening $\vartheta \in (0, 2\pi]$ with sides γ_0 and γ_1 given by

$$\gamma_0 = \{0 \le x_1 < \infty,\ x_2 = 0\},$$
$$\gamma_1 = \{x_1 = r\cos\vartheta,\ x_2 = r\sin\vartheta,\ 0 \le r = \sqrt{x_1^2 + x_2^2} < \infty\}$$

in a Cartesian coordinate system $\{x_1, x_2\}$. We consider the elliptic boundary value problem

$$\begin{aligned} &-\Delta u + su = f(x), \quad x \in d_\vartheta, \\ &\left(\frac{\partial u}{\partial n} + h_i \frac{\partial u}{\partial r}\right)\bigg|_{\gamma_i} = \varphi_i(r), \quad i = 0, 1, \end{aligned} \tag{1.1}$$

where n is the exterior normal to γ_i, h_0 and h_1 are given real constants, and s is a complex parameter with $\Re s \equiv a^2 \ge 0$.

Problem (1.1) arises from the parabolic initial-boundary value problem

$$\begin{aligned} &v_t - \Delta v = f(x,t), \quad x \in d_\vartheta,\ t > 0, \\ &v(x,0) = 0, \quad \left(\frac{\partial v}{\partial n} + h_i \frac{\partial v}{\partial r}\right)\bigg|_{\gamma_i} = \varphi_i(r,t), \quad i = 0, 1, \end{aligned} \tag{1.2}$$

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after taking the Laplace transform with respect to t. We think that problem (1.1) is of interest in itself and not only in relation to the parabolic case, as studied in [3]. Here we present a complete discussion of the elliptic problem, which generalizes the results of [3]. We obtain estimates of the solution of problem (1.1) which are uniform with respect to s in weighted Sobolev spaces introduced by V. A. Kondrat'ev for investigation of elliptic boundary value problems in domains with angular and conical points at the boundary. In these spaces the distance $|x|$ from the origin, with an appropriate exponent, is the weight.

The spaces in which the solution exists depend on the sign of $h_0 + h_1$. We denote these spaces by $H^k_\mu(d_\vartheta)$ (k is a non-negative integer, $\mu \in \mathbb{R}$) and define them as completions of the set of complex-valued infinitely differentiable functions with compact support vanishing near the origin in the norms

$$\|u\|_{H^k_\mu(d_\vartheta)} = \bigg(\sum_{|j|\le k}\int_{d_\vartheta} |D^j u(x)|^2 |x|^{2\mu-2k+2|j|}\,dx\bigg)^{1/2}.$$

We denote the space $H^0_\mu(d_\vartheta)$ by $L_{2,\mu}(d_\vartheta)$ and set

$$\|u\|^2_{L_{2,\mu}(d_\vartheta)} = \int_{d_\vartheta} |u|^2|x|^{2\mu}\,dx.$$

It is well known that the space of traces of functions from $H^{k+1}_\mu(d_\vartheta)$ on γ_i (and in general on an arbitrary half-line $\gamma = \{x_1 = r\cos\omega,\ x_2 = r\sin\omega\}$, $\omega \in [0,\vartheta]$) is the space $H^{k+1/2}_\mu(\gamma)$ with the norm

$$\|u\|_{H^{k+1/2}_\mu(\gamma)} = \bigg(\sum_{j=0}^{k}\int_\gamma |D^j u(r)|^2 r^{2\mu-2k-1+2j}\,dr + \|u\|^2_{L^{k+1/2}_\mu(\gamma)}\bigg)^{1/2},$$

where

$$\|u\|^2_{L^{k+1/2}_\mu(\gamma)} = \int_0^\infty r^{2\mu}\,dr \int_0^r |D^k u(r+\varrho) - D^k u(r)|^2\,\frac{d\varrho}{\varrho^2}.$$

We shall also work in the spaces $W^k_{2,\mu}(d_\vartheta)$, $k \ge 1$, and in the corresponding spaces of traces $W^{k-1/2}_{2,\mu}(\gamma)$ with the norms

$$\|u\|^2_{W^k_{2,\mu}(d_\vartheta)} = \sum_{0\le|j|\le k} \|D^j u\|^2_{L_{2,\mu}(d_\vartheta)}$$

and

$$\|u\|^2_{W^{k-1/2}_{2,\mu}(\gamma)} = \sum_{j=0}^{k-1} \|D^j u\|^2_{L_{2,\mu}(\gamma)} + \|u\|^2_{L^{k-1/2}_\mu(\gamma)},$$

respectively.

Our main results are as follows.

THEOREM 1.1. *Let* $\Re s \geq 0$, $\mu \geq 0$, $\beta_i = \arctan h_i \in (-\pi/2, \pi/2)$, $h_0 + h_1 > 0$ *and*

$$0 < 1 + k - \mu < \frac{\beta_0 + \beta_1}{\vartheta}. \tag{1.3}$$

For every $f \in H^k_\mu(d_\vartheta) \cap W^k_{2,\mu}(d_\vartheta)$ *and* $\varphi_i \in H^{k+1/2}_\mu(\gamma_i) \cap W^{k+1/2}_{2,\mu}(\gamma_i)$, $i = 0, 1$, *problem* (1.1) *has a unique solution* $u \in H^{k+2}_\mu(d_\vartheta) \cap W^{k+2}_{2,\mu}(d_\vartheta)$, *and*

$$\sum_{l=0}^{k+2} |s|^{k+2-l} \|u\|^2_{H^l_\mu(d_\vartheta)} \tag{1.4}$$
$$\leq c_1 \Big[\sum_{l=0}^{k} |s|^{k-l} \|f\|^2_{H^l_\mu(d_\vartheta)} + \sum_{i=0}^{1} \Big(\sum_{l=0}^{k} |s|^{k+1/2-l} \|\varphi_i\|^2_{H^l_\mu(\gamma)} + \|\varphi_i\|^2_{H^{k+1/2}_\mu(\gamma_i)} \Big) \Big].$$

THEOREM 1.2. *If* $h_0 + h_1 \leq 0$ *and*

$$0 < 1 + k - \mu < \frac{\pi + \beta_0 + \beta_1}{\vartheta}, \tag{1.5}$$

then for every $f \in W^k_{2,\mu}(d_\vartheta)$ *and* $\varphi_i \in W^{k+1/2}_{2,\mu}(\gamma_i)$, $i = 0, 1$, *problem* (1.1) *has a unique solution* $u \in W^{k+2}_{2,\mu}(d_\vartheta)$, *and*

$$\sum_{l=0}^{k+2} |s|^{k+2-l} \sum_{|j|=l} \|D^j u\|^2_{L_{2,\mu}(d_\vartheta)} \tag{1.6}$$
$$\leq c_2 \Big[\sum_{l=0}^{k} |s|^{k-l} \sum_{|j|=l} \|D^j f\|^2_{L_{2,\mu}(d_\vartheta)}$$
$$+ \sum_{i=0}^{1} \Big(\sum_{l=0}^{k} |s|^{k+1/2-l} \|D^l \varphi_i\|^2_{L_{2,\mu}(\gamma_i)} + \|\varphi_i\|^2_{L^{k+1/2}_\mu(\gamma_i)} \Big) \Big].$$

Results of this type are proved in [3] only for $k = 0$ and $\mu \in (0, 1)$. Here (see Sections 3–5) a new, complete exposition of existence and uniqueness results and a priori estimates are given for $\mu \in [0, 1)$ and $k \geq 0$.

In Section 2 we formulate a corollary of Kondrat'ev's general results which plays a fundamental role in our arguments. Then we prove auxiliary estimates in the space $W^k_{2,\mu}(d_\vartheta)$ ($h_0 + h_1 \leq 0$) by the construction of special auxiliary functions.

In Section 6 we prove Theorems 1.1 and 1.2 for every integer $k > 0$ and $\mu \geq 0$. We think that this extension is important since by choosing an appropriate μ satisfying either (1.3) or (1.5), it is possible, for fixed ϑ, h_0 and h_1, to obtain a greater regularity for the solution.

In Section 7 some applications of these results to the parabolic case are given.

REMARK. We can also consider problem (1.1) in an n-dimensional dihedral angle $D_\vartheta = d_\vartheta \times \mathbb{R}^{n-2}$. After taking the Fourier transform with respect to the variables tangential to the edge $x_1 = x_2 = 0$ the problem reduces to (1.1) with the parameter $s + |\xi|^2$ instead of s (here $\xi = (\xi_3, \dots, \xi_n)$ are the dual variables of the Fourier transform). The above theorems hold true for the transformed problem (with the parameter $|s|$ in the estimates (1.4) and (1.6) replaced by $|s| + |\xi|^2$), and only obvious modifications of the proofs are necessary.

Elliptic boundary value problems in domains with angular and conical points at the boundary were studied in a pioneering paper of V. A. Kondrat'ev [5] and in a series of fundamental papers of V. G. Maz'ya and B. A. Plamenevskiĭ (see, for instance, [6] and the bibliography there). Boundary value problems for the equation $-\Delta u + su = f$ were considered in [2], [7], [8]. In particular, in [2], [7] problems with the boundary conditions

$$\frac{\partial u}{\partial n}\bigg|_{\gamma_1} = \varphi_1, \quad \left(\frac{\partial u}{\partial n} + h_0 \frac{\partial u}{\partial r} - \sigma u\right)\bigg|_{\gamma_0} = \varphi_0$$

were studied, and it was made clear that the spaces in which the solution exists depend on the sign of h_0.

2. Auxiliary propositions

In this section we are concerned mainly with the problem

$$\begin{gathered} -\Delta u(x) = f(x) \quad (x \in d_\vartheta), \\ \left(\frac{\partial u}{\partial n} + h_0 \frac{\partial u}{\partial r}\right)\bigg|_{\gamma_0} = \Phi_0, \quad \left(\frac{\partial u}{\partial n} + h_1 \frac{\partial u}{\partial r}\right)\bigg|_{\gamma_1} = \Phi_1. \end{gathered} \tag{2.1}$$

It is well known that the homogeneous problem ($f = 0$, $\Phi_0 = 0$, $\Phi_1 = 0$) has solutions of the form

$$u = r^\lambda U(\varphi), \quad r = (x_1^2 + x_2^2)^{1/2}, \quad \varphi = \arctan x_2/x_1, \tag{2.2}$$

for $\lambda = \lambda_m = \frac{\beta_0 + \beta_1}{\vartheta} + \frac{\pi}{\vartheta} m$, $m = 0, \pm 1, \pm 2, \dots$, and for $\lambda = 0$. The corresponding $U(\varphi)$ are defined by

$$U_m(\varphi) = a \cos(\lambda_m \varphi - \beta_0), \quad U(\varphi) = a \quad \text{in the case } \lambda = 0.$$

They are computed as "eigenvalues" and "eigenfunctions" of the problem

$$\frac{d^2 U}{d\varphi^2} + \lambda^2 U = 0, \quad \left(\frac{dU}{d\varphi} - h_0 \lambda U\right)\bigg|_{\varphi=0} = 0, \quad \left(\frac{dU}{d\varphi} + h_1 \lambda U\right)\bigg|_{\varphi=\vartheta} = 0. \tag{2.3}$$

Moreover, if $h_0 + h_1 = 0$, then in the case $\lambda = 0$ this problem has an "associated" function to which there corresponds the solution

$$u = b(\log r + h_0 \varphi)$$

of the homogeneous problem (2.1).

Along with (2.1), we consider the penalized problem

$$
\begin{aligned}
&-\Delta u(x) = f(x) \quad (x \in d_\vartheta),\\
&\left(\frac{\partial u}{\partial n} + h_0 \frac{\partial u}{\partial r} + \varepsilon \frac{u}{r}\right)\Big|_{\gamma_0} = \Phi_0, \quad \left(\frac{\partial u}{\partial n} + h_1 \frac{\partial u}{\partial r}\right)\Big|_{\gamma_1} = \Phi_1,
\end{aligned} \tag{2.4}
$$

and the corresponding homogeneous problem. The latter has solutions of the form (2.2) if λ satisfies the equation

$$
\tan(\lambda\vartheta - \beta_1) - \tan\beta_0 = \varepsilon\lambda^{-1}. \tag{2.5}
$$

It is easy to show that this equation has only real solutions. Indeed, if $\lambda = \lambda' + i\lambda''$, then the relation

$$
\begin{aligned}
0 &= \Im[\tan(\lambda\vartheta - \beta_1) - \tan\beta_0 - \varepsilon\lambda^{-1}]\\
&= \frac{\cosh\lambda''\vartheta \sinh\lambda''\vartheta}{\cos^2(\lambda'\vartheta + \beta_1) + \sinh^2\lambda''\vartheta} + \frac{\lambda''\varepsilon}{|\lambda|^2}
\end{aligned}
$$

implies $\lambda'' = 0$. We are interested in finding out how the "eigenvalue" $\lambda = 0$ is changed when ε becomes positive. If $\beta_0 + \beta_1 > 0$ (or, what is the same, $h_0 + h_1 > 0$), then the left-hand side of (2.5) is negative for $0 < \lambda < \vartheta^{-1}(\beta_0 + \beta_1)$, so (2.5) has no solutions in the interval $[0, \vartheta^{-1}(\beta_0 + \beta_1)]$. However, if $\beta_0 + \beta_1 \le 0$, then the interval $(0, \lambda_1) = (0, \vartheta^{-1}(\pi + \beta_0 + \beta_1))$ contains one solution of (2.5) close to zero for small ε.

Now we formulate a corollary of general results of V. A. Kondrat'ev [5] which will play a fundamental role in our arguments.

Theorem 2.1. 1. *For every $f \in H^k_\mu(d_\vartheta)$ and $\Phi_i \in H^{k+1/2}_\mu(\gamma_i)$, $i = 1, 2$, problem (2.1) (resp. (2.4)) is uniquely solvable in $H^{k+2}_\mu(d_\vartheta)$ and its solution satisfies the inequality*

$$
\|u\|^2_{H^{k+2}_\mu(d_\vartheta)} \le c_1\left(\|f\|^2_{H^k_\mu(d_\vartheta)} + \sum_{i=0}^{1} \|\Phi_i\|^2_{H^{k+1/2}_\mu(\gamma_i)}\right), \tag{2.6}
$$

provided that $1+k-\mu$ is not an "eigenvalue", i.e. $1+k-\mu \ne 0$ and $1+k-\mu \ne \lambda_m$ (resp. $1+k-\mu$ is not a solution of (2.5)).

2. *If $f \in H^k_\mu(d_\vartheta) \cap H^{k'}_{\mu'}(d_\vartheta)$ and $\Phi_i \in H^{k+1/2}_\mu(\gamma_i) \cap H^{k'+1/2}_{\mu'}(\gamma_i)$, $i = 0, 1$, and there are no "eigenvalues" between $1 + k - \mu$ and $1 + k' - \mu'$, then the solutions $u \in H^{k+2}_\mu(d_\vartheta)$ and $u' \in H^{k'+2}_{\mu'}(d_\vartheta)$ of (2.1) resp. (2.4) coincide.*

3. *If there only is the "eigenvalue" $\lambda = 0$ of problem (2.3) between $1 + k - \mu$ and $1 + k' - \mu'$, then*

$$
u - u' = a + b(\log r + h_0\varphi), \quad a, b = \text{const},
$$

and $b = 0$ in the case $h_0 + h_1 \ne 0$.

4. *If* $f \in L_{2,\mu}(d_\vartheta)$ *and* $\Phi_i \in H^{1/2}_\mu(\gamma_i)$, $i = 0,1$, *then any solution of* (2.1) *or* (2.4) *belonging to* $H^1_{\mu-1}(d_\vartheta)$ *has the second generalized derivatives* $D^j u \in L_{2,\mu}(d_\vartheta)$, $|j| = 2$, *and*

$$\|D^2 u\|^2_{L_{2,\mu}(d_\vartheta)} \le c_2\bigg(\|u\|^2_{H^1_{\mu-1}(d_\vartheta)} + \|f\|^2_{L_{2,\mu}(d_\vartheta)} + \sum_{i=0}^{1} \|\Phi_i\|^2_{H^{1/2}_\mu(\gamma_i)}\bigg). \tag{2.7}$$

Here we have used the notation

$$\|D^2 u\|^2_{L_{2,\mu}(d_\vartheta)} \equiv \sum_{|j|=2} \|D^j u\|^2_{L_{2,\mu}(d_\vartheta)}.$$

We shall also need estimates of solutions of problem (2.1) in the spaces $W^{k+2}_{2,\mu}(d_\vartheta)$ similar to those obtained in [8] for the Neumann problem ($h_0 = h_1 = 0$).

Theorem 2.2. *Let* $h_0 + h_1 \le 0$, $\mu \in [0,1)$ *and*

$$0 < 1 + k - \mu < \frac{\pi + \beta_0 + \beta_1}{\vartheta}.$$

For every $f \in W^k_{2,\mu}(d_\vartheta)$ *and* $\Phi_i \in W^{k+1/2}_{2,\mu}(\gamma_i)$, $i = 0,1$, *problem* (2.1) *has a solution* $u(x)$ *with* $D^j u \in W^k_{2,\mu}(d_\vartheta)$, $|j| = 2$, *satisfying the inequality*

$$\sum_{|j|=2} \|D^j u\|^2_{W^k_{2,\mu}(d_\vartheta)} \le c_3\bigg(\|f\|^2_{W^k_{2,\mu}(d_\vartheta)} + \sum_{i=0}^{1} \|\Phi_i\|^2_{W^{k+1/2}_{2,\mu}(\gamma_i)}\bigg) \equiv c_3 F_k. \tag{2.8}$$

Any solution $u \in W^{k+2}_{2,\mu}(d_\vartheta)$ *of problem* (2.1) *satisfies* (2.8).

As in [8], this theorem reduces to the preceding one by the construction of a special auxiliary function.

Proposition 2.1. *For all* f, Φ_0, Φ_1 *satisfying the hypotheses of Theorem* 2.2 *there exists a function* $w \in W^{k+2}_{2,\mu}(d_\vartheta)$, $\mu \in [0,1)$, *such that* $g \equiv \Delta w + f \in H^k_\mu(d_\vartheta)$, $\psi_i \equiv \Phi_i - \frac{\partial w}{\partial n} - h_i \frac{\partial w}{\partial r} \in H^{k+1/2}_\mu(\gamma_i)$, *and*

$$\|w\|^2_{W^{k+2}_{2,\mu}(d_\vartheta)} + \|g\|^2_{H^k_\mu(d_\vartheta)} + \sum_{i=0}^{1} \|\Phi_i\|^2_{H^{k+1/2}_\mu(\gamma_i)} \le c_4 F_k.$$

Now (2.1) reduces to the problem

$$-\Delta v = g, \quad \left(\frac{\partial v}{\partial n} + h_i \frac{\partial v}{\partial r}\right)\bigg|_{\gamma_i} = \psi_i, \quad i = 0,1,$$

for the function $v = u - w$. Since $g \in H^k_\mu(d_\vartheta)$ and $\psi_i \in H^{k+1/2}_\mu(\gamma_i)$, Theorem 2.1 can be applied.

The construction of w (which is different for $\mu > 0$ and for $\mu = 0$) relies on the following proposition.

Proposition 2.2. *Let $f(x)$ and $\Phi_i(r), i = 0, 1,$ be homogeneous polynomials of degrees $l-2$ and $l-1 \geq 0$, respectively:*

$$f(x) = \sum_{l_1+l_2=l-2} f_{l_1,l_2} x_1^{l_1} x_2^{l_2}, \quad \Phi_i = A_i r^{l-1}, \quad i = 0, 1.$$

If $l \neq \lambda_m$, then problem (2.1) *has a unique solution which is also a homogeneous polynomial of degree l:*

$$u(x) = \sum_{l_1+l_2=l} u_{l_1,l_2} x_1^{l_1} x_2^{l_2}.$$

Proof. It is easily seen that under the above hypotheses problem (2.1) has a unique homogeneous solution of the form (2.2) with $\lambda = l$. But it can only be a polynomial: indeed, $v = D^j u$, $|j| = l-1$, is a harmonic function of the form $v = rV(\varphi)$ with V satisfying the equation $V'' + V = 0$, hence, $V = a\cos\varphi + b\sin\varphi$, i.e. $v = ax_1 + bx_2$, which proves the proposition.

Further arguments are similar to those in [7]. For $\mu > 0$ the construction of w is simpler and it reduces to finding a polynomial

$$P(x) = \sum_{1\leq |j| \leq k} p_{j_1 j_2} \frac{x_1^{j_1}}{j_1!} \frac{x_2^{j_2}}{j_2!}$$

such that

$$-\Delta P = \sum_{|j|\leq k-2} \frac{x_1^{j_1}}{j_1!} \frac{x_2^{j_2}}{j_2!} D^j f|_{x=0},$$

$$\left(\frac{\partial P}{\partial n} + h_i \frac{\partial P}{\partial r}\right)\Big|_{\gamma_i} = \sum_{j=0}^{k-1} \frac{r^j}{j!} \frac{d^j \Phi_i}{dr^j}\Big|_{r=0}, \quad i = 0, 1$$

(the existence of $P(x)$ follows from Proposition 2.2). The auxiliary function equals $w(x) = P(x)\zeta(x)$, where $\zeta \in C_0^\infty(\mathbb{R}^2)$ is equal to 1 for $|x| \leq 1/2$ and to zero for $|x| \geq 1$.

For $\mu = 0$, w has the form

$$w(x) = \sum_{j=1}^{k+1} w^{(j)}(x),$$

where $w^{(j)} \in W_2^{k+2}(d_\vartheta) \cap H_0^j(d_\vartheta)$ satisfy the conditions

$$f^{(j)} \equiv \Delta w^{(j)} + f^{(j-1)} \in H_0^{j-1}(d_\vartheta), \quad j = 1, \dots, k+1, \quad f^{(0)} = f,$$

$$\Phi_i^{(j)} \equiv \left(\Phi_i^{j-1} \frac{\partial w^{(j)}}{\partial n} + h_i \frac{\partial w^{(j)}}{\partial r}\right)\Big|_{\gamma_i} \in H_0^{j-1/2}(\gamma_i), \quad i = 0, 1, \ j = 1, \dots, k+1,$$

$$\Phi_i^{(0)} = \Phi_i$$

and the inequality

$$\|w^{(j)}\|^2_{W_2^{k+2}(d_\vartheta)} + \|f^{(j)}\|^2_{H_0^{j-1}(d_\vartheta)} + \sum_{i=0}^{1} \|\Phi_i^{(j)}\|^2_{H_0^{j-1/2}(\gamma_i)} \le c_5 F_k. \tag{2.9}$$

The construction of $w^{(k)}$ is also carried out with the help of Proposition 2.2 (see [8], §3).

Remark. As pointed out in [8] (see the end of §3), it is possible to introduce a positive parameter into the norms in inequalities (2.8) and (2.9). In particular, along with (2.8) we have the inequality

$$\begin{aligned} \sum_{2\le|j|\le k+2} & b^{k+2-|j|}\|D^j u\|^2_{L_{2,\mu}(d_\vartheta)} \\ & \le c_6\bigg[\sum_{|j|\le k} b^{k-|j|}\|D^j f\|^2_{L_{2,\mu}(d_\vartheta)} \\ & \quad + \sum_{i=0}^{1}\bigg(\|\Phi_i\|^2_{L_\mu^{k+1/2}(\gamma_i)} + \sum_{j=0}^{k} b^{k+1/2-|j|}\|D^j\Phi_i\|^2_{L_{2,\mu}(\gamma_i)}\bigg)\bigg], \end{aligned} \tag{2.10}$$

where c_6 is a constant independent of the parameter $b > 0$.

To conclude this section, we quote several useful inequalities involving $L_{2,\mu}$-norms of the functions given in d_ϑ or on γ_i (see [2], [7], [8]). We mean, first of all, well known estimates of traces of functions from $H^1_\mu(d_\vartheta)$ or $W^1_{2,\mu}(d_\vartheta)$ on γ_i, i.e.

$$\|u\|_{H_\mu^{1/2}(\gamma_i)} \le c_7\|u\|_{H^1_\mu(d_\vartheta)}, \tag{2.11}$$

$$\|u\|_{L_\mu^{1/2}(\gamma_i)} \le c_8\|\nabla u\|_{L_{2,\mu}(d_\vartheta)}, \tag{2.12}$$

interpolation inequalities

$$\|u\|_{L_{2,\mu'}(d_\vartheta)} \le c_9\|\nabla u\|^{\mu-\mu'}_{L_{2,\mu}(d_\vartheta)}\|u\|^{1-\mu+\mu'}_{L_{2,\mu}(d_\vartheta)}, \tag{2.13}$$

$$\|u\|_{L_{2,\nu}(\gamma_i)} \le c_{10}\|\nabla u\|^{1/2+\mu-\nu}_{L_{2,\mu}(d_\vartheta)}\|u\|^{1/2-\mu+\nu}_{L_{2,\mu}(d_\vartheta)}, \tag{2.14}$$

where $\mu \in [0,1]$, $\mu' \in [\mu-1,\mu]$ for $\mu > 0$, $\mu' \in (-1,0]$ for $\mu = 0$, $\nu \in [\mu-1/2,\mu]$ for $\mu > 0$, $\nu \in (-1/2,0]$ for $\mu = 0$, and a variant of the Hardy inequality

$$\|u\|_{L_{\mu-1/2}(\gamma_i)} \le c_{11}\|u\|_{L_\mu^{1/2}(\gamma_i)}. \tag{2.15}$$

Finally, for all positive a and $R > a^{-1}$ we have the estimate (see [2])

$$a\int_{\gamma_i(a^{-1},R)} |u(r)|^2 r^{2\mu}\,dr \le c_{12}\int_{d_\vartheta(a^{-1},R)} (|\nabla u|^2 + a^2|u|^2)|x|^{2\mu}\,dx, \tag{2.16}$$

where $\gamma_i(a^{-1},R)$ and $d_\vartheta(a^{-1},R)$ are the intersections of γ_i and d_ϑ, respectively, with the domain $a^{-1} < |x| < R$, and the constant c_{12} is independent of a and R.

3. A priori estimates

In the next two sections we prove the following propositions.

Proposition 3.1. *Let* $h_0 + h_1 > 0$, $\Re s = a^2 \geq 0$, $\mu \in [0,1)$ *and*

$$0 < 1 - \mu < \frac{\beta_0 + \beta_1}{\vartheta}, \quad \beta_i = \arctan h_i \in (-\pi/2, \pi/2), \quad i = 0, 1.$$

Any solution $u \in H^2_\mu(d_\vartheta) \cap W^2_{2,\mu}(d_\vartheta)$ *of problem* (1.1) *satisfies the inequality*

$$\begin{aligned}(3.1)\quad & \|u\|^2_{H^2_\mu(d_\vartheta)} + |s| \cdot \|u\|^2_{H^1_\mu(d_\vartheta)} + |s|^2 \|u\|^2_{L_{2,\mu}(d_\vartheta)} + |s|^{1-\mu} \|\nabla u\|^2_{L_2(d_\vartheta)} \\ & + |s|^{2-\mu} \|u\|^2_{L_2(d_\vartheta)} \\ & \leq c_1 \Big[\|f\|^2_{L_{2,\mu}(d_\vartheta)} + \sum_{i=1}^{2} (\|\varphi_i\|^2_{H^{1/2}_\mu(\gamma_i)} + |s|^{1/2} \|\varphi_i\|^2_{L_{2,\mu}(\gamma_i)}) \Big].\end{aligned}$$

Proposition 3.2. *Let* $h_0 + h_1 \leq 0$, $\Re s = a^2 \geq 0$, $\mu \in [0,1)$ *and*

$$0 < 1 - \mu < \frac{\pi + \beta_0 + \beta_1}{\vartheta}.$$

Any solution $u \in W^2_{2,\mu}(d_\vartheta)$ *of problem* (1.1) *satisfies the inequality*

$$\begin{aligned}(3.2)\quad & \|D^2 u\|^2_{L_{2,\mu}(d_\vartheta)} + |s| \cdot \|\nabla u\|^2_{L_{2,\mu}(d_\vartheta)} + |s|^2 \|u\|^2_{L_{2,\mu}(d_\vartheta)} + |s|^{1-\mu} \|\nabla u\|^2_{L_2(d_\vartheta)} \\ & + |s|^{2-\mu} \|u\|^2_{L_2(d_\vartheta)} \\ & \leq c_2 \Big[\|f\|^2_{L_{2,\mu}(d_\vartheta)} + \sum_{i=1}^{2} (\|\varphi_i\|^2_{L^{1/2}_\mu(\gamma_i)} + |s|^{1/2} \|\varphi_i\|^2_{L_{2,\mu}(\gamma_i)}) \Big].\end{aligned}$$

These propositions are proved in several steps, made for both cases simultaneously.

Step 1: *The estimate of* $\|\nabla u\|_{L_2(d_\vartheta)}$. We multiply equation (1.1) by $\overline{u}$, integrate over d_ϑ and equate the real parts of both sides of the resulting equation. Taking account of the boundary conditions we obtain after integration by parts

$$\begin{aligned}(3.3)\quad & \int_{d_\vartheta} (|\nabla u|^2 + a^2 |u|^2)\, dx + \Re \sum_{i=0}^{1} h_i \int_{\gamma_i} \frac{\partial u}{\partial r} \overline{u}\, dr \\ & = \Re \Big(\int_{d_\vartheta} f \overline{u}\, dx + \sum_{i=0}^{1} \int_{\gamma_i} \varphi_i \overline{u}\, dr \Big) \equiv \Re l(\overline{u}).\end{aligned}$$

We observe that

$$\begin{aligned}\Re \sum_{i=0}^{1} h_i \int_{\gamma_i} \frac{\partial u}{\partial r} \overline{u}\, dr &= \frac{1}{2} \sum_{i=0}^{1} h_i \int_{\gamma_i} \Big(\frac{\partial u}{\partial r} \overline{u} + \frac{\partial \overline{u}}{\partial r} u \Big) dr \\ &= -\frac{1}{2} (h_0 + h_1) |u(0)|^2 \geq 0\end{aligned}$$

(since in the case $h_0 + h_1 > 0$ we have $u \in H^2_\mu(d_\vartheta)$ and, as a consequence, $u(0) = 0$). The functional $l(\overline{u})$ can be estimated by the Hölder inequality and by (2.13), (2.14):

$$\begin{aligned}|l(\overline{u})| &\le \|f\|_{L_{2,\mu}(d_\vartheta)}\|u\|_{L_{2,-\mu}(d_\vartheta)} + \sum_{i=0}^{1}\|\varphi_i\|_{L_{2,\nu}(\gamma_i)}\|u\|_{L_{2,-\nu}(\gamma_i)} \\ &\le c_3\|f\|_{L_{2,\mu}(d_\vartheta)}\|\nabla u\|^{\mu}_{L_2(d_\vartheta)}\|u\|^{1-\mu}_{L_2(d_\vartheta)} \\ &\quad + c_3\sum_{i=0}^{1}\|\varphi_i\|_{L_{2,\nu}(\gamma_i)}\|\nabla u\|^{\beta}_{L_2(d_\vartheta)}\|u\|^{1-\beta}_{L_2(d_\vartheta)},\end{aligned}$$

where $\nu = \max(0, \mu - 1/2)$ and $\beta = 1/2 + \nu$. Next, we multiply (3.3) by $|s|^{1-\mu}$ and make use of the elementary inequalities

$$|s|^{1-\mu}\|\nabla u\|^{\mu}_{L_2(d_\vartheta)}\|u\|^{1-\mu}_{L_2(d_\vartheta)} \le (|s|^{1-\mu}\|\nabla u\|^2_{L_2(d_\vartheta)} + |s|^{2-\mu}\|u\|^2_{L_2(d_\vartheta)})^{1/2},$$
$$|s|^{3/4-\mu/2-\nu/2}\|\nabla u\|^{\beta}_{L_2(d_\vartheta)}\|u\|^{1-\beta}_{L_2(d_\vartheta)} \le (|s|^{1-\mu}\|\nabla u\|^2_{L_2(d_\vartheta)} + |s|^{2-\mu}\|u\|^2_{L_2(d_\vartheta)})^{1/2}$$

and, in the case $\mu \in (0, 1/2)$, of the estimate

$$\begin{aligned}|s|^{1/4-\mu/2}\|\varphi_i\|_{L_2(\gamma_i)} &\le |s|^{1/4-\mu/2}\|\varphi_i\|^{1-2\mu}_{L_{2,\mu}(\gamma_i)}\|\varphi_i\|^{2\mu}_{L_{2,\mu-1/2}(\gamma_i)} \\ &\le c_4(|s|^{1/2}\|\varphi_i\|^2_{L_{2,\mu}(\gamma_i)} + \|\varphi_i\|^2_{L_{2,\mu-1/2}(\gamma_i)})^{1/2}\end{aligned}$$

(for $\mu = 0$ the final inequality is evident). By (2.15), this leads to

$$|s|^{1-\mu}|l(\overline{u})| \le c_5F^{1/2}\bigg[|s|^{1-\mu}\int_{d_\vartheta}(|\nabla u|^2 + |s|\cdot|u|^2)\,dx\bigg]^{1/2} \tag{3.4}$$

and

$$|s|^{1-\mu}\int_{d_\vartheta}(|\nabla u|^2 + a^2|u|^2)\,dx \le c_6(F + F^{1/2}A_1^{1/2}), \tag{3.5}$$

where

$$\begin{aligned}F &= \|f\|^2_{L_{2,\mu}(d_\vartheta)} + \sum_{i=0}^{1}(\|\varphi_i\|^2_{L^{1/2}_\mu(\gamma_i)} + |s|^{1/2}\|\varphi_i\|^2_{L_{2,\mu}(\gamma_i)}), \\ A_1 &= |s|^{2-\mu}\int_{d_\vartheta}|u|^2\,dx.\end{aligned} \tag{3.6}$$

Step 2: *The estimate of* $\|\nabla u\|_{L_{2,\mu}(d_\vartheta)}$. Suppose that $\mu > 0$. Multiplying (1.1) by $\overline{u}|x|^{2\mu}$ and integrating, we obtain from (3.3) the equation

$$\begin{aligned}\int_{d_\vartheta}(|\nabla u|^2 + a^2|u|^2)|x|^{2\mu}\,dx + \Re\sum_{i=0}^{1}h_i\int_{\gamma_i}\frac{\partial u}{\partial r}\overline{u}r^{2\mu}\,dr \\ + \Re\int_{d_\vartheta}\nabla u\cdot\nabla|x|^{2\mu}\overline{u}\,dx = \Re l(\overline{u}|x|^{2\mu}).\end{aligned} \tag{3.7}$$

It is easily seen that

$$\Re\int_{\gamma_i}\frac{\partial u}{\partial r}\overline{u}r^{2\mu}\,dr=-\mu\int_{\gamma_i}|u|^2r^{2\mu-1}\,dr\le\mu\|u\|^{2\gamma}_{L_{2,\mu}(\gamma_i)}\|u\|^{2(1-\gamma)}_{L_{2,-\nu}(\gamma_i)},$$

where $\nu\in(\max(0,1/2-\mu),1/2)$ and $\gamma=1-1/[2(\nu+\mu)]$.

We estimate the other terms in (3.7) also by the Hölder inequality and arrive at

$$\begin{aligned}\int_{d_\vartheta}(|\nabla u|^2+a^2|u|^2)|x|^{2\mu}\,dx&\\ \le 2\mu\|\nabla u\|_{L_{2,\mu}(d_\vartheta)}\|u\|_{L_{2,\mu-1}(d_\vartheta)}&\\ +\mu\sum_{i=0}^{1}|h_i|\cdot\|u\|^{2\gamma}_{L_{2,\mu}(\gamma_i)}\|u\|^{2(1-\gamma)}_{L_{2,-\nu}(\gamma_i)}+\|f\|_{L_{2,\mu}(d_\vartheta)}\|u\|_{L_{2,\mu}(d_\vartheta)}&\\ +\sum_{i=0}^{1}\|\varphi_i\|_{L_{2,\mu}(\gamma_i)}\|u\|_{L_{2,\mu}(\gamma_i)}.&\end{aligned}$$

Now, we multiply both sides of this inequality by $|s|$ and make use of the estimates (they follow from (2.13), (2.14) and from the Young inequality)

$$\begin{aligned}|s|^{1/2}\|u\|_{L_{2,\mu-1}(d_\vartheta)}&\le c_7|s|^{1/2}\|\nabla u\|^{1-\mu}_{L_2(d_\vartheta)}\|u\|^{\mu}_{L_2(d_\vartheta)}\\ &\le c_8(F+F^{1/2}A_1^{1/2})^{(1-\mu)/2}A_1^{\mu/2},\\ |s|^{3/4}\|u\|_{L_{2,\mu}(\gamma_i)}&\le c_9|s|^{3/4}\|\nabla u\|^{1/2}_{L_{2,\mu}(d_\vartheta)}\|u\|^{1/2}_{L_{2,\mu}(d_\vartheta)}\le c_9A^{1/2},\\ |s|^{3/4-\mu/2-\nu/2}\|u\|_{L_{2,-\nu}(\gamma_i)}&\le c_{10}(|s|^{(1-\mu)/2}\|\nabla u\|_{L_2(d_\vartheta)})^{\beta}(|s|^{1-\mu/2}\|u\|_{L_2(d_\vartheta)})^{1-\beta}\\ &\le c_{11}(F+F^{1/2}A_1^{1/2})^{1/2},\end{aligned}$$

where we have set $\beta=1/2+\nu$ and

$$A=|s|\int_{d_\vartheta}(|\nabla u|^2+|s|\cdot|u|^2)|x|^{2\mu}\,dx+|s|^{2-\mu}\int_{d_\vartheta}|u|^2\,dx.$$

This leads to

$$\begin{aligned}&|s|\int_{d_\vartheta}(|\nabla u|^2+a^2|u|^2)|x|^{2\mu}\,dx\\ &\le c_{12}[F^{1/2}A^{1/2}+F^{1/4}A^{3/4}+A^{\gamma'}(F+F^{1/2}A^{1/2})^{1-\gamma'}]\\ &\le c_{13}[F^{1/2}A^{1/2}+F^{1/4}A^{3/4}+F^{1-\gamma'}A^{\gamma'}+F^{(1-\gamma')/2}A^{(1+\gamma')/2}]\equiv H[F,A],\end{aligned}\tag{3.8}$$

with $\gamma'=\max(\gamma,\mu/2)\in(0,1)$.

Step 3: *The estimate of A.* After multiplication of (1.1) by $\overline{u}|x|^{2\mu}(1 - i\operatorname{sign}\Im s)$ and integration we obtain

$$
\begin{aligned}
(3.9)\qquad (a^2+|\Im s|)\int_{d_\vartheta}|u|^2|x|^{2\mu}\,dx &+ \int_{d_\vartheta}|\nabla u|^2|x|^{2\mu}\,dx \\
&+ \Re(1-i\operatorname{sign}\Im s)\int_{d_\vartheta}\nabla u\cdot\nabla|x|^{2\mu}\overline{u}\,dx \\
&+ \Re(1-i\operatorname{sign}\Im s)\sum_{i=0}^{1}h_i\int_{\gamma_i}\frac{\partial u}{\partial r}\overline{u}r^{2\mu}\,dr \\
&= \Re(1-i\operatorname{sign}\Im s)l(\overline{u}|x|^{2\mu}).
\end{aligned}
$$

The right-hand side and the volume integral on the left-hand side are estimated as above, but in the integrals over γ_i we cannot get rid of $\partial u/\partial r$ by integration by parts. We multiply (3.9) by $|s|$ and estimate these integrals by the Hölder inequality and by (2.14) as follows:

$$
\begin{aligned}
|s|\bigg|\Re(1-i\operatorname{sign}\Im s)\sum_{i=0}^{1}\int_{\gamma_i}\frac{\partial u}{\partial r}\overline{u}r^{2\mu}\,dr\bigg|
&\le c_{14}|s|\bigg\|\frac{\partial u}{\partial r}\bigg\|_{L_{2,\mu}(\gamma_i)}\|u\|_{L_{2,\mu}(\gamma_i)} \\
&\le c_{15}|s|\cdot\|D^2u\|^{1/2}_{L_{2,\mu}(d_\vartheta)}\|\nabla u\|_{L_{2,\mu}(d_\vartheta)}\|u\|^{1/2}_{L_{2,\mu}(d_\vartheta)}.
\end{aligned}
$$

But $|s|\cdot\|\nabla u\|^2_{L_{2,\mu}(d_\vartheta)}$ is already estimated (see (3.7)), so the right-hand side does not exceed $c_{16}\|D^2u\|^{1/2}_{L_{2,\mu}(d_\vartheta)}A^{1/4}H^{1/2}[F,A]$, and we easily obtain

$$
(3.10)\qquad A\le c_{17}H[F,A]+c_{16}\|D^2u\|^{1/2}_{L_{2,\mu}(d_\vartheta)}A^{1/4}H^{1/2}[F,A].
$$

Step 4: *Estimate of D^2u and the end of proof.* We consider u as a solution of the problem

$$
(3.11)\qquad
\begin{aligned}
&-\Delta u=-su+f\equiv f_1\quad (x\in d_\vartheta),\\
&\left(\frac{\partial u}{\partial n}+h_i\frac{\partial u}{\partial r}\right)\bigg|_{\gamma_i}=\varphi_i,\quad i=0,1,
\end{aligned}
$$

and apply Theorem 2.1 or Theorem 2.2, for $k=0$. This gives

$$
\begin{aligned}
\|u\|^2_{H^2_\mu(d_\vartheta)}&\le c_{18}\bigg(\|f_1\|^2_{L_{2,\mu}(d_\vartheta)}+\sum_{i=0}^{1}\|\varphi_i\|^2_{H^{1/2}_\mu(\gamma_i)}\bigg)\\
&\le 2c_{18}\bigg(\|f\|^2_{L_{2,\mu}(d_\vartheta)}+\sum_{i=0}^{1}\|\varphi_i\|^2_{H^{1/2}_\mu(\gamma_i)}+A\bigg)
\end{aligned}
$$

and

$$\|D^2u\|^2_{L_{2,\mu}(d_\vartheta)} \le c_{19}\Big(\|f\|^2_{L_{2,\mu}(d_\vartheta)} + \sum_{i=0}^{1}\|\varphi_i\|^2_{L_\mu^{1/2}(\gamma_i)} + A\Big) \tag{3.12}$$

in the first and in the second case, respectively.

These inequalities together with (3.10) make it possible to estimate A by the right-hand side of (3.1) or (3.2). Then by (3.5), (3.10) and (3.12) we prove (3.2). In order to prove (3.1) it is sufficient to estimate $|s|\cdot\|u\|^2_{H^1_\mu(d_\vartheta)}$; we have

$$\begin{aligned}|s|\cdot\|u\|^2_{H^1_\mu(d_\vartheta)} &\le A + |s|\cdot\|u\|^2_{L_{2,\mu-1}(d_\vartheta)} \le A + |s|\cdot\|u\|_{L_{2,\mu}(d_\vartheta)}\|u\|_{L_{2,\mu-2}(d_\vartheta)}\\ &\le A + A^{1/2}\|u\|_{H^2_\mu(d_\vartheta)} \le c_{20}\Big(\|f\|^2_{L_{2,\mu}(d_\vartheta)} + \sum_{i=0}^{1}\|\varphi_i\|^2_{H_\mu^{1/2}(\gamma_i)}\Big),\end{aligned}$$

which completes the proof of (3.1).

4. The solvability of problem (1.1)

In this section we establish the existence of the solutions of problem (1.1) estimated in §3. It suffices to do this for smooth data f, φ_0, φ_1 with compact supports.

We consider a penalized problem

$$\begin{aligned}&-\Delta u_\varepsilon + s u_\varepsilon = f,\\ &\Big(\frac{\partial u_\varepsilon}{\partial n} + h_0\frac{\partial u_\varepsilon}{\partial r} + \frac{\varepsilon}{r}\Big)\Big|_{\gamma_0} = \varphi_0, \quad \Big(\frac{\partial u_\varepsilon}{\partial n} + h_1\frac{\partial u_\varepsilon}{\partial r}\Big)\Big|_{\gamma_1} = \varphi_1,\end{aligned} \tag{4.1}$$

for small positive ε under the hypothesis $\Re s = a^2 > 0$ (once the regularity of the solution of problem (1.1) is established, we can apply the above a priori estimates and let a tend to zero if necessary). We define a weak solution of this problem as a function $u_\varepsilon \in W_2^1(d_\vartheta) \cap H_0^1(d_\vartheta)$ satisfying the integral identity

$$Q_\varepsilon[u_\varepsilon, \eta] = l(\eta), \tag{4.2}$$

where

$$Q_\varepsilon[u_\varepsilon,\eta] = \int_{d_\vartheta}(\nabla u_\varepsilon\cdot\nabla\eta + s u_\varepsilon\eta)\,dx - \sum_{i=0}^{1} h_i\int_{\gamma_i} u_\varepsilon\frac{\partial\eta}{\partial r}\,dr + \varepsilon\int_{\gamma_0} u_\varepsilon\eta\,\frac{dr}{r},$$

for every smooth η with compact support vanishing near the origin. Since

$$\int_{\gamma_i} u_\varepsilon(r)\frac{\partial\eta(r)}{\partial r}\,dr = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varrho\widehat{u}_\varepsilon(-\varrho)\widehat{\eta}(\varrho)\,d\varrho,$$

where $\widehat{\eta}(\varrho) = \int_0^\infty \eta(r)e^{-i\varrho r}dr$ is the Fourier transform of the function η extended by zero into the half-axis $r<0$, the form $Q_\varepsilon[u,\eta]$ can be extended by continuity to all $u,\eta\in W_2^1(d_\vartheta)\cap H_0^1(d_\vartheta)$. In addition,

$$\Re Q_\varepsilon[u,u]=\int_{d_\vartheta}(|\nabla u|^2+a^2|u|^2)\,dx+\varepsilon\int_{\gamma_0}|u|^2\,\frac{dr}{r}\geq c_1(\varepsilon)(\|u\|^2_{H_0^1(d_\vartheta)}+\|u\|^2_{W_2^1(d_\vartheta)})$$

(the last inequality is due to V. A. Kondrat'ev, see [5]). From the estimates of §3 it is clear that $l(\eta)$ is a linear continuous functional on $W_2^1(d_\vartheta)$, so the existence of a unique generalized solution follows from the Lax–Milgram theorem. From the regularity theorems for the solutions of elliptic boundary value problems it follows that $u_\varepsilon\in W_2^2(\omega)$ in every bounded subdomain ω of d_ϑ, maybe adjacent to the boundary but bounded away from the origin. To clarify the regularity properties of a generalized solution, we need to estimate it uniformly with respect to ε.

Setting $\eta=a^{2-2\mu}\overline{u}_\varepsilon$, $\mu\in[0,1)$, in (4.2), taking the real part of both sides, and then estimating the right-hand side precisely as above (see Step 1) we arrive at estimate (3.5) with a^2 instead of $|s|$, i.e. at

$$a^{2-2\mu}\bigg(\int_{d_\vartheta}(|\nabla u_\varepsilon|^2+a^2|u_\varepsilon|^2)\,dx+\varepsilon\int_0^\infty|u_\varepsilon|^2\,\frac{dr}{r}\bigg)\leq c_2F, \tag{4.3}$$

where F is given in (3.6).

Next, we show that $u_\varepsilon,\nabla u_\varepsilon\in L_{2,\mu}(d_\vartheta)$.

Proposition 4.1. *The function u_ε belongs to $L_{2,\mu}(d_\vartheta)$ together with its gradient, and*

$$a^2\int_{d_\vartheta}(|\nabla u_\varepsilon|^2+a^2|u_\varepsilon|^2)|x|^{2\mu}\,dx+\varepsilon a^2\int_{\gamma_0}|u_\varepsilon|^2r^{2\mu-1}\,dr\leq c_3F. \tag{4.4}$$

Proof. We set $\eta=a^2\overline{u}_\varepsilon\min(|x|^{2\mu},R^{2\mu})$ in (4.2) with arbitrarily large $R>0$ (because of the presence of the integral over the boundary it would be better to take $\eta=(1-\zeta_\varrho(x))\overline{u}_\varepsilon\min(|x|^{2\mu},R^{2\mu})$, where $\zeta_\varrho(x)=\zeta(x\varrho^{-1})$, $\zeta\in C_0^\infty(\mathbb{R}^n)$, $\zeta(x)=1$ near 0, and then to pass to the limit as $\varrho\to 0$; see [3] for details). After integration by parts we obtain

$$\begin{aligned}a^2\int_{d_{\vartheta,R}}(|\nabla u_\varepsilon|^2+a^2|u_\varepsilon|^2)|x|^{2\mu}\,dx-a^2\Re\sum_{i=0}^1\mu h_i\int_{\gamma_{i,R}}|u_\varepsilon|^2r^{2\mu-1}\,dr\\ +a^2\varepsilon\int_0^\infty|u_\varepsilon|^2r^{2\mu-1}\,dr+a^2\Re\int_{d_{\vartheta,R}}\nabla u_\varepsilon\cdot\nabla|x|^{2\mu}\overline{u}_\varepsilon\,dx\\ \leq a^2\Re l(\overline{u}_\varepsilon\min(|x|^{2\mu},R^{2\mu})),\end{aligned} \tag{4.5}$$

where $d_{\vartheta,R}$ and $\gamma_{i,R}$ are the intersections of d_ϑ and γ_i with the ball $|x|<R$.

Now we repeat the arguments of Step 2 in §3, but, since the boundedness of the $L_{2,\mu}$-norm of ∇u_ε in the whole d_ϑ is not yet proved, we should not let this norm arise in the process of estimates. By (2.13) and the Young inequality, we have

$$a^2\left|\int_{d_{\vartheta,R}} \nabla u_\varepsilon\cdot\nabla|x|^{2\mu}\overline{u}_\varepsilon\,dx\right|$$
$$\le 2\mu a^2\biggl(\int_{d_{\vartheta,R}}|\nabla u_\varepsilon|^2|x|^{2\mu}\,dx\biggr)^{1/2}\|u_\varepsilon\|_{L_{2,\mu-1}(d_\vartheta)}$$
$$\le 2\mu\biggl(a^2\int_{d_{\vartheta,R}}|u_\varepsilon|^2|x|^{2\mu}\,dx\biggr)^{1/2}\biggl(a^{2-2\mu}\int_{d_\vartheta}(|\nabla u_\varepsilon|^2+a^2|u_\varepsilon|^2)\,dx\biggr)^{1/2};$$

moreover, using estimate (2.16) in $d_{\vartheta,R}\setminus d_{\vartheta,a^{-1}}$, we obtain

$$\begin{aligned} a^3\int_{\gamma_{i,R}}|u_\varepsilon|^2r^{2\mu}\,dr &\le a^{3-2\mu}\int_{\gamma_{i,a^{-1}}}|u_\varepsilon|^2dr+a^3\int_{\gamma_{i,R}\setminus\gamma_{i,a^{-1}}}|u_\varepsilon|^2r^{2\mu}\,dr\\ &\le a^{3-2\mu}\|u_\varepsilon\|_{L_2(d_\vartheta)}\|u_\varepsilon\|_{L_2(d_\vartheta)}\\ &\quad+c_3a^2\int_{d_{\vartheta,R}\setminus d_{\vartheta,a^{-1}}}(|\nabla u_\varepsilon|^2+a^2|u_\varepsilon|^2)|x|^{2\mu}\,dx \end{aligned}$$

and

$$a^2\int_{\gamma_{i,R}}|u_\varepsilon|^2r^{2\mu-1}\,dr$$
$$\le a^2\biggl(\int_{\gamma_{i,R}}|u_\varepsilon|^2r^{2\mu}\,dr\biggr)^{\gamma}\biggl(\int_{\gamma_i}|u_\varepsilon|^2r^{-2\nu}\,dr\biggr)^{1-\gamma}$$
$$\le c_4\biggl(a^2\int_{d_{\vartheta,R}}(|\nabla u_\varepsilon|^2+a^2|u_\varepsilon|^2)|x|^{2\mu}\,dx+a^{2-2\mu}\int_{d_\vartheta}(|\nabla u_\varepsilon|^2+a^2|u_\varepsilon|^2)\,dx\biggr),$$

where, as above, $\gamma=1-1/[2(\mu+\nu)]$. The right-hand side of (4.5) can be estimated in the same way, and we arrive at the inequality

$$a^2\int_{d_{\vartheta,R}}(|\nabla u_\varepsilon|^2+a^2|u_\varepsilon|^2)|x|^{2\mu}\,dx+\varepsilon a^2\int_{\gamma_{0,R}}|u_\varepsilon|^2r^{2\mu-1}\,dr\le c_5F,$$

which implies (4.4).

5. Bounds for second derivatives

Now we show that the solution of problem (2.4) has a bounded norm $\|D^2u\|_{L_{2,\mu}(d_\vartheta)}$.

Proposition 5.1. *If the hypotheses of Proposition* 3.1 *or* 3.2 *are satisfied, then problem* (1.1) *has a solution* $u\in H^2_\mu(d_\vartheta)\cap W^2_{2,\mu}(d_\vartheta)$ *or* $u\in W^2_{2,\mu}(d_\vartheta)$, *respectively.*

Proof. We consider the cases $h_0+h_1>0$ and $h_0+h_1\le 0$ separately.

Case 1: $h_0 + h_1 > 0$. In this case we show that the solution of the penalized problem (4.1) belongs to $H^2_\mu(d_\vartheta)$. Clearly, $u_{\varepsilon,R} = u_\varepsilon \zeta_R(x)$, where $\zeta_R(x) = \zeta(x/R)$, belongs to $H^1_0(d_\vartheta)$ and is a solution of the problem (2.4) with the right-hand sides

$$f_R \equiv (f - su_\varepsilon)\zeta_R - 2\nabla\zeta_R \cdot \nabla u_\varepsilon - u_\varepsilon \Delta\zeta_R \tag{5.1}$$

and

$$\Phi_{i,R} \equiv \left(\varphi_0\zeta_R + h_i u_\varepsilon \frac{\partial \zeta_R}{\partial r}\right)\bigg|_{\gamma_0}, \quad i = 0, 1. \tag{5.2}$$

Since the supports of f and Φ_i are compact, we have $f\zeta_R = f$ and $\varphi_i\zeta_R = \varphi_i$ if R is large enough, and

$$\begin{aligned} \|f_R\|_{L_{2,\mu'}(d_\vartheta)} &\le \|f\|_{L_{2,\mu'}(d_\vartheta)} + |s| \cdot \|u_\varepsilon\|_{L_{2,\mu'}(d_\vartheta)} \\ &\quad + c_1\left(\frac{1}{R}\|u_\varepsilon\|_{L_{2,\mu'}(d_\vartheta)} + \frac{1}{R^2}\|u_\varepsilon\|_{L_{2,\mu'}(d_\vartheta)}\right), \\ \|\Phi_{i,R}\|_{H^{1/2}_{\mu'}(\gamma_i)} &\le \|\varphi_i\|_{H^{1/2}_{\mu'}(\gamma_i)} + c_2\|u_\varepsilon\zeta'_R\|_{H^{1/2}_{\mu'}(\gamma_i)}, \end{aligned}$$

for all $\mu' \in [\mu, 1]$. Hence, by (4.3), the $H^1_0(d_\vartheta)$-norm of $u_{\varepsilon,R}$ is bounded by a constant independent of R, and by Theorem 2.1, $u_{\varepsilon,R} \in H^2_1(d_\vartheta)$. On the other hand, the same problem (2.4) has a solution $w_{\varepsilon,R} \in H^2_\mu(d_\vartheta)$. Since the interval $(0, 1-\mu)$ contains no solutions of equation (2.5), $w_{\varepsilon,R} = u_{\varepsilon,R} \in H^2_\mu(d_\vartheta)$. For small ε the estimate

$$\|u_{\varepsilon,R}\|_{H^2_\mu(d_\vartheta)} \le c_3\left(\|f_R\|_{L_{2,\mu}(d_\vartheta)} + \left\|\Phi_{0,R} - \frac{\varepsilon}{r}u_{\varepsilon,R}\right\|_{H^{1/2}_\mu(\gamma_0)} + \|\Phi_{1,R}\|_{H^{1/2}_\mu(\gamma_1)}\right)$$

implies

$$\begin{aligned} &\|u_{\varepsilon,R}\|_{H^2_\mu(d_\vartheta)} \\ &\quad \le c_4\left(\|f\|_{L_{2,\mu}(d_\vartheta)} + |s| \cdot \|u_\varepsilon\|_{L_{2,\mu}(d_\vartheta)} + \sum_{i=0}^{1}\|\varphi\|_{H^{1/2}_\mu(\gamma_i)} + \|u_\varepsilon\nabla\zeta_R\|_{H^1_\mu(d_\vartheta)}\right). \end{aligned}$$

Taking the limit as $R \to \infty$ we conclude that $u_\varepsilon \in H^2_\mu(d_\vartheta)$, and

$$\|u_\varepsilon\|_{H^2_\mu(d_\vartheta)} \le c_4\left(\|f\|_{L_{2,\mu}(d_\vartheta)} + |s| \cdot \|u_\varepsilon\|_{L_{2,\mu}(d_\vartheta)} + \sum_{i=0}^{1}\|\varphi_i\|_{H^{1/2}_\mu(\gamma_i)}\right).$$

But the $L_{2,\mu}(d_\vartheta)$-norm of u_ε has already been estimated uniformly with respect to ε, so we see that problem (1.1) also has a solution $u \in H^2_\mu(d_\vartheta)$.

Case 2: $h_0 + h_1 \le 0$. In this case the above arguments fail. We cannot affirm that $w_{\varepsilon,R} = u_{\varepsilon,R}$, since the interval $(0, (\pi + \beta_0 + \beta_1)\vartheta^{-1})$ contains a solution of equation (2.5). Therefore we pass to the limit as $\varepsilon \to 0$. The limiting function $u \in W^1_2(d_\vartheta)$ is a generalized solution of problem (1.1). It satisfies identity (4.2) with $\varepsilon = 0$ for every smooth $\eta(x)$ vanishing near the origin with a compact support, and inequalities (4.3)–(4.5) (also with $\varepsilon = 0$). Moreover, $D^2 u \in L_2(\omega)$ for each bounded $\omega \subset d_\vartheta$ with $\operatorname{dist}(\omega, 0) > 0$. Let us show that $D^2 u \in L_{2,\mu}(d_\vartheta)$. Assume first that $\mu > 0$. The function $u_R = u\zeta_R$ has the same differentiability properties as u, and it is a solution of the problem

$$(5.3)\qquad \begin{aligned} &-\Delta u_R = f_R, \\ &\left(\frac{\partial u_R}{\partial n} + h_i \frac{\partial u_R}{\partial r}\right)\bigg|_{\gamma_i} = \Phi_{i,R}, \end{aligned}$$

where f_R and $\Phi_{i,R}$ are defined in terms of u according to formulas (5.1), (5.2). Clearly, $f_R \in L_{2,\mu}(d_\vartheta) \cup L_{2,1+\lambda}(d_\vartheta)$, $\Phi_{i,R} \in H_\mu^{1/2}(\gamma_i) \cup H_{1+\lambda}^{1/2}(\gamma_i)$ with a small $\lambda > 0$, and $u_R \in H_\lambda^1(d_\vartheta)$. Hence, by Theorem 2.1, $u_R \in H_{1+\lambda}^2(d_\vartheta)$. On the other hand, problem (5.3) has a solution $w_R \in H_\mu^2(d_\vartheta)$. Since the interval $(-\lambda, 1-\mu)$ contains the "eigenvalue" $\lambda = 0$, we conclude in the case $h_0 + h_1 < 0$ that

$$(5.4)\qquad u_R - w_R = \text{const} = u_R(0)$$

and that

$$\|D^2 u_R\|_{L_{2,\mu}(d_\vartheta)} \le c_5\left(\|f_R\|_{L_{2,\mu}(d_\vartheta)} + \sum_{i=0}^{1} \|\Phi_{i,R}\|_{H_\mu^{1/2}(\gamma_i)}\right).$$

The right-hand side is uniformly bounded for large R, so $D^2 u \in L_{2,\mu}(d_\vartheta)$.

If $h_0 + h_1 = 0$, then

$$u_R - w_R = a_R + b_R(\log r + h_0\varphi),$$

but the last term (if it is different from zero) has an unbounded Dirichlet integral in every neighbourhood of the origin, therefore $b_R = 0$. Hence, (5.4) holds and the same conclusion as above can be made.

Let us turn to the case $\mu = 0$. Since the supports of f and φ_i are compact, it follows that $f \in L_2(d_\vartheta) \cap L_{2,\mu}(d_\vartheta)$, $\varphi_i \in W_2^{1/2}(\gamma_i) \cap W_{2,\mu}^{1/2}(\gamma_i)$ for all $\mu \in (0,1)$, and, as we have seen, $u \in W_{2,\mu}^2(d_\vartheta)$. Further, let $w \in W_2^2(d_\vartheta)$ be such that

$$(5.5)\qquad \psi_i \equiv \left(\varphi_i - \frac{\partial w}{\partial n} - h_i \frac{\partial w}{\partial r}\right)\bigg|_{\gamma_i} \in H_0^{1/2}(\gamma_i)$$

and

$$(5.6)\qquad \|w\|_{W_2^2(d_\vartheta)}^2 + \sum_{i=0}^{1} \|\psi_i\|_{H_0^{1/2}(\gamma_i)}^2 \le c_6 \sum_{i=0}^{1} \|\varphi_i\|_{W_2^{1/2}(\gamma_i)}^2$$

(see §2). Without restriction of generality we can assume that the support of w is compact (since multiplication of w by the cut-off function ζ_R does not destroy

(5.5), (5.6)), so $w \in W_2^2(d_\vartheta) \cap W_{2,\mu}^2(d_\vartheta)$. It is easily seen that $u = w + v$, where v is a solution of the problem

$$-\Delta v = f - su + \Delta w, \quad \left(\frac{\partial v}{\partial n} + h_i \frac{\partial v}{\partial r}\right)\bigg|_{\gamma_i} = \psi_i, \quad i = 0, 1.$$

As $f - su + \Delta w \in L_2(d_\vartheta) \cap L_{2,\mu}(d_\vartheta)$ and $\psi_i \in H_0^{1/2}(\gamma_i) \cap H_\mu^{1/2}(\gamma_i)$, we can conclude by Theorem 2.1 that $v \in H_0^2(d_\vartheta) \cap H_\mu^2(d_\vartheta)$. Hence, $u \in W_2^2(d_\vartheta) \cap W_{2,\mu}(d_\vartheta)$, and the proof of Proposition 5.1 is complete.

Remark. In the case $h_0 + h_1 \le 0$ we have proved that problem (1.1) with $\Re s = a^2 > 0$ has a generalized solution $u \in W_2^1(d_\vartheta)$. This solution is unique, or, what is the same, each generalized solution of the homogeneous problem vanishes. This can be established by setting $\eta = u\zeta(x, \delta)$ and letting $\delta \to 0$. Here

$$\zeta(x, \delta) = \psi\left(\log \frac{\log|x|}{\log \delta}\right),$$

where $\psi \in C_0^\infty(\mathbb{R})$, $\psi(t) = 0$ for $t \ge 1$, $\psi(t) = 1$ for $t \le 0$. With this choice of η we easily arrive at the inequality

$$\int_{d_\vartheta \cap \{|x| > \delta\}} (|\nabla u|^2 + a^2|u|^2)\, dx \le z(\delta)$$

with $z(\delta) \to 0$ as $\delta \to 0$ (see Theorem 3.4 of [3] for details). In the proof of this inequality the condition $h_0 + h_1 \le 0$ is used. For $h_0 + h_1 > 0$ the proof fails, and the existence of a unique generalized solution of problem (1.1) in $W_2^1(d_\vartheta)$ is problematic. However, the uniqueness of the solution $u \in H_\mu^2(d_\vartheta)$ obtained above follows from the a priori estimate (3.1).

6. The case $k > 0$, $\mu \ge 0$

We proceed to the proof of Theorems 1.1 and 1.2 for all $k > 0$ and consider two cases: $\mu \in [0, 1)$ and $\mu \ge 1$.

Step 1: $\mu \in [0, 1)$. Let us start with Theorem 1.1. It is easily seen that every $f \in W_{2,\mu}^k(d_\vartheta) \cap H_\mu^k(d_\vartheta)$ belongs to the union of the spaces $W_{2,\mu}^l(d_\vartheta) \cap H_\mu^l(d_\mu)$, $l = 0, \ldots, k$; in addition, $\varphi_i \in W_{2,\mu}^{k+1/2}(\gamma_i) \cap H_\mu^{k+1/2}(\gamma_i)$ implies $\varphi_i \in \bigcap_{l=0}^k W_{2,\mu}^{l+1/2}(\gamma_i) \cap H_\mu^{l+1/2}(\gamma_i)$. As shown above, problem (1.1) has a solution $u \in W_{2,\mu}^2(d_\vartheta) \cap H_\mu^2(d_\vartheta)$ which we consider as a solution of (3.11). As $f_1 \in W_{2,\mu}^1(d_\vartheta) \cap H_\mu^1(d_\vartheta)$, according to Theorem 2.1, $u \in H_\mu^3(d_\vartheta)$, hence, $u \in W_{2,\mu}^3(d_\vartheta) \cap H_\mu^3(d_\vartheta)$. If $k > 1$, we can repeat this argument and show that $u \in W_{2,\mu}^4(d_\vartheta) \cap H_\mu^4(d_\vartheta)$ etc.

By (2.6),

$$\begin{aligned} (6.1) \qquad & |s|^{k-l} \|u\|^2_{H_\mu^{l+2}(d_\vartheta)} \\ & \le c_1 |s|^{k-l} \left(\|f\|^2_{H_\mu^l(d_\vartheta)} + |s|^2 \|u\|^2_{H_\mu^l(d_\vartheta)} + \sum_{i=0}^1 \|\varphi_i\|^2_{H_\mu^{l+1/2}(\gamma_i)} \right), \quad l = 0, \ldots, k. \end{aligned}$$

These inequalities and (3.1) imply (1.2).

Theorem 1.2 is proved in a similar way. Proposition 3.2 guarantees that $u \in W^2_{2,\mu}(d_\vartheta)$, hence, $f_1 \in W^1_{2,\mu}(d_\vartheta)$. By Theorem 2.2 applied to problem (3.11), $u \in W^3_{2,\mu}(d_\vartheta)$ etc. Inequality (2.10) with $b = |s|^{1/2}$ implies

$$(6.2)\qquad |s|^{k-l} \sum_{|j|=l+2} \|D^j u\|^2_{L_{2,\mu}(d_\vartheta)} \le c_2\Big[\sum_{|j|\le l} |s|^{k-|j|}(\|D^j f\|^2_{L_{2,\mu}(d_\vartheta)} + |s|^2\|D^j u\|^2_{L_{2,\mu}(d_\vartheta)}) + \sum_{i=0}^{1}\Big(\sum_{m=0}^{l} |s|^{l+1/2-m}\|D^m_r \varphi_i\|^2_{L_{2,\mu}(\gamma_i)} + \|\varphi_i\|^2_{L^{l+1/2}_{\mu}(\gamma_i)}\Big)\Big]$$

for $l = 0, \dots, k$. These inequalities and (3.2) yield (1.6).

Step 2: $\mu \ge 1,\ k > 0$. We start the consideration of this case with Theorem 1.2. Observe that, by the Hardy inequality

$$(6.3)\qquad \|f\|^2_{L_{2,\mu-j}(d_\vartheta)} \le c_3 \sum_{|m|=j} \|D^m f\|^2_{L_{2,\mu}(d_\vartheta)},$$

f belongs to $W^{k'}_{2,\mu'}(d_\vartheta)$ and the estimate

$$\|f\|^2_{W^{k'}_{2,\mu'}(d_\vartheta)} \le c_3\|f\|^2_{W^k_{2,\mu}(d_\vartheta)}$$

holds with $\mu' = \mu - [\mu] \in [0,1)$ and $k' = k - [\mu]$ (the condition $1 + k - \mu > 0$ guarantees non-negativity of k'). We also have the inequality

$$\|\varphi_i\|^2_{W^{k'+1/2}_{2,\mu'}(\gamma_i)} \le c_4\|\varphi_i\|^2_{W^{k+1/2}_{2,\mu}(\gamma_i)}.$$

Therefore problem (1.1) has a solution $u \in W^{k'+2}_{2,\mu'}(d_\vartheta)$, and

$$(6.4)\qquad \sum_{l=0}^{r+2} |s|^{r+2-l} \sum_{|j|=l} \|D^j u\|^2_{L_{2,\mu'}(d_\vartheta)} \le c_5\Big[\sum_{l=0}^{r} |s|^{r-l} \sum_{|j|=l} \|D^j f\|^2_{L_{2,\mu'}(d_\vartheta)} + \sum_{i=0}^{1}\Big(\sum_{l=0}^{r} |s|^{r+1/2-l}\|D^l \varphi_i\|^2_{L_{2,\mu'}(\gamma_i)} + \|\varphi_i\|^2_{L^{r+1/2}_{2,\mu'}(\gamma_i)}\Big)\Big], \quad r = 0, \dots, k'.$$

Next, we show that this solution belongs to $W^{k+2}_{2,\mu}(d_\vartheta)$ and we estimate its norm. For this we need the following auxiliary proposition whose proof will be given in the Appendix.

Proposition 6.1. *The solution of problem* (2.1) *satisfies the inequality*

$$\sum_{|j|=l+2}\|D^j u\|^2_{L_{2,\mu}(d_\vartheta)} \le c_6\bigg[\sum_{|j|=l}\|D^j f\|^2_{L_{2,\mu}(d_\vartheta)}+\sum_{|j|=l+1}\|D^j u\|^2_{L_{2,\mu-1}(d_\vartheta)} + \sum_{i=0}^{1}\|D^l_r\Phi_i\|^2_{L^{1/2}_\mu(\gamma_i)}\bigg],\quad \mu\ge 1. \tag{6.5}$$

First of all, we obtain an a priori estimate for $I(\mu)\equiv\|\nabla u\|^2_{L_{2,\mu}(d_\vartheta)}+|s|\cdot\|u\|^2_{L_{2,\mu}(d_\vartheta)}$ using equations (3.7) and (3.9) (our arguments here apply also to Theorem 1.1). Equation (3.7) implies

$$\begin{aligned}\int_{d_\vartheta}(|\nabla u|^2+a^2|u|^2)|x|^{2\mu}\,dx &\le c_7\bigg(\int_{d_\vartheta}|u|^2|x|^{2\mu-2}\,dx+\sum_{i=0}^{1}\int_{\gamma_i}|u|^2r^{2\mu-1}\,dr \\ &\quad+\|f\|_{L_{2,\mu}(d_\vartheta)}\|u\|_{L_{2,\mu}(d_\vartheta)}+\sum_{i=0}^{1}\|\varphi_i\|_{L_{2,\mu}(\gamma_i)}\|\nabla u\|^{1/2}_{L_{2,\mu}(d_\vartheta)}\|u\|^{1/2}_{L_{2,\mu}(d_\vartheta)}\bigg).\end{aligned}$$

Multiplying both sides of this inequality by $|s|$ and making use of the estimate

$$\begin{aligned}\|u\|^2_{L_{2,\mu-1/2}(\gamma_i)} &\le c_8\|\nabla u\|_{L_{2,\mu-1/2}(d_\vartheta)}\|u\|_{L_{2,\mu-1/2}(d_\vartheta)} \\ &\le c_9\|\nabla u\|^{1/2}_{L_{2,\mu}(d_\vartheta)}\|u\|^{1/2}_{L_{2,\mu}(d_\vartheta)}\|\nabla u\|^{1/2}_{L_{2,\mu-1}(d_\vartheta)}\|u\|^{1/2}_{L_{2,\mu-1}(d_\vartheta)},\end{aligned}$$

we obtain

$$\begin{aligned}|s|\cdot\|\nabla u\|^2_{L_{2,\mu}(d_\vartheta)} &\le c_{10}|s|^{1/2}I(\mu)^{1/2}\bigg(\|f\|^2_{L_{2,\mu}(d_\vartheta)}+|s|^{1/2}\sum_{i=0}^{1}\|\varphi_i\|^2_{L_{2,\mu}(\gamma_i)}\bigg)^{1/2} \\ &\quad+c_{10}[|s|^{1/2}I(\mu)^{1/2}I(\mu-1)^{1/2}+I(\mu-1)].\end{aligned}\tag{6.6}$$

Equation (3.9) yields

$$\begin{aligned}I(\mu)\le c_{11}\bigg(&\|u\|^2_{L_{2,\mu-1}(d_\vartheta)}+\|D^2u\|^{1/2}_{L_{2,\mu}(d_\vartheta)}\|\nabla u\|_{L_{2,\mu}(d_\vartheta)}\|u\|^{1/2}_{L_{2,\mu}(d_\vartheta)} \\ &+\|f\|_{L_{2,\mu}(d_\vartheta)}\|u\|_{L_{2,\mu}(d_\vartheta)}+\sum_{i=0}^{1}\|\varphi_i\|_{L_{2,\mu}(\gamma_i)}\|\nabla u\|^{1/2}_{L_{2,\mu}(d_\vartheta)}\|u\|^{1/2}_{L_{2,\mu}(d_\vartheta)}\bigg).\end{aligned}$$

We multiply this inequality by $|s|$ and estimate the norm of the second derivatives by Proposition 6.1 applied to problem (3.11). After elementary computations we arrive at

$$|s|I(\mu) \le c_{12}(F_0(\mu) + |s| \cdot \|u\|^2_{L_{2,\mu-1}(d_\vartheta)} + |s|^{3/2}\|\nabla u\|_{L_{2,\mu}(d_\vartheta)}\|u\|_{L_{2,\mu}(d_\vartheta)}),$$

and, taking account of (6.6), at

$$|s|I(\mu) \le c_{13}[F_0(\mu) + I(\mu - 1)], \tag{6.7}$$

where

$$F_0(\mu) \equiv \|f\|^2_{L_{2,\mu}(d_\vartheta)} + \sum_{i=0}^{1}(\|\varphi_i\|^2_{L^{1/2}_\mu(\gamma_i)} + |s|^{1/2}\|\varphi_i\|^2_{L_{2,\mu}(\gamma_i)}).$$

In the proof of (6.7) we have used the boundedness of the integral $I(\mu)$, which can be justified precisely as in §4 under the assumption $\Re s = a^2 > 0$ (it does not restrict generality). Indeed, inequality (4.5) with $\varepsilon = 0$ and $\mu \ge 1$ yields

$$\begin{aligned} a^2 &\int_{d_\vartheta}(|\nabla u|^2 + a^2|u|^2)V_R^2\,dx \\ &\le c_{14}a^2\bigg(\int_{d_{\vartheta,R}}|u|^2|x|^{2\mu-2}\,dx + \sum_{i=0}^{1}\int_{\gamma_{i,R}}|u|^2r^{2\mu-1}\,dr \\ &\quad + \|f\|_{L_{2,\mu}(d_\vartheta)}\|uV_R\|_{L_2(d_\vartheta)} + \sum_{i=0}^{1}\|\varphi_i\|_{L_{2,\mu}(\gamma_i)}\|uV_R\|^{1/2}_{L_2(d_\vartheta)}\|\nabla(uV_R)\|^{1/2}_{L_2(d_\vartheta)}\bigg), \end{aligned}$$

where $V_R \equiv \min(|x|^\mu, R^\mu)$. Since by (2.16),

$$\begin{aligned} a^2 &\int_{\gamma_{i,R}}|u|^2r^{2\mu-1}\,dr \\ &\qquad \le a\int_{\gamma_{i,a^{-1}}}|u|^2r^{2\mu-2}\,dr + c_{15}a\int_{d_{\vartheta,R}\setminus d_{\vartheta,a^{-1}}}(|\nabla u|^2 + a^2|u|^2)|x|^{2\mu-1}\,dx \\ &\qquad \le c_{16}a\|\nabla u\|_{L_{2,\mu-1}(d_\vartheta)}\|u\|_{L_{2,\mu-1}(d_\vartheta)} \\ &\qquad\quad + c_{16}a\bigg[\int_{d_{\vartheta,R}}(|\nabla u|^2 + a^2|u|^2)|x|^{2\mu}\,dx\bigg]^{1/2} \\ &\qquad\quad \times\bigg[\int_{d_\vartheta}(|\nabla u|^2 + a^2|u|^2)|x|^{2\mu-2}\,dx\bigg]^{1/2}, \end{aligned}$$

we obtain

$$a^2\int_{d_\vartheta}(|\nabla u|^2 + a^2|u|^2)V_R^2\,dx \le c_{17}F_0(\mu) + c_{17}\int_{d_\vartheta}(|\nabla u|^2 + a^2|u|^2)|x|^{2\mu-2}\,dx.$$

Hence, if $\nabla u, u \in L_{2,\mu-1}(d_\vartheta)$, then $\nabla u, u \in L_{2,\mu}(d_\vartheta)$, and (6.7) holds. As we have already shown, $\nabla u, u \in L_{2,\mu'}(d_\vartheta)$, so we can conclude that $\nabla u, u \in$

$\bigcap_{j=0}^{[\mu]} L_{2,\mu-j}(d_\vartheta)$; moreover, iterating (6.7) and making use of (6.3) and (6.4) we obtain

$$\begin{aligned}(6.8)\quad &\sum_{j=0}^{[\mu]}(|s|^{1+k-j}\|\nabla u\|^2_{L_{2,\mu-j}(d_\vartheta)}+|s|^{2+k-j}\|u\|^2_{L_{2,\mu-j}(d_\vartheta)})\\ &=\sum_{j=0}^{[\mu]}|s|^{1+k-j}I(\mu-j)\le c_{18}\bigg[\sum_{j=0}^{[\mu]}|s|^{k-j}F_0(\mu-j)+|s|^{1+k'}I(\mu')\bigg]\le c_{19}F,\end{aligned}$$

where F is the sum of the norms on the right-hand side of (1.6).

Let us estimate $|s|^{k-l-j}\sum_{|m|=l+2}\|D^m u\|^2_{L_{2,\mu-j}(d_\vartheta)}$. If $l+j\le[\mu]-1$, then we can do it with the help of Proposition 6.1 applied to problem (3.11). By (6.5), the solution of this problem satisfies the inequality

$$\begin{aligned}&|s|^{k-l-j}\sum_{|m|=l+2}\|D^m u\|^2_{L_{2,\mu-j}(d_\vartheta)}\\ &\le c_{20}\bigg(|s|^{k-l-j}\sum_{|m|=l+1}\|D^m u\|^2_{L_{2,\mu-j-1}(d_\vartheta)}+|s|^{2+k-l-j}\sum_{|m|=l}\|D^m u\|^2_{L_{2,\mu-j}(d_\vartheta)}+F\bigg).\end{aligned}$$

To estimate the norms of u on the right-hand side, we apply (6.5) l more times to obtain

$$\begin{aligned}(6.9)\quad |s|^{k-l-j}\sum_{|m|=l+2}\|D^m u\|^2_{L_{2,\mu-j}(d_\vartheta)}&\le c_{21}\bigg(\sum_{j'=j}^{[\mu]}|s|^{1+k-j'}I(\mu-j')+F\bigg)\\ &\le c_{22}F.\end{aligned}$$

If we repeat this procedure in the case $l+j\ge[\mu]$, then additional terms of the type

$$|s|^{k'-r}\sum_{|m|=2+r}\|D^m u\|^2_{L_{2,\mu'}(d_\vartheta)},\quad r=0,\dots,l+j-[\mu],$$

appear on the right-hand side of (6.9). Since these terms are already estimated in Step 1, (6.9) holds for every l such that $l+j\le k$. Theorem 1.2 is proved.

Let us turn to Theorem 1.1. It follows from the above results that problem (1.1) has a solution $u\in H^{k'+2}_{\mu'}(d_\vartheta)\cap W^{k'+2}_{2,\mu'}(d_\vartheta)$ satisfying the inequality

$$\begin{aligned}(6.10)\quad &\sum_{l=0}^{r}|s|^{r+2-l}\|u\|^2_{H^l_{\mu'}(d_\vartheta)}\\ &\le c_{23}\bigg[\sum_{l=0}^{r}|s|^{r-l}\|f\|^2_{H^l_{\mu'}(d_\vartheta)}\\ &\quad+\sum_{i=0}^{1}\bigg(\sum_{l=0}^{r}|s|^{r+1/2-l}\|\varphi_i\|^2_{H^l_{\mu'}(\gamma_i)}+\|\varphi_i\|^2_{H^{r+1/2}_{\mu'}(\gamma_i)}\bigg)\bigg],\quad r=0,\dots,k'.\end{aligned}$$

In addition, inequality (6.8) holds, where F stands for the sum of the norms on the right-hand side of (1.4). The concluding part of the proof is the same as in Theorem 1.2. We use Proposition 6.1 applied to problem (3.11), which yields

$$(6.11)\quad |s|^{k-l-j}\|u\|^2_{H^{l+2}_{\mu-j}(d_\vartheta)} \le c_{24}(|s|^{k-l-j}\|u\|_{H^{l+1}_{\mu-j-1}(d_\vartheta)} + |s|^{2+k-l-j}\|u\|^2_{H^l_{\mu-j}(d_\vartheta)} + F).$$

Using only these inequalities and estimate (6.8), we obtain

$$(6.12)\quad \sum_{l=0}^{1+[\mu]} |s|^{k+2-l}\|u\|^2_{H^l_\mu(d_\vartheta)} \le c_{25}\Big(\sum_{j=0}^{[\mu]}|s|^{1+k-j}I(\mu-j) + F + |s|^{1+k'}\|u\|^2_{L_{2,\mu'-1}(d_\vartheta)}\Big) \le c_{26}(F + |s|^{1+k'}\|u\|^2_{L_{2,\mu'-1}(d_\vartheta)}).$$

By (6.8) and (6.10),

$$(6.13)\quad |s|^{1+k'}\|u\|^2_{L_{2,\mu'-1}(d_\vartheta)} \le (|s|^{2+k'}\|u\|^2_{L_{2,\mu'}(d_\vartheta)})^{(k'+1)/(k'+2)}(\|u\|^2_{L_{2,\mu'-k'-2}(d_\vartheta)})^{1/(k'+2)} \le c_{27}F,$$

so

$$\sum_{l=0}^{1+[\mu]} |s|^{k+2-l}\|u\|^2_{H^l_\mu(d_\vartheta)} \le c_{28}F.$$

When we estimate $\sum_{l=2+[\mu]}^{k+2} |s|^{k+2-l}\|u\|^2_{H^l_\mu(d_\vartheta)}$ in the same manner, there appear additional terms of the type $|s|^{k'-r}\|u\|^2_{H^{2+r}_{\mu'}(d_\vartheta)}$ on the right-hand side. But they are already estimated in (6.10), hence, we arrive at (1.4) and complete the proof.

7. Applications

The results proved in Theorems 1.1 and 1.2 for problem (1.1) allow us to prove similar results for the parabolic problem (1.2). Actually, (1.1) can be obtained from (1.2) by means of the Laplace transform with respect to t. Thus, with the aid of the inverse Laplace transform we obtain the following results:

Theorem 7.1. *Let* $\mu \ge 0$, $\beta_i = \arctan h_i \in (-\pi/2, \pi/2)$, $h_0 + h_1 > 0$ *and*

$$(7.1)\qquad 0 < 1 + k - \mu < \frac{\beta_0 + \beta_1}{\vartheta}.$$

For every $f \in H^{k,k/2}_{0,\mu}(d_{\vartheta,T}) \cap W^{k,k/2}_{0,\mu}(d_{\vartheta,T})$ *and* $\varphi_i \in H^{k+1,(k+1)/2}_{0,\mu}(\gamma_{i,T}) \cap W^{k+1,(k+1)/2}_{0,\mu}(\gamma_{i,T})$, $i = 0, 1$, *problem* (1.2) *has a unique solution* $v \in H^{k+2,(k+2)/2}_{0,\mu}(d_{\vartheta,T}) \cap W^{k+2,(k+2)/2}_{0,\mu}(d_{\vartheta,T})$ *and*

$$\begin{aligned}(7.2)\quad &\sum_{l=0}^{k+2} \|v\|^2_{H^{l,l/2}_{0,\mu}(d_{\vartheta,T})} \\ &\leq c_1 \Bigg[\sum_{l=0}^{k} \|f\|^2_{H^{l,l/2}_{0,\mu}(d_{\vartheta,T})} + \sum_{i=0}^{1} \Bigg[\sum_{l=0}^{k} \|\varphi_i\|^2_{H^{l,l/2}_{0,\mu}(\gamma_{i,T})} + \|\varphi_i\|^2_{H^{k+1/2,k/2+1/4}_{0,\mu}(\gamma_{i,T})} \Bigg]\Bigg].\end{aligned}$$

Theorem 7.2. *If* $h_0 + h_1 \leq 0$ *and*

$$(7.3)\qquad 0 < 1 + k - \mu < \frac{\pi + \beta_0 + \beta_1}{\vartheta},$$

then for every $f \in W^{k,k/2}_{0,\mu}(d_{\vartheta,T})$ *and* $\varphi_i \in W^{k+1,(k+1)/2}_{0,\mu}(\gamma_{i,T})$, $i = 0, 1$, *problem* (1.2) *has a unique solution* $v \in W^{k+2,(k+2)/2}_{0,\mu}(d_{\vartheta,T})$ *and*

$$(7.4)\quad \|v\|_{W^{k+2,(k+2)/2}_{0,\mu}(d_{\vartheta,T})} \leq c_2 \Bigg[\|f\|^2_{W^{k,k/2}_{0,\mu}(d_{\vartheta,T})} + \sum_{i=0}^{1} \|\varphi_i\|^2_{W^{k+1/2,k/2+1/4}_{0,\mu}(\gamma_{i,T})} \Bigg].$$

The spaces $H^{k,k/2}_{0,\mu}(d_{\vartheta,T})$ and $W^{k,k/2}_{0,\mu}(d_{\vartheta,T})$ are the natural extensions of $H^k_\mu(d_\vartheta)$ and $W^k_\mu(d_\vartheta)$ respectively (see [3], [7], [8] for definitions).

Notice that estimates (1.4), (1.6), (7.2) and (7.4) are more general than the corresponding estimates in [3]. Actually, estimates (2.2) and (2.4) in [3] are stated only for $\mu \in (0, 1)$, while estimates (7.2) and (7.4) hold for any non-negative weight μ.

In this section we anticipate an interesting application of the above results: that is, the construction of the Green function for the heat equation in an angle with oblique boundary conditions. In [4] we prove that there exists a Green function for problem (1.2). This function is given in the following form:

$$(7.5)\qquad G(x, y, t) = \Gamma(x - y, t)\psi(x, y, t) + G'(x, y, t);$$

here $x, y \in d_\vartheta$, $t > 0$, $\Gamma(x, t) = (4\pi t)^{-1} e^{-|x|^2/(4t)}$ is the fundamental solution of the heat equation, $\psi(x, y, t)$ is an infinitely differentiable function of its arguments, equal to one for small $|x - y|$ and t, equal to zero when $|x - y|$ and/or t is large and also y is near the vertex (i.e. for $|y| = 0$).

The function $G'(x, y, t)$, for any fixed $y \in d_\vartheta$, is defined as the solution in weighted parabolic spaces of the problem

$$\begin{aligned}&\partial_t G' - \Delta_x G' = 2\nabla_x \Gamma \cdot \nabla_x \psi + \Gamma(\Delta_x \psi - \partial_t \psi), && x \in d_\vartheta,\ 0 < t \leq T,\\ &G'|_{t=0} = 0, && x \in d_\vartheta,\\ &\frac{\partial}{\partial n} G' + h_i \frac{\partial}{\partial r} G' = -\left(\frac{\partial(\Gamma\psi)}{\partial n} + h_i \frac{\partial(\Gamma\psi)}{\partial r} \right), && x \in \gamma_i,\ 0 < t \leq T,\ i = 0, 1.\end{aligned}$$

We prove that, for some $\mu \in \mathbb{R}^+$, $G'(x,\cdot,t)$ belongs (at least) either to $H^{2,1}_{0,\mu}(d_{\vartheta,T}) \cap W^{2,1}_{0,\mu}(d_{\vartheta,T})$, for $h_0+h_1>0$, or to $W^{2,1}_{0,\mu}(d_{\vartheta,T})$, for $h_0+h_1\le 0$.

This allows us to prove the fundamental property of G, i.e., that the solution of (1.2) for $\phi_i=0$ and $f\in L_{2,\mu}(d_{\vartheta,T})$ can be represented in the form

$$v(x,t)=\int_0^t d\tau\int_{d_\vartheta} G(x,y,t-\tau)f(y,\tau)\,dy.$$

The following estimates for the derivatives of $G(x,y,t)$ are proved.

For $x,y\in d_\vartheta$, $t>0$ and any $\alpha\equiv(\alpha_1,\alpha_2)$, $\beta\equiv(\beta_1,\beta_2)$ and a,

$$\begin{aligned}|D_x^\alpha D_y^\beta D_t^a G(x,y,t)| &\le \frac{c(\alpha,\beta,a,\vartheta)}{(|x-y|^2+t)^{(2+|\alpha|+|\beta|+2a)/2}}\\ &\quad\times\left(\frac{|x|}{|x|+|x-y|+t^{1/2}}\right)^{\lambda_1(|\alpha|)-|\alpha|}\left(\frac{|y|}{|y|+|x-y|+t^{1/2}}\right)^{\lambda_2(|\beta|)-|\beta|},\end{aligned}$$

where

$$\lambda_1(|\alpha|)=\begin{cases}\min(|\alpha|,(\beta_0+\beta_1)/\vartheta-\varepsilon) & \text{if } h_0+h_1>0,\\ \min(|\alpha|,(\pi+\beta_0+\beta_1)/\vartheta-\varepsilon) & \text{if } h_0+h_1\le 0,\end{cases}$$

with some $\varepsilon>0$, and similarly for $\lambda_2(|\beta|)$.

Appendix: the proof of Proposition 6.1

We split d_ϑ into the domains $\omega_q\equiv\{r_q\le|x|<2r_q\}$, $r_q=2^q$, $q=0,\pm1,\pm2,\ldots$, and observe that the solution of problem (2.1) satisfies the estimate

$$\text{(A.1)}\qquad \begin{aligned}\sum_{|m|=l+2}\|D^m u\|^2_{L_2(\omega_q)} &\equiv \|D^{l+2}u\|^2_{L_2(\omega_q)}\\ &\le c_1\Big(r_q^{-2}\|D^{l+1}u\|^2_{L_2(\Omega_q)}+\|D^l f\|^2_{L_2(\Omega_q)}+\sum_{i=0}^1\|D^l\Phi_i\|^2_{L^{1/2}(\Sigma_{i,q})}\Big)\end{aligned}$$

where $\Omega_q=\omega_{q-1}\cup\omega_q\cup\omega_{q+1}$, $\Sigma_{i,q}=\{x\in\gamma_i : r_q/2\le|x|<4r_q\}$ and

$$\|\Phi\|^2_{L^{1/2}(\Sigma_{i,q})}=\int_{\Sigma_{i,q}}\int_{\Sigma_{i,q}}|\Phi(\varrho)-\Phi(r)|^2\,\frac{d\varrho\,dr}{|\varrho-r|^2}.$$

After multiplication of (A.1) by $r_q^{2\mu}$ and summation with respect to q from $-\infty$ to ∞ we obtain (6.5).

To prove (A.1), we consider the equation

$$\text{(A.2)}\qquad \Delta u=f(z),\quad z\in B_{2r}\equiv\{|z|<2r\},$$

and the boundary value problem

$$\text{(A.3)}\qquad \begin{aligned} &\Delta u = f(z), \quad z \in B_{2r}^+ \equiv \{|z| < 2r,\ z_2 > 0\},\\ &\left(\frac{\partial u}{\partial z_2} - h\frac{\partial u}{\partial z_1}\right)\Big|_{z_2=0} = \Phi(z_1). \end{aligned}$$

We write (A.2) in the form

$$\Delta v = f(z), \quad v = u - \frac{1}{|B_{2r}|}\int_{B_{2r}} u(z)\,dz,$$

and use the well known estimate

$$\|D^2 v\|^2_{L_2(B_r)} \le c_2(\|f\|^2_{L_2(B_{2r})} + r^{-2}\|\nabla v\|^2_{L_2(B_{2r})} + r^{-4}\|v\|^2_{L_2(B_{2r})}),$$

which implies, by the Poincaré inequality,

$$\|D^2 u\|^2_{L_2(B_r)} \le c_3(\|f\|^2_{L_2(B_{2r})} + r^{-2}\|\nabla u\|^2_{L_2(B_{2r})}).$$

Applying this inequality to the lth derivatives of u we obtain

$$\text{(A.4)}\qquad \|D^{l+2}u\|^2_{L_2(B_r)} \le c_3(\|D^l f\|^2_{L_2(B_{2r})} + r^{-2}\|D^{l+1}u\|^2_{L(B_{2r})}).$$

Similar estimates hold for the solution of (A.3). We have (see, for instance, [1])

$$\|D^2 u\|^2_{L_2(B_r^+)} \le c_4(r^{-2}\|\nabla u\|^2_{L_2(B_{2r}^+)} + \|f\|^2_{L_2(B_{2r}^+)} + \|\Phi\|^2_{L^{1/2}(K_{2r})}),$$

where $K_r = \{|z_1| < r\}$. We may also differentiate (A.3) with respect to z_1 and obtain the same kind of estimate for $D^l_{z_1}u$. The derivative $w = \partial^l u/\partial z_1^{l-1}\partial z_2$ satisfies the relations

$$\Delta w = \frac{\partial^l f}{\partial z_1^{l-1}\partial z_2}, \quad \frac{\partial w}{\partial z_2}\Big|_{z_2=0} = \left(\frac{\partial^{l-1} f}{\partial z_1^{l-1}} - \frac{\partial^{l+1} u}{\partial z_1^{l+1}}\right)\Big|_{z_2=0},$$

hence,

$$\begin{aligned} \|D^2 w\|^2_{L_2(B_r^+)} &\le c_5\bigg(r^{-2}\|\nabla w\|^2_{L_2(B_{3r/2}^+)} + \bigg\|\frac{\partial^l f}{\partial z_1^{l-1}\partial z_2}\bigg\|^2_{L_2(B_{3r/2}^+)} \\ &\qquad + \bigg\|\frac{\partial^{l-1} f}{\partial z_1^{l-1}}\bigg\|^2_{L^{1/2}(K_{3r/2})} + + \bigg\|\frac{\partial^{l+1} u}{\partial z_1^{l+1}}\bigg\|^2_{L_2(K_{3r/2})}\bigg) \\ &\le c_6\bigg(r^{-2}\|\nabla w\|^2_{L_2(B_{3r/2}^+)} + \|D^l f\|^2_{L_2(B_{3r/2}^+)} + \bigg\|D^2\frac{\partial^l u}{\partial z_1^l}\bigg\|^2_{L_2(B_{3r/2}^+)}\bigg). \end{aligned}$$

The last term has been just estimated, so we obtain an estimate for $\partial^l u/\partial z_1^{l-1}\partial z_2$. Other lth derivatives of u are estimated in a similar way, and as a result we get the inequality

$$\begin{aligned} \text{(A.5)}\quad &\|D^{l+2}u\|^2_{L_2(B_r^+)} \\ &\qquad \le c_7(r^{-2}\|D^{l+1}u\|^2_{L_2(B_{2r}^+)} + \|D^l f\|^2_{L_2(B_{2r}^+)} + \|D^l\Phi\|^2_{L^{1/2}(K_{2r})}). \end{aligned}$$

The domain ω_q can be covered by a finite number of balls with radii of order r_q located in the interior of ω_q and half-balls adjacent to γ_0 or to γ_1, and estimate (6.5) can be obtained by summing (A.4) and (A.5) in these domains. Hence, the proposition is proved.

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