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NONTRIVIAL SOLUTIONS FOR ASYMPTOTICALLY LINEAR VARIATIONAL INEQUALITIES

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Dedicated to Louis Nirenberg on the occasion of his 70th birthday

1. Introduction

The aim of this paper is to extend to variational inequalities a result of Amann and Zehnder's, refined by Chang (see [1, 6, 14]), concerning the existence of nontrivial solutions for a semilinear elliptic boundary value problem, when the nonlinearity is asymptotically linear, is zero at zero and its derivative has a suitable jump between zero and infinity. In the constrained problem studied here (see Theorem 4.1) the presence of the "obstacle" enters into the discussion and what determines the required "jump of behaviour" is a combination of the nonlinearity and the obstacle.

Following the ideas of [1] we use the Conley index and show that the index of zero as invariant set in the associated parabolic flow is different from the index of the maximal invariant set in a suitable large ball. Then there exists an invariant set larger than $\{0\}$, so, by the variational nature of the flow, there exists a second invariant point. For computing these indices it is natural to use a continuation argument, passing to some "limit flows". This requires proving an index continuation result for flows with moving domains, which is done in Section 2, generalizing the work of Rybakowski [13, 15].

In Section 3 the result described above is proved in a general abstract setting. We use a nonsmooth variational approach which consists in viewing solutions as

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lower critical points of a functional of the type $f(u) = a(u,u) + B(u) + I_K$, where a is a coercive quadratic form, B is asymptotically quadratic and I_K is the indicator function of a closed convex set. By means of the theory of evolution equations for maximal monotone operators (in particular for the subdifferential of lower semicontinuous functions, see [5, 10]) we can construct the flow of the steepest descent curves associated with f and regard the critical points as rest points for the flow. Thus we can apply the results of Section 1 for finding nontrivial critical points via the Conley index.

In Section 4 the application described at the beginning is treated; we finally point out that, although we just consider the Laplace operator, everything can be repeated for a general strictly elliptic operator, without significant changes.

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2. A continuation theorem for flows with variable domains

In all what follows (X, d) and (Σ, d') will be two fixed metric spaces. We recall the concept of flow (see e.g. [8, 13]).

Definition 2.1. Let $\omega : X \to]0, \infty]$ be a lower semicontinuous function and set $D_\omega = \{(x,t) \in X \times [0, \infty[\mid t < \omega(x)\}$. Let $\Phi : D_\omega \to X$ be a continuous map with the properties:

(a) $\Phi(x, 0) = x$ for $x \in X$;
(b) if $x \in X$, $t < \omega(x)$, $s < \omega(\Phi(x,t))$, then $t+s < \omega(x)$ and $\Phi(\Phi(x,t), s) = \Phi(x, t+s)$.

In this situation we say that (X, ω, Φ) is a *local unilateral flow* (briefly a *flow*) on X. If $\omega \equiv \infty$, we omit ω and write (X, Φ).

Definition 2.2. Suppose that for all σ in Σ we are given a flow $(X_\sigma, \omega_\sigma, \Phi_\sigma)$ on a subset X_σ of X, where X_σ is endowed with the metric d inherited from X. We say that $((X_\sigma, \omega_\sigma, \Phi_\sigma))_{\sigma \in \Sigma}$ is a *continuous family of local unilateral flows* (briefly of flows) in X if for all $(\sigma_n)_n, \sigma_0$ in Σ such that $\sigma_n \to \sigma_0$, and for all $(x_n)_n, x_0$ in X such that $x_n \in X_{\sigma_n}$ for all n and $x_n \to x_0$ one has $x_0 \in X_{\sigma_0}$, and furthermore for all $(t_n)_n, t_0$ in $[0, \infty[$ such that $t_n \to t_0$ and $t_0 < \omega_{\sigma_0}(x_0)$ one has eventually $t_n < \omega_{\sigma_n}(x_n)$ and $\Phi_{\sigma_n}(x_n, t_n) \to \Phi_{\sigma_0}(x_0, t_0)$.

If $\omega_\sigma \equiv \infty$ for all σ's, then we omit ω_σ and write $((X_\sigma, \Psi_\sigma))_{\sigma \in \Sigma}$.

If $X_\sigma = X$ independently of σ Definition 2.2 is the usual one, given in [8, 13].

Remark 2.3. Let $((X_\sigma, \omega_\sigma, \Phi_\sigma))_{\sigma \in \Sigma}$ be as above, let U be an open subset of X and, for σ in Σ and x in $X_\sigma \cap U$, set

$$\omega_\sigma^{(U)} = \sup\{t < \omega_\sigma(x) \mid \Phi_\sigma(x,t') \in U \ \forall t' \in [0,t]\},$$
$$\Phi_\sigma^{(U)}(x,t) = \Phi(x,t) \quad \text{if } t < \omega_\sigma^{(U)}(x).$$

It can be easily checked that $\{(X_\sigma \cap U, \omega_\sigma^{(U)}, \Phi_\sigma^{(U)})\}_{\sigma \in \Sigma}$ is a continuous family of flows in X.

We now introduce some notations which are customary in the context of the Conley index theory (see [13]). For σ in Σ and a subset N of X we denote by $S_\sigma(N)$ the maximal invariant set in the flow $(X_\sigma, \omega_\sigma, \Phi_\sigma)$ which is contained in $N \cap X_\sigma$. We say that N is a *σ-isolating neighbourhood* (briefly N is *σ-isolating*) if $S_\sigma(N) \subset \operatorname{int}(N \cap X_\sigma)$ in the relative X_σ topology (since $S_\sigma(N) \subset X_\sigma$ this corresponds to requiring that $S_\sigma \subset \operatorname{int}(N)$); if this is the case we say that $S = S_\sigma(N)$ is a *σ-isolated invariant set* (briefly S is *σ-isolated*). For a σ-invariant set S in the flow $(X_\sigma, \omega_\sigma, \Phi_\sigma)$ we can consider the Conley index of S, which we denote by $\mathcal{I}_\sigma(S)$ $(= \mathcal{I}_\sigma(S_\sigma(N))$ for some σ-isolating neighbourhood N).

In the remainder of this section, σ_0 in Σ and a subset N of X will be fixed and satisfy the following assumptions (which are taken from [13]):

- (N.1) N is σ_0-isolating and closed.
- (N.2) There exists a neighbourhood W of σ_0 in Σ such that for all $(\sigma_n)_n, \sigma$ in W with $\sigma_n \to \sigma$, for all $(x_n)_n$ in N with $x_n \in X_{\sigma_n}$ for each n, and for all $(t_n)_n$ in $[0,\infty[$ with

$$t_n < \omega_{\sigma_n}(x_n), \quad \Phi_{\sigma_n}(x_n,[0,t_n]) \subset N \quad \forall n, \quad t_n \to \infty,$$

there exist $(x_{n_k})_k$ and x in X_σ such that $\Phi_{\sigma_{n_k}}(x_{n_k}, t_{n_k}) \to x$.
- (N.3) There exists a neighbourhood W of σ_0 (which we can suppose to be the same as in (N.2)) such that for all σ in W,

$$\forall x \in N \cap X_\sigma \quad \omega_\sigma(x) < \infty \Rightarrow \exists t < \omega_\sigma(x) \text{ such that } \Phi_\sigma(x,t) \notin N.$$

The following properties can be easily proved.

Proposition 2.4. *If* (N.1) *and* (N.2) *hold, then*

- (a) *$S_\sigma(N)$ is compact for all σ in W;*
- (b) *there exists a neighbourhood W' of σ_0 such that N is σ-isolating for all σ in W'.*

To prove a continuation theorem for the index an additional assumption is necessary, which is a sort of continuity of the sets X_σ, at least in a neighbourhood of $S_{\sigma_0}(N)$. We assume the following condition.

(C) There exist an open neighbourhood U of $S_{\sigma_0}(N)$ in X, a neighbourhood W of σ_0 (that we can suppose to be the same as in (N.1) and (N.2)) and a continuous map $\Psi : W \times U \times [0,1] \to X$ such that:
   (a) $\Psi(\sigma, x, 0) = x$ for $\sigma \in W$ and $x \in U$;
   (b) $\Psi(\sigma, \Psi(\sigma, x, t), s) = \Psi(\sigma, x, t + s)$ for $\sigma \in W$, $x \in U$ and $t, s \in [0,1]$ such that $\Psi(\sigma, x, t) \in U$ and $t + s \leq 1$;
   (c) $\Psi(\sigma, x, t) = x$ for $\sigma \in W$, $x \in U \cap X_\sigma$ and $t \in [0,1]$;
   (d) $\Psi(\sigma, x, 1) \in X_\sigma$ for $\sigma \in W$ and $x \in U$.

Remark 2.5. If assumption (C) holds, then the following property is true: for all $(\sigma_n)_n, \sigma$ in W with $\sigma_n \to \sigma$, for all $(x_n)_n, x$ in U with $x_n \to x$, and for all $(t_n)_n, t$ in $[0,1]$ with $t_n \to t$, if $\Psi(\sigma_n, x_n, t_n) \in X_{\sigma_n}$, then $\Psi(\sigma, x, t) \in X_\sigma$ (in particular, taking $t_n = t = 0$ and $\sigma_n = \sigma$ fixed, one finds that $X_\sigma \cap U$ is closed in U, for all σ in W).

Proof. It suffices to note that the properties of Ψ imply

$$\forall \sigma \in W,\ \forall x \in U,\ \forall t \in [0,1] \quad \Psi(\sigma, x, t) \in X_\sigma \Leftrightarrow \Psi(\sigma, x, t) = \Psi(\sigma, x, 1)$$

and exploit the continuity of Ψ. $\square$

Now we are ready to prove the main theorem of this section.

Theorem 2.6. *Let σ_0 and N satisfy* (N.1)–(N.3) *and* (C). *Then there exists a neighbourhood W_0 of σ_0 such that*

$$\mathcal{I}_\sigma(S_\sigma(N)) = \mathcal{I}_{\sigma_0}(S_{\sigma_0}(N)) \quad \forall \sigma \in W_0.$$

Proof. If $S_{\sigma_0}(N) = \emptyset$, it is easy to prove that $S_\sigma = \emptyset$ for σ in a neighbourhood of σ_0, so the theorem is true in this case. Assume $S_{\sigma_0}(N) \neq \emptyset$ and let U, W, Ψ be as in (C). For σ in W and x in U set

$$t_1(\sigma, x) = \sup\{t \in [0,1] \mid \Psi(\sigma, x, t') \in U \ \forall t' \in [0,t]\},$$
$$t_2(\sigma, x) = \min\{t \in [0,1] \mid \Psi(\sigma, x, t) \in X_\sigma\}$$

(the fact that $t_2(\sigma, x)$ is a minimum is a consequence of Remark 2.5). We claim that t_1 is lower semicontinuous and t_2 is continuous in $W \times U$. The lower semicontinuity of t_1 and t_2 follows easily from the properties of Ψ and the fact that U and $U \setminus X_\sigma$ are open. The upper semicontinuity of t_2 is an immediate consequence of Remark 2.5.

Now we can set, for σ in W and x in U,

$$\widetilde{\omega}_\sigma(x) = \begin{cases} t_1(\sigma, x) & \text{if } \Psi(\sigma, x, t_1(\sigma, x)) \notin U, \\ t_2(\sigma, x) + \omega_\sigma(\Psi(\sigma, x, 1)) & \text{otherwise.} \end{cases}$$

Note that in the second case $t_1(\sigma, x) = 1$, since U is open; also note that $\widetilde{\omega}_\sigma(x) > 0$ for all σ in W and x in U. We claim that the map $(\sigma, x) \mapsto \widetilde{\omega}_\sigma(x)$ is lower semicontinuous. To see this we take σ in W, x in U and $t < \widetilde{\omega}_\sigma(x)$; we distinguish two cases:

1. $\widetilde{\omega}_\sigma(x) = t_1(\sigma, x)$: then by definition $t_2(\sigma, x) \geq t_1(\sigma, x)$, which implies that for (σ', x') in a neighbourhood of (σ, x) we have both $t_1(\sigma', x') > t$ and $t_2(\sigma', x') > t$ and therefore $\widetilde{\omega}_{\sigma'}(x') > t$;
2. $\widetilde{\omega}_\sigma(x) = t_2(\sigma, x) + \omega_\sigma(\Psi(\sigma, x, 1))$: then $\Psi(\sigma, x, [0,1]) \subset U$ and since Ψ is continuous, taking (σ', x') in a neighbourhood of (σ, x) we have $\Psi(\sigma', x', [0,1]) \subset U$ and $\widetilde{\omega}_{\sigma'}(x') = t_2(\sigma', x') + \omega_{\sigma'}(\Psi(\sigma', x', 1))$; then the semicontinuity follows from the semicontinuity of $(\sigma, x) \mapsto \omega_\sigma(x)$ and the continuity of t_2 and Ψ.

Finally, for σ in W we define $\widetilde{D}_\sigma = \{(x,t) \in U \times [0, \infty[\mid t < \widetilde{\omega}_\sigma(x)\}$ and $\widetilde{\Phi}_\sigma : \widetilde{D}_\sigma \to U$ by

$$\widetilde{\Phi}_\sigma(x,t) = \begin{cases} \Psi(\sigma, x, t) & \text{if } t < t_2(\sigma, x), \\ \Phi_\sigma(\Psi(\sigma, x, 1), t - t_2(\sigma, x)) & \text{otherwise.} \end{cases}$$

It is easy to check that $((U, \widetilde{\omega}_\sigma, \widetilde{\Phi}_\sigma))_{\sigma \in W}$ is a continuous family of flows having the same fixed domain U. It is also immediate that a subset S of U is invariant with respect to $\widetilde{\Phi}_\sigma$ if and only if $S \subset X_\sigma$ and S is invariant with respect to Φ_σ.

Now we take a closed set $\widetilde{N}$ such that $\widetilde{N} \subset U$ and $S_{\sigma_0}(N) = S_{\sigma_0}(\widetilde{N})$ (this can be done using the compactness of $S_{\sigma_0}(N)$ and the fact that U is a neighbourhood of $S_{\sigma_0}(N)$). We can suppose, possibly reducing W, that $\widetilde{N}$ is σ-isolating and that $S_\sigma(\widetilde{N}) = S_\sigma(N)$ for all σ in W (use (N.2)).

It is also straightforward that σ_0 and $\widetilde{N}$ satisfy (N.1)–(N.3) in the family of flows $((U, \widetilde{\omega}_\sigma, \widetilde{\Phi}_\sigma))_{\sigma \in W}$: then we can apply the results of [13] concerning the continuation of the Conley index to obtain

$$\widetilde{\mathcal{I}}_\sigma(\widetilde{S}_\sigma(\widetilde{N})) = \widetilde{\mathcal{I}}_{\sigma_0}(\widetilde{S}_{\sigma_0}(\widetilde{N})) \quad \forall \sigma \in W$$

(possibly reducing W), where $\widetilde{\mathcal{I}}_\sigma$ denotes the Conley index with respect to the flow $(U, \widetilde{\omega}_\sigma, \widetilde{\Phi}_\sigma)$. To conclude the proof we just need to show that

$$\widetilde{\mathcal{I}}_\sigma(\widetilde{S}_\sigma(\widetilde{N})) = \mathcal{I}_\sigma(S_\sigma(N)) \quad \forall \sigma \in W.$$

To this end let $\sigma \in W$ be given and let (N_1, N_2) be an index pair in the isolating neighbourhood $\widetilde{N} \cap X_\sigma$, relative to the flow $(X_\sigma, \omega_\sigma, \Phi_\sigma)$ (see [13]). Set

$$\widetilde{N}_1 = \{x \in \widetilde{N} \mid \Psi(\sigma, x, t) \in \widetilde{N} \ \forall t \in [0,1],\ \Psi(\sigma, x, 1) \in N_1\},$$
$$\widetilde{N}_2 = \{x \in \widetilde{N} \mid \Psi(\sigma, x, t) \in \widetilde{N} \ \forall t \in [0,1],\ \Psi(\sigma, x, 1) \in N_2\}$$

(one could as well take $\widetilde{N}_2 = N_2$). We prove that $(\widetilde{N}_1, \widetilde{N}_2)$ is an index pair in $\widetilde{N}$, relative to the flow $(U, \widetilde{\omega}_\sigma, \widetilde{\Phi}_\sigma)$. It is clear that $\widetilde{N}_1$ and $\widetilde{N}_2$ are closed and positively invariant in $\widetilde{N}$. It is also evident that the exit set for $\widetilde{N}_1$ is contained in $\widetilde{N}_2$. What remains to be proved is $\widetilde{S}_\sigma(\widetilde{N}) \subset \operatorname{int}(\widetilde{N}_1)$. If this were false we could find a sequence $(x_n)_n$ in U such that $x_n \notin \widetilde{N}_1$ for all n and converging to some x in $S_\sigma(\widetilde{N})$. On the other hand, for n large, $\Psi(\sigma, x_n, [0,1]) \subset \widetilde{N}$ (since $S_\sigma(\widetilde{N}) \subset \operatorname{int}(\widetilde{N})$) and therefore $\Psi(\sigma, x_n, 1) \notin N_1$. Passing to the limit we obtain $x \in S_\sigma(\widetilde{N}) \setminus \operatorname{int}(N_1)$, which contradicts the properties of N_1. So $(\widetilde{N}_1, \widetilde{N}_2)$ is an index pair in $\widetilde{N}$. Now taking the map $H : (\widetilde{N}_1, \widetilde{N}_2) \times [0,1] \to (\widetilde{N}_1, \widetilde{N}_2)$ defined by $H(x,t) = \Psi(\sigma, x, t)$, we conclude that the pair $(\widetilde{N}_1, \widetilde{N}_2)$ has the same homotopy type as (N_1, N_2). This implies that $\widetilde{N}_1/\widetilde{N}_2$ has the homotopy type of N_1/N_2, and so the indexes are the same. Thus the theorem is proved. □

3. Nontrivial solutions for asymptotically linear variational inequalities

Let H, L be two Hilbert spaces such that $H \subset L$ and the embedding $i : H \to L$ is compact. Let $a : H \times H \to \mathbb{R}$ be a symmetric bilinear quadratic form such that there exist two constants $C \geq 0$ and $\nu > 0$ with the properties:

$$a(u,v) \leq C\|u\|_H\|v\|_H \quad \forall u, v \in H, \tag{1}$$

$$a(u,u) \geq \nu\|u\|_H^2 \qquad \forall u \in H. \tag{2}$$

Furthermore, let $B : L \to \mathbb{R}$ be a differentiable function such that

$$\|B'(u) - B'(v)\|_L \leq M\|u - v\|_L \quad \forall u, v \in L, \quad B'(0) = 0 = B(0),$$

for a suitable constant M (the condition $B(0) = 0$ is not very relevant, one can always subtract $B(0)$ without affecting anything of what follows). Finally, we consider a convex set K, closed in H, such that $0 \in K$.

We are interested in finding nontrivial solutions of the variational inequality

$$\begin{cases} a(u, v-u) + \langle B'(u), v-u\rangle_L \geq 0 \quad \forall v \in K, \\ u \in K. \end{cases} \tag{3}$$

The main result is the following.

Theorem 3.1. *Let a, B, K be as above and suppose that there exist two linear symmetric operators on L, which we denote by $B''(0), B''(\infty) : L \to L$, such that*

$$\lim_{\|u\|_L \to 0} \frac{\|B'(u) - B''(0)u\|_L}{\|u\|_L} = 0, \qquad \lim_{\|u\|_L \to \infty} \frac{\|B'(u) - B''(\infty)u\|_L}{\|u\|_L} = 0 \tag{4}$$

(then $\|B''(0)\|_{L,L} \leq M$ and $\|B''(\infty)\|_{L,L} \leq M$) and define

$$b_0(u,v) = \langle B''(0)u, v\rangle_L, \quad b_\infty(u,v) = \langle B''(\infty)u, v\rangle_L.$$

Moreover, set

$$K_0 = H\text{-}closure\ of \bigcup_{\sigma > 0}\{u \mid \sigma u \in K\}, \quad K_\infty = \bigcap_{\sigma > 0}\{u \mid \sigma u \in K\}, \tag{5}$$

and denote by $(\lambda_n^{(0)})_n, (\lambda_n^{(\infty)})_n$ *the eigenvalues of the forms* $a+b_0$, $a+b_\infty$ *respectively, setting for convenience* $\lambda_0^{(0)} = \lambda_0^{(\infty)} = -\infty$. *Assume that there exist two distinct integers* i_0, i_∞ *and two linear spaces* H_0, H_∞ *such that* $\dim(H_0) = i_0$, $\dim(H_\infty) = i_\infty$, $\lambda_{i_0}^{(0)} < 0 < \lambda_{i_0+1}^{(0)}$, $\lambda_{i_\infty}^{(\infty)} < 0 < \lambda_{i_\infty+1}^{(\infty)}$ *and*

$$H_0 \subset K_0, \qquad \sup_{u\in H_0, \|u\|_H=1} \{a(u,u) + b_0(u,u)\} < 0, \tag{6}$$

$$H_\infty \subset K_\infty, \qquad \sup_{u\in H_\infty, \|u\|_H=1} \{a(u,u) + b_\infty(u,u)\} < 0. \tag{7}$$

Then there exists a solution u *of* (3) *such that* $u \neq 0$.

To prove Theorem 3.1 we introduce some additional notations: for $\sigma \in \,]0,\infty[$ we set

$$B_\sigma(u) = \frac{1}{\sigma^2} B(\sigma u), \quad B_0(u) = \frac{1}{2} b_0(u,u), \quad B_\infty(u) = \frac{1}{2} b_\infty(u,u), \tag{8}$$

$$K_\sigma = \{u \in H \mid \sigma u \in K\}. \tag{9}$$

We also define the functionals $f_\sigma : L \to \mathbb{R} \cup \{\infty\}$ by

$$f_\sigma(u) = \begin{cases} \frac{1}{2} a(u,u) + B_\sigma(u) & \text{if } u \in K_\sigma, \\ \infty & \text{if } u \in L \setminus K_\sigma. \end{cases} \tag{10}$$

Now we have some lemmas.

Lemma 3.2. *The following facts are true:*

(a) *For all* σ_1, σ_2 *in* $[0,\infty]$ *with* $\sigma_1 \le \sigma_2$, $K_{\sigma_2} \subset K_{\sigma_1}$.
(b) *For every* σ_0 *in* $]0,\infty]$, $K_{\sigma_0} = \bigcap_{\sigma<\sigma_0} K_\sigma$.
(c) *For every* σ_0 *in* $[0,\infty[$, $K_{\sigma_0} = H$*-closure of* $\bigcup_{\sigma>\sigma_0} K_\sigma$.
(d) *If* $\sigma_n \to \sigma_0$ *in* $[0,\infty]$, $u_n \overset{H}{\rightharpoonup} u_0$ *and* $u_n \in K_{\sigma_n}$ *for all* n, *then* $u_0 \in K_{\sigma_0}$.
(e) *If* $\sigma_n \to \sigma_0$ *in* $[0,\infty]$ *and* $u_0 \in K_{\sigma_0}$, *then there exists* $(u_n)_n$ *such that* $u_n \in K_{\sigma_n}$ *for all* n *and* $u_n \overset{H}{\to} u_0$.
(f) *Denote by* $P_\sigma : H \to K_\sigma$ *the projection onto* K_σ; *then the map* $(\sigma,u) \mapsto P_\sigma(u)$ *is continuous on* $[0,\infty] \times H$.

Proof. (a) is trivial. We prove (b): for all $\sigma < \sigma_0$ we have $K_{\sigma_0} \subset K_\sigma$ so $K_{\sigma_0} \subset \bigcap_{\sigma<\sigma_0} K_\sigma$; conversely, if $u \in \bigcap_{\sigma<\sigma_0} K_\sigma$, then $\sigma u \in K$ for all $\sigma < \sigma_0$ and therefore $\sigma_0 u \in K$, that is, $u \in K_{\sigma_0}$, since K is closed (this for the case $\sigma_0 < \infty$; otherwise the conclusion follows from the definition of K_∞).

We prove (c): for all $\sigma > \sigma_0$, $K_\sigma \subset K_{\sigma_0}$ so (H-closure of $\bigcup_{\sigma>\sigma_0} K_\sigma$) $\subset K_{\sigma_0}$; conversely, if $u \in K_{\sigma_0}$, then for all $\sigma > \sigma_0$, $(\sigma_0/\sigma)u \in K_\sigma$ and therefore $u \in H$-closure of $\bigcup_{\sigma>\sigma_0} K_\sigma$ (this for the case $\sigma_0 > 0$; otherwise the conclusion follows from the definition of K_0).

To prove (d) and (e) it is sufficient to treat the two cases:

(i) $\sigma_n < \sigma_0$ for all n,
(ii) $\sigma_n > \sigma_0$ for all n.

We prove (d) in case (i): if $\sigma < \sigma_0$, then eventually $\sigma_n > \sigma$ so $u_n \in K_\sigma$ and therefore $u \in K_\sigma$; this implies $u \in \bigcap_{\sigma<\sigma_0} K_\sigma = K_{\sigma_0}$. In case (ii), (d) is obvious, since $K_{\sigma_n} \subset K_{\sigma_0}$.

To prove (e) in case (i) it suffices to take $u_n = u_0$ for all n, since $K_{\sigma_0} \subset K_{\sigma_n}$. We prove (e) in case (ii): first we take a sequence u'_k such that $u'_k \xrightarrow{H} u_0$ and $u'_k \in \sigma'_k$ for some $\sigma'_k > \sigma_0$ (we are using (c)). Now we can choose an increasing sequence $(n_k)_k$ such that $\sigma_n < \sigma'_k$ for all $n \geq n_k$. Then we can define u_n by $u_n = u'_k$ for $n_k \leq n < n_{k+1}$; it is easy to check that $u_n \xrightarrow{H} u_0$ and $u_n \in K_{\sigma_n}$ for all n.

We prove (f). Let $\sigma_n \to \sigma_0$ in $[0,\infty]$; we first show that $P_{\sigma_n}(u) \xrightarrow{H} P_{\sigma_0}(u)$ for all u in H. Since $\|P_{\sigma_n}(u)\|_H \leq \|u\|_H$ $(0 \in K_{\sigma_n})$, we can find $(n_k)_k$ and w in H such that $P_{\sigma_{n_k}} \xrightarrow{H} w$; by (d), $w \in K_{\sigma_0}$. Now using (e) we can find $(v_k)_k$ such that $v_k \in K_{\sigma_{n_k}}$ for all k and $v_k \xrightarrow{H} P_{\sigma_0}(u)$. Then

$$\begin{aligned}\|u-w\|_H &\leq \liminf_{k\to\infty} \|u - P_{\sigma_{n_k}}(u)\|_H \leq \limsup_{k\to\infty} \|u - P_{\sigma_{n_k}}(u)\|_H \\ &\leq \lim_{k\to\infty} \|u - v_k\|_H = \|u - P_{\sigma_0}(u)\|_H.\end{aligned}$$

Hence $w = P_{\sigma_0}(u)$ and $\|u - P_{\sigma_{n_k}}(u)\|_H \to \|u - P_{\sigma_0}(u)\|_H$. This implies that $P_{\sigma_{n_k}}(u) \xrightarrow{H} P_{\sigma_0}(u)$. Since the previous argument can be repeated for every subsequence of $(\sigma_n)_n$, we have $P_{\sigma_n}(u) \xrightarrow{H} P_{\sigma_0}(u)$. Finally, if $u_n \to u_0$, we have

$$\begin{aligned}\|P_{\sigma_n}(u_n) - P_{\sigma_0}(u_0)\|_H &\leq \|P_{\sigma_n}(u_n) - P_{\sigma_n}(u_0)\|_H + \|P_{\sigma_n}(u_0) - P_{\sigma_0}(u_0)\|_H \\ &\leq \|u_n - u_0\|_H + \|P_{\sigma_n}(u_0) - P_{\sigma_0}(u_0)\|_H \to 0,\end{aligned}$$

which gives the conclusion.

LEMMA 3.3. *Let $\sigma_n \to \sigma_0$ in $[0,\infty]$ and $u_n \xrightarrow{L} u_0$. Then $B'_{\sigma_n}(u_n) \xrightarrow{L} B'_{\sigma_0}(u_0)$ and $B_{\sigma_n}(u_n) \to B_{\sigma_0}(u_0)$.*

PROOF. The assertions are clear when $\sigma_0 \in]0,\infty[$. We carry out the proof in the case $\sigma_0 = 0 < \sigma_n$ for all n. We have

$$\begin{aligned}\|B'_{\sigma_n}(u_n) - B''_0(0)(u_0)\|_L &= \left\| \frac{1}{\sigma_n} B'(\sigma_n u_n) - B''(0)(u_0) \right\|_L \\ &\leq \frac{1}{\sigma_n} \|B'(\sigma_n u_n) - B'(\sigma_n u_0)\|_L \\ &\quad + \frac{1}{\sigma_n} \|B'(\sigma_n u_0) - B''(0)(\sigma_n u_0)\|_L \\ &\leq M\|u_n - u_0\|_L + \|u_0\|_L \frac{\|B'(\sigma_n u_0) - B''(0)(\sigma_n u_0)\|_L}{\|\sigma_n u_0\|_L} \\ &\to 0.\end{aligned}$$

The proof of the case $\sigma_0 = \infty$ is similar. For the second assertion just notice that

$$B_{\sigma_n}(u_n) = \int_0^1 \langle B'_{\sigma_n}(tu_n), u_n\rangle_L \, dt$$

and $\langle B'_{\sigma_n}(tu_n), u_n\rangle_L \to \langle B'_{\sigma_0}(tu_0), u_0\rangle_L$, $|\langle B'_{\sigma_n}(tu_n), u_n\rangle_L| \le M\|u_n\|_L^2$ for all t in $[0,1]$.

Lemma 3.4. *The following assertions are true:*

(a) $\mathcal{D}(f_\sigma) = K_\sigma$ *for all* σ *in* $[0,\infty]$.

(b) *If* $\sigma \in [0,\infty], u \in K_\sigma$ *and* $\alpha \in L$, *then* (*using the* L *norm*)

$$\alpha \in \partial^- f_\sigma(u) \Leftrightarrow a(u, v-u) + \langle B'_\sigma(u), v-u\rangle_L \ge \langle \alpha, v-u\rangle_L \quad \forall v \in K_\sigma.$$

(c) *For all* u, v *in* K_σ *and* α *in* $\partial^- f(u)$,

$$f_\sigma(v) \ge f_\sigma(u) + \langle \alpha, v-u\rangle_L - M\|v-u\|_L^2.$$

(d) *For all* σ *in* $[0,\infty]$ *and for all* u_0 *in* K_σ *there exists an absolutely continuous curve* $\mathcal{U} : [0,\infty[\, \to L$ *such that* $\mathcal{U}(0) = u_0$ *and*

$$\begin{cases} \mathcal{U}(t) \in K_\sigma & \forall t \ge 0, \\ a(\mathcal{U}(t), v - \mathcal{U}(t)) & \\ \quad + \langle B'_\sigma(\mathcal{U}(t)) + \mathcal{U}'(t), v - \mathcal{U}(t)\rangle_L \ge 0 & \forall v \in K_\sigma, \text{a.e. } t \ge 0, \\ f_\sigma(\mathcal{U}(t_1)) - f_\sigma(\mathcal{U}(t_2)) = \int_{t_1}^{t_2} \|\mathcal{U}'(t)\|_L^2 \, dt & \forall t_1, t_2 \ge 0, t_1 \le t_2. \end{cases} \tag{11}$$

Furthermore, if we set $\Phi_\sigma(u,t) = \mathcal{U}(t)$, *then* $(K_\sigma, \Phi_\sigma)_{\sigma\in[0,\infty]}$ *is a continuous family of semiflows, according to Definition* 2.2, *with respect to the* H *norm* (*that is,* $X = H$ *in Definition* 2.2).

Proof. (a) is trivial. The proofs of (c) and (d) are formally identical to the proofs of Section 3 in [7]. The first part of (d) corresponds to the existence of the solution for the evolution problem associated with f_σ, which follows in a standard way from (c) (see e.g. [10, 12]). To prove the continuous dependence on σ, u, t we show that, if $\sigma_n \to \sigma_0$ in $[0,\infty]$, then

$$f_{\sigma_0} = \Gamma^-(L) \lim_{n\to\infty} f_{\sigma_n}$$

(for the notion of Γ convergence we refer to [2, 9]). For this let $u_n \xrightarrow{L} u_0$ and $\sup_n f_{\sigma_n}(u_n) < \infty$. Then $u_n \in K_{\sigma_n}$ for all n and, by (1) and Lemma 3.3, u_n is bounded in H, so we can suppose that $u_n \overset{H}{\rightharpoonup} u_0$. By Lemma 3.2(d), we deduce $u_0 \in K_{\sigma_0}$. Moreover, using Lemma 3.3 we get $B_{\sigma_n}(u_n) \to B_{\sigma_0}(u_0)$ and by the weak lower semicontinuity of $u \mapsto a(u,u)$ in H we also have $a(u_0,u_0) \le \liminf_n a(u_n,u_n)$. Collecting all these things we obtain

$$f_{\sigma_0}(u_0) \le \liminf_{n\to\infty} f_{\sigma_n}(u_n),$$

which is the first part of Γ convergence.

On the other hand, take u_0 in K_{σ_0}. By Lemma 3.2(d) there exists $(u_n)_n$ such that $u_n \in K_{\sigma_n}$ for all n and $u_n \xrightarrow{H} u_0$. From the convergence in H we get

$$f_{\sigma_0}(u_0) = \lim_{n\to\infty} f_{\sigma_n}(u_n),$$

which is the second part of Γ convergence.

Furthermore, since any sequence $(u_n)_n$ with $f_{\sigma_n}(u_n)$ bounded is bounded in H, hence relatively compact in L, it follows that $(f_{\sigma_n})_n$ is asymptotically locally equicoercive as defined in [10]. Using the results in Section 4 of [10] we get, for all $t_n \to t_0$ in $[0,\infty]$,

$$\begin{aligned} u_n \xrightarrow{H} u_0 &\Rightarrow u_n \xrightarrow{L} u_0,\ f_{\sigma_n}(u_n) \to f_{\sigma_0}(u_0) \\ &\Rightarrow \Phi_{\sigma_n}(u_n,t_n) \xrightarrow{L} \Phi_0(u_0,t_0),\ f_{\sigma_n}(\Phi_{\sigma_n}(u_n,t_n)) \to f_{\sigma_0}(\Phi_0(u_0,t_0)) \\ &\Rightarrow \Phi_{\sigma_n}(u_n,t_n) \xrightarrow{H} \Phi_0(u_0,t_0), \end{aligned}$$

which implies (d).

Lemma 3.5. *Let $\sigma_n \to \sigma_0$ in $[0,\infty]$, $(u_n)_n$, $(\alpha_n)_n$ be such that $u_n \in K_{\sigma_n}$ for all n, $\|u_n\|_H$ is bounded, $u_n \xrightarrow{L} u_0$, $\alpha_n \xrightarrow{L} \alpha_0$ and for all n,*

$$\begin{cases} a(u_n, v-u_n) + \langle B'_{\sigma_n}(u_n) + \alpha_n, v-u_n\rangle_L \ge 0 \quad \forall v \in K_{\sigma_n}, \\ u_n \in K_{\sigma_n}. \end{cases}$$

Then $u_n \xrightarrow{H} u_0$ and

$$\begin{cases} a(u_0, v-u_0) + \langle B'_{\sigma_0}(u_0) + \alpha_0, v-u_0\rangle_L \ge 0 \quad \forall v \in K_{\sigma_0} \\ u_0 \in K_{\sigma_0}. \end{cases}$$

Proof. It is clear that $u_n \xrightarrow{H} u_0$, $f_{\sigma_n}(u_n)$ is bounded and $\alpha_n \in \partial^- f_{\sigma_n}(u_n)$. Using Remark 1.14 and Theorem 1.17 of [10], we see that $f_{\sigma_n}(u_n) \to f_{\sigma_0}(u_0)$ and $\alpha_0 \in \partial^- f_{\sigma_0}(u_0)$, so the limit inequality is fulfilled. In particular, using (3.3), we get $a(u_n,u_n) \to a(u_0,u_0)$, hence $u_n \xrightarrow{H} u_0$, since $(u,v) \mapsto a(u,v)$ is an inner product equivalent to $(u,v) \mapsto \langle u,v\rangle_H$.

Proposition 3.6. *The family of semiflows $(K_\sigma, \Phi_\sigma)_{\sigma\in[0,\infty]}$ satisfies the compactness assumption* (N.2) *of Section* 1, *with respect to any bounded closed subset N of H.*

Proof. We prove that for all σ in $[0,\infty]$, for all u_0 in K_σ and $t > 0$, if $\mathcal{U}$ is the solution of (11) with starting point u_0 then

$$\|\mathcal{U}'_+(t)\|_L \le \left(\frac{f_\sigma(u_0) - f_\sigma(\mathcal{U}(t))}{t}\right)^{1/2} e^{Mt}. \tag{12}$$

To see this let $0 \le t \le t'$; using 3.4 with $u = \mathcal{U}(t)$, $v = \mathcal{U}(t')$ and $\alpha = -\mathcal{U}'_+(t)$ (which exists for all $t > 0$: see [10]) we get

$$f_\sigma(\mathcal{U}(t')) - f_\sigma(\mathcal{U}(t)) \ge -\|\mathcal{U}'_+(t)\|_L\|\mathcal{U}(t') - \mathcal{U}(t)\|_L - M\|\mathcal{U}(t') - \mathcal{U}(t)\|_L^2.$$

Since

$$f_\sigma(\mathcal{U}(t')) - f_\sigma(\mathcal{U}(t)) = -\int_t^{t'} \|\mathcal{U}(\tau)\|_L^2\, d\tau \le -\frac{\big(\int_t^{t'} \|\mathcal{U}'(\tau)\|_L\, d\tau\big)^2}{t'-t}$$

and

$$\|\mathcal{U}(t') - \mathcal{U}(t)\|_L \le \int_t^{t'} \|\mathcal{U}'(\tau)\|_L\, d\tau,$$

setting $p(t) = \int_t^{t'} \|\mathcal{U}'(\tau)\|_L\, d\tau$, we have

$$p'(t)p(t) \le \left(M - \frac{1}{t'-t}\right)p(t)^2.$$

With standard calculations this yields

$$\frac{p(t)}{t'-t} \le \frac{p(0)}{t}e^{Mt}$$

and letting $t' \to t^+$,

$$\|\mathcal{U}'_+(t)\|_L \le \frac{1}{t}\int_0^t \|\mathcal{U}'(\tau)\|_L\, d\tau \le \left(\frac{f_\sigma(u_0) - f_\sigma(\mathcal{U}(t))}{t}\right)^{1/2} e^{Mt}.$$

Now let N be a closed bounded subset of H, let $\sigma_n \to \sigma_0$ in $[0,\infty]$, $(u_n)_n$ in H be such that $u_n \in N \cap K_{\sigma_n}$, let $t_n \to \infty$ and let $\mathcal{U}_n([0,t_n]) \subset N$, where $\mathcal{U}_n$ are the corresponding curves with starting point u_n. By (12), $\mathcal{U}'_n(t_n)$ is bounded in L, since $\mathcal{U}_n(t_n - 1)$ are bounded. We can suppose, considering a subsequence, that $\mathcal{U}_n(t_n) \overset{H}{\rightharpoonup} u_0$ and $\mathcal{U}'_n(t_n) \overset{L}{\rightharpoonup} \alpha$ for suitable u_0 in K_{σ_0} and α in L. Using 3.5 we get $\mathcal{U}(t_n) \overset{H}{\to} u_0$.

Remark 3.7. For any σ in $[0,\infty]$ the flow (K_σ, Φ_σ) has the following property: if S is an invariant set with respect to (K_σ, Φ_σ), then either S is a single point or it contains at least two rest points (namely points which are lower critical for f_σ). In particular, any isolated rest point is an isolated invariant set.

Proof. Let S be invariant. Since S is compact, there exist u_1, u_2 in S such that f_σ attains its maximum (resp. minimum) at u_1 (resp. u_2). It is easy to see that u_1 and u_2 are rest points and that, if $u_1 = u_2$, then $S = \{u_1\}$.

Proposition 3.8. *Condition* (C) *of Section* 1 *is satisfied for every closed bounded subset* N *of* H.

Proof. It suffices to take a bounded neighbourhood U of N and define

$$\Psi(\varrho, u, t) = u + ((tD) \wedge \|P_\varrho(u) - u\|_H)\frac{P_\varrho(u) - u}{\|P_\varrho(u) - u\|_H}$$

for all ϱ in $[0,\infty]$, u in U and t in $[0,1]$, where D is the diameter of U. Using Lemma 3.2(f) we find that Ψ is continuous; the other properties required are very easy to check.

Proposition 3.9. *Let $a : H \times H \to \mathbb{R}$ be as in the previous context and let $\widetilde{b} : L \times L \to \mathbb{R}$ be a symmetric bilinear form continuous on L. Let $\widetilde{K}$ be a closed convex cone in H. Define $\widetilde{a} = a + \widetilde{b}$ and denote by $(\widetilde{\lambda}_n)_n$ the eigenvalues associated with $\widetilde{a}$, $\widetilde{\lambda}_0 = -\infty$. Assume that there exist an integer i and a linear subspace $\widetilde{H}$ such that $\dim(\widetilde{H}) = i$, $\widetilde{\lambda}_i < 0 < \widetilde{\lambda}_{i+1}$ and*

$$\widetilde{H} \subset \widetilde{K}, \qquad \sup_{u \in \widetilde{H},\, \|u\|_H = 1} \widetilde{a}(u,u) < 0.$$

Then

(a) *There are no solutions of the problem*

$$\begin{cases} \widetilde{a}(u, v-u) \geq 0 & \forall v \in \widetilde{K}, \\ u \in \widetilde{K}, \end{cases} \tag{13}$$

except the trivial one.

(b) *The Conley index $\widetilde{\mathcal{I}}(\{0\})$ of 0 as an isolated invariant set in the flow associated with (13) (using the functional $\widetilde{f}(u) = \widetilde{a}(u,u) + I_{\widetilde{K}}(u)$) is equal to $\mathcal{S}_i$, the i-dimensional sphere (more precisely, to the pair $(\mathcal{S}_i, p)$ with $p \in \mathcal{S}_i$).*

Proof. (a) Assume by contradiction that a solution u of (13) different from 0 exists. Taking $v = u + \widetilde{u}$ for $\widetilde{u}$ in $\widetilde{H}$ and $v = 0, 2u$ (all are in $\widetilde{K}$ since $\widetilde{K}$ is a cone) we have

$$\widetilde{a}(u,\widetilde{u}) = 0 \quad \forall \widetilde{u} \in \widetilde{H}, \quad \widetilde{a}(u,u) = 0. \tag{14}$$

Then $u \notin \widetilde{H}$ so, setting $\widehat{H} = \widetilde{H} \oplus \mathrm{span}(u)$, we have $\dim(\widehat{H}) = i+1$ and, using (14) we get $\widetilde{a}(\widehat{u},\widehat{u}) \leq 0$ for all $\widehat{u}$ in $\widehat{H}$. But this contradicts $\widetilde{\lambda}_{i+1} \geq 0$.

(b) Let $H' = \{u' \in H \mid \widetilde{a}(u',\widetilde{u}) = 0 \ \forall \widetilde{u} \in \widetilde{H}\}$. With standard arguments we can define a continuous linear operator $\widetilde{P} : H \to \widetilde{H}$ such that $u - \widetilde{P}u \in H'$ for all u in H. For $\varrho \geq 1$ and u in H we set $\widetilde{a}_\varrho(u,u) = \widetilde{a}(\widetilde{P}u,\widetilde{P}u) + \varrho\widetilde{a}(u-\widetilde{P}u, u-\widetilde{P}u)$ and define the functionals $\widetilde{f}_\varrho, \widetilde{f}_\infty : \widetilde{K} \to \mathbb{R} \cup \{\infty\}$ by $\widetilde{f}_\varrho(u) = \frac{1}{2}\widetilde{a}(u,u)$ and $\widetilde{f}_\infty(u) = \frac{1}{2}\widetilde{a}(u,u) + I_{\widetilde{H}}$ (for u in $\widetilde{K}$). It is easy to check that, for ϱ fixed, each $\widetilde{f}_\varrho$ generates a semiflow $(\widetilde{K}, \widetilde{\phi}_\varrho)$ or $(\widetilde{H}, \widetilde{\phi}_\infty)$ which satisfies all the assumptions required to consider the index.

Now let $c > 0$ be fixed and set $\widetilde{K}^c_\varrho = \{u \in \widetilde{K} \mid \widetilde{f}_\varrho(u) \leq c\}$ for $\varrho < \infty$ and $\widetilde{K}^c_\infty = \widetilde{H}$. Since $\widetilde{f}_\varrho$ decreases on the ϱ-flow, we can consider the flows $(\widetilde{K}^c_\varrho, \widetilde{\phi}_\varrho)$. Using the same arguments of the previous lemmas we can easily show that the latter flows form a continuous family of flows (while the former do not) and that the compactness assumption (N2) of Section 1 is satisfied. Finally, we claim that also condition (C) of Section 1 is satisfied. For this take $U = B(0,R) \cap \widetilde{K}$ and define

$$t(\varrho,u) = \begin{cases} \|u-\widetilde{P}u\|_H\left(1-\sqrt{\dfrac{c-\widetilde{a}(\widetilde{P}u,\widetilde{P}u)}{\varrho\widetilde{a}(\widetilde{P}u-u,\widetilde{P}u-u)}}\right)\vee 0 & \text{if } \varrho<\infty, \\ \|u-\widetilde{P}u\|_H & \text{if } \varrho=\infty, \end{cases}$$

$$\widetilde{\Psi}(\varrho,u,t) = u + ((tR)\wedge t(\varrho,u))\frac{\widetilde{P}u-u}{\|\widetilde{P}u-u\|_H}$$

($t(\varrho,u)$ is "the first point on the segment between u and $\widetilde{P}u$ such that $\widetilde{f}_\varrho(u) \le c$"). It is simple to check that $\widetilde{\Psi}$ is continuous and satisfies (C) of Section 1. It is straightforward to see that the index of $\{0\}$ in $(\widetilde{K},\widetilde{\Phi}_1)$ is the same if computed in $(\widetilde{K}^c,\widetilde{\Phi}_1)$, since for r small, $B(0,r)\cap\widetilde{K}\subset\widetilde{K}^c$. So we consider the flows $(\widetilde{K}^c_\varrho,\widetilde{\phi}_\varrho)$.

Using (a) we deduce that 0 is the unique critical point of $\widetilde{f}_\varrho$ for all ϱ's in $[0,\infty]$; then, by the continuation argument proved in Section 1, the ϱ-index of 0 is the same for all ϱ's. But the index is easily computed if $\varrho=\infty$ since $\widetilde{H}$ has dimension i and all the boundary of (for example) the unit ball is made up by exit points. Then, with the obvious notations,

$$\widetilde{\mathcal{I}}(\{0\}) = \widetilde{\mathcal{I}}_1(\{0\}) = \widetilde{\mathcal{I}}_\infty(\{0\}) = B_i/\partial B_i = \mathcal{S}_i$$

and the desired assertion is proved.

Proof of Theorem 3.1. We argue by contradiction and suppose 0 to be the unique solution of (3), that is, the unique critical point for f_1.

Using the assumption (7) and Lemma 3.9 we find that the ball $B(0,1)$ is an isolating neighbourhood for 0 in the flow (K_∞,Φ_∞) and $\mathcal{I}_\infty(\{0\}) = \mathcal{S}_{i_\infty}$. By Theorem 1.6 there exists $\overline{\sigma}$ such that $B(0,1)$ is σ-isolating for all $\sigma\in[\overline{\sigma},\infty]$. We set $R=\overline{\sigma}$ and claim that $B(0,R)$ is σ-isolating for all $\sigma\in[1,\infty]$. For, let $\sigma\in[1,\infty]$ and $o:\mathbb{R}\to B(0,R)$ be a bilateral orbit in $B(0,R)$, relative to the σ-flow. If we define $\widetilde{o}:\mathbb{R}\to L^2(\Omega)$ by $\widetilde{o}(t) = \frac{1}{\overline{\sigma}}o(t)$, we can easily see, using the definition of f_σ, that $\widetilde{o}$ is a bilateral orbit for the $\sigma\overline{\sigma}$-flow, lying in $B(0,1)$. Then $\widetilde{o}$ is in $\operatorname{int}(B(0,1))$; because $B(0,1)$ is $\sigma\overline{\sigma}$-isolating; hence o is contained in $\operatorname{int}(B(0,R))$: this means that $B(0,R)$ is σ-isolating.

In this way we also see that, for all σ in $[1,\infty]$, $\mathcal{I}_\sigma(S_\sigma(B(0,R))) = \mathcal{S}_{i_\infty}$; then $\mathcal{I}_1(\{0\}) = \mathcal{S}_{i_\infty}$, since $S_1(B(0,R)) = \{0\}$ for $R>0$, 0 being the unique invariant in the 1-flow.

Using similar arguments we can prove, on the other hand, that, for a suitable $r>0$, $B(0,r)$ is σ-isolating for all σ in $[0,1]$ and $\mathcal{I}_\sigma(S_\sigma(B(0,r))) = \mathcal{S}_{i_0}$ for all σ in $[0,1]$. But, as before, this implies $\mathcal{I}_1(\{0\}) = \mathcal{S}_{i_0}$, which is contradictory, since $i_0\ne i_\infty$. □

4. An application to obstacle problems

Let Ω be a bounded open subset of $\mathbb{R}^N$. Let $\varphi_1 : \Omega \to [-\infty, 0]$ and $\varphi_2 : \Omega \to [0, \infty]$ be two functions such that φ_1 is quasi-upper semicontinuous and φ_2 is quasi-lower semicontinuous (see [3]). We consider the convex set

$$K = \{u \in W_0^{1,2}(\Omega) \mid \varphi_1(x) \le \widetilde{u}(x) \le \varphi_2(x) \text{ for quasi-every } x \text{ in } \Omega\},$$

where for every u in $W_0^{1,2}(\Omega)$, $\widetilde{u}$ is the quasi-everywhere continuous function defined quasi-everywhere by

$$\widetilde{u}(x) = \lim_{r \to 0} \frac{1}{\operatorname{meas}(B(x,r))} \int_{B(x,r)} u(\xi)\, d\xi$$

(see [16]). We also consider a function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $s \mapsto g(x,s)$ is of class C^1 for almost all x in Ω, $x \mapsto g(x,s)$ is measurable for all s in $\mathbb{R}$ and

(1) there exists $M > 0$ such that $|g_s'(x,s)| \le M$ for $x \in \Omega$ and $s \in \mathbb{R}$;
(2) $g(x,0) = 0$ for $x \in \Omega$;
(3) there exists a function $m_\infty : \Omega \to \mathbb{R}$ such that $m_\infty(x) = \lim_{s\to\infty} g_s'(x,s)$ uniformly with respect to x.

For convenience we set

$$m_0(x) = g_s'(x,0). \tag{4}$$

Furthermore, we set

$$\begin{aligned} F_1^0 &= \{x \in \Omega \mid \phi_1(x) = 0\}, & F_2^0 &= \{x \in \Omega \mid \phi_2(x) = 0\},\\ F_1^\infty &= \{x \in \Omega \mid \phi_1(x) = -\infty\}, & F_2^\infty &= \{x \in \Omega \mid \phi_2(x) = \infty\}, \end{aligned}$$

and denote by $(\lambda_n^{(0)})_n$, $(\lambda_n^{(\infty)})_n$, $(\mu_n^{(0)})_n$, $(\mu_n^{(\infty)})_n$ the eigenvalues of $-\Delta$ (the Laplace operator) in the following closed subspaces of $H = W_0^{1,2}(\Omega)$:

$$\begin{aligned} H_0 &= \{u \in H \mid \widetilde{u} = 0 \text{ for quasi-every } x \text{ in } F_1^0 \cap F_2^0\},\\ H_\infty &= \{u \in H \mid \widetilde{u} = 0 \text{ for quasi-every } x \text{ outside } F_1^\infty \cup F_2^\infty\},\\ H_0' &= \{u \in H \mid \widetilde{u} = 0 \text{ for quasi-every } x \text{ in } F_1^0 \cup F_2^0\},\\ H_\infty' &= \{u \in H \mid \widetilde{u} = 0 \text{ for quasi-every } x \text{ outside } F_1^\infty \cap F_2^\infty\}, \end{aligned}$$

where of course $\lambda_i^{(0)} \le \mu_i^{(0)} \le \lambda_i^{(\infty)} \le \mu_i^{(\infty)}$; for convenience we agree that $\lambda_0^{(0)} = \mu_0^{(0)} = \lambda_0^{(\infty)} = \mu_0^{(\infty)} = -\infty$ (we are assuming that H_∞' is not trivial, so there are “a lot of points” at which $\phi_1 = -\infty$ and $\phi_2 = \infty$).

We also introduce $G : \Omega \times \mathbb{R} \to \mathbb{R}$ given by $G(x,s) = \int_0^s g(x,\sigma)\, d\sigma$.

Theorem 4.1. *Let φ_1, φ_2, K and g be as above. Suppose there exist $i, j \in \mathbb{N}$ such that $i \neq j$ and*

$$\mu_j < \inf_{x\in\Omega} m_0(x) \leq \sup_{x\in\Omega} m_0(x) < \lambda_{j+1},$$
$$\mu_i < \inf_{x\in\Omega} m_\infty(x) \leq \sup_{x\in\Omega} m_\infty(x) < \lambda_{i+1}.$$

Then there exists a nontrivial solution of the variational inequality

$$\begin{cases} \displaystyle\int_\Omega DuD(v-u)\,dx - \int_\Omega g(x,u)(v-u)\,dx \geq 0 \quad \forall v \in K, \\ u \in K, \end{cases} \tag{5}$$

in addition to the trivial one ($u \equiv 0$).

Proof. We use Theorem 3.1 with $H = W_0^{1,2}(\Omega)$, $L = L^2(0,T;H)$, and

$$a(u,v) = \int_\Omega Du(x)Dv(x)\,dx, \quad B(u) = \int_\Omega G(x,u(x))\,dx,$$
$$B''(0)u = m_0 u, \quad B''(\infty)u = m_\infty u.$$

It is clear that $B'(u) = g(\cdot, u)$ so the variational inequality (3) corresponds to (5). Using the assumptions on g it is simple to check that B fulfills the requirements of Section 2. Moreover, it is also simple to see that

$$K_0 = \{u \in W_0^{1,2}(\Omega) \mid \widetilde{u} \geq 0 \text{ q.e. on } F_1^0, u \leq 0 \text{ q.e. on } F_2^0\},$$
$$K_\infty = \{u \in W_0^{1,2}(\Omega) \mid \widetilde{u} \geq 0 \text{ q.e. outside } F_1^0, u \leq 0 \text{ q.e. outside } F_2^0\}.$$

So $H_0' \subset K_0$ and $H_\infty' \subset K_\infty$; if we consider the spaces

$$\widetilde{H}_0 = \operatorname{span}(e_1', \ldots, e_j'), \quad \widetilde{H}_\infty = \operatorname{span}(f_1', \ldots, f_i'),$$

where $(e_n')_n$ and $(f_n')_n$ denote the eigenfunctions of $-\Delta$ on H_0', H_∞' respectively, then it is clear that $\widetilde{H}_0$ and $\widetilde{H}_\infty$ satisfy the assumptions (6) and (7) of Theorem 3.1, with $i_0 = j$ and $i_\infty = i$. Then the conclusion follows from Theorem 3.1.

A simpler version of the previous theorem is the following.

Corollary 4.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$ and $F \subset \Omega$ a closed set. Let $\varphi_1, \varphi_2 : F \to \mathbb{R}$ be two functions such that $\varphi_1 < 0 < \varphi_2$ in F, φ_1 is upper semicontinuous and φ_2 is lower semicontinuous, and consider the convex set*

$$K = \{u \in W_0^{1,2}(\Omega) \mid \varphi_1 \leq u \leq \varphi_2 \textit{ on } F \textit{ in the } W_0^{1,2}(\Omega)\textit{-sense}\}$$

(that is, $u \in K$ if and only if u is the $W^{1,2}(\Omega)$ limit of a sequence $(u_n)_n$ of functions which are Lipschitz continuous, with support in Ω and such that $\phi_1 \leq u_n \leq \phi_2$ in Ω). Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that $s \mapsto g(x,s)$ is of class C^1 for almost all x in Ω, $x \mapsto g(x,s)$ is measurable for all s in $\mathbb{R}$ and

- *there exists* $M > 0$ *such that* $|g'_s(x,s)| \le M$ *for* $x \in \Omega$ *and* $s \in \mathbb{R}$*;*
- $g(x,0) = 0$ *for* $x \in \Omega$*;*
- *the limits* $m_0 = \lim_{s\to 0} g'_s(x,s)$ *and* $m_\infty = \lim_{s\to\infty} g'_s(x,s)$ *exist uniformly with respect to* x.

Denote by $(\lambda_n)_n$ *the eigenvalues of* $-\Delta$ *in* $H = W_0^{1,2}(\Omega)$ *and by* $({\lambda'}_n)_n$ *the eigenvalues of* $-\Delta$ *in the closed linear subspace* $H' = W_0^{1,2}(\Omega \setminus F)$ *with* $\lambda_0 = \lambda'_0 = -\infty$*. Suppose there exist* $i, j \in \mathbb{N}$ *such that* $i \ne j$*,* $\lambda_j < m_0 < \lambda_{j+1}$ *and* $\lambda'_i < m_\infty < \lambda'_{i+1}$*. Then there exists a nontrivial solution of the variational inequality* (5) *in addition to the trivial one* ($u \equiv 0$)*.*

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