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TRAJECTORY ATTRACTORS FOR REACTION-DIFFUSION SYSTEMS

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Dedicated to Louis Nirenberg

Introduction

Recently, the problem how to describe the limit behaviour of solutions of evolution equations for which the Cauchy problem can have non-unique solution arouses much interest (see, for example, [3], [5], [6], [8], [14]). Trajectory attractors can help to solve such problems.

In the present paper we study trajectory attractors for the non-autonomous reaction-diffusion system

$$(1) \qquad \partial_t u = a\Delta u - f_0(u,t) + g_0(x,t), \quad u|_{\partial\Omega} = 0 \quad (\text{or } \partial u/\partial\nu|_{\partial\Omega} = 0)$$

where $u = u(x,t) = (u^1,\dots,u^N)$, $x \in \Omega \Subset \mathbb{R}^n$, $t \geq 0$, $f_0(v,s) = (f_0^1,\dots,f_0^N)$, $(v,s) \in \mathbb{R}^N \times \mathbb{R}_+$, $g_0(x,s) = (g_0^1,\dots,g_0^N)$, $x \in \Omega$, $s \geq 0$. We assume that the matrix a and the functions f_0, g_0 satisfy some general conditions (see Section 2). These conditions provide the existence of a solution u of the Cauchy problem for the system (1) ($u|_{t=0} = u_0$, $u_0 \in H = (L_2(\Omega))^N$). However, this solution can be non-unique because we do not suppose any Lipschitz conditions for $f_0(v,s)$ with respect to v.

The pair of functions $(f_0(v,s), g_0(x,s)) = \sigma_0(s)$ is called the symbol of equation (1). To construct a trajectory attractor for (1), we consider the family of

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all such systems with shifted symbols $\sigma_0(s+h) = (f_0(v, s+h), g_0(x, s+h))$, $h \geq 0$. We also include in this family every symbol $\sigma(s) = (f(v,s), g(x,s))$ that is the limit of some sequence $\{\sigma_0(s+h_m)\}_{m\in\mathbb{N}}$, $h_m \geq 0$, in an appropriate topological space Ξ_+. The topology in Ξ_+ provides the solvability of the equation of the form (1) with symbol $\sigma(s) = (f(v,s), g(x,s))$. The family $\{\sigma(s)\}$ of such symbols is said to be the *hull* $\mathcal{H}_+(\sigma_0)$ of the function σ_0 in Ξ_+, i.e. $\mathcal{H}_+(\sigma_0) = [\{(f_0(v, s+h),\ g_0(x, s+h)) \mid h \geq 0\}]_{\Xi_+}$. Here $[\,\cdot\,]_{\Xi_+}$ means the closure in Ξ_+. We assume that $\mathcal{H}_+(\sigma_0)$ is compact in Ξ_+.

To every equation of the form (1) with a symbol $\sigma(s) = (f(v,s), g(x,s)) \in \mathcal{H}_+(\sigma_0)$, there corresponds a trajectory space $\mathcal{K}_\sigma^+ = \{u(x,s), s \geq 0\}$ that consists of all solutions $u(x,s) = u(s)$ of this equation in a weak sense. Here we replace t by s. Consider the united trajectory space $\mathcal{K}_\Sigma^+ = \bigcup_{\sigma\in\mathcal{H}_+(\sigma_0)} \mathcal{K}_\sigma^+$. The translation semigroup $\{T(t) \mid t \geq 0\}$ acts on $\mathcal{K}_\Sigma^+$: $T(t)u(s) = u(t+s)$. Evidently, $T(t)\mathcal{K}_\sigma^+ \subseteq \mathcal{K}_{T(t)\sigma}^+$, therefore $T(t)\mathcal{K}_\Sigma^+ \subseteq \mathcal{K}_\Sigma^+$ for all $t \geq 0$. Now, we embed the set $\mathcal{K}_\Sigma^+$ into an appropriate topological space Θ_+ such that $\mathcal{K}_\Sigma^+$ is closed in Θ_+. The topology in Θ_+ is a local weak convergence topology on every segment $[t_1, t_2] \subseteq \mathbb{R}_+$ (see Section 2).

A global attractor (in the topology Θ_+) of the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}_\Sigma^+$ is said to be a *trajectory attractor* $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ of the family of equations (1) with symbols $\sigma \in \mathcal{H}_+(\sigma_0)$. More, precisely, the set $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ is compact in Θ_+, it is strictly invariant with respect to $\{T(t)\}$: $T(t)\mathcal{A}_{\mathcal{H}_+(\sigma_0)} = \mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ for $t \geq 0$, and the set $T(t)B$ is attracted to $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ in the topology Θ_+ as $t \to \infty$ for every bounded set $B \subset \mathcal{K}_\Sigma^+$. To define bounded sets in $\mathcal{K}_\Sigma^+$ we use a Banach space $\mathcal{F}_+ \subseteq \Theta_+$.

In Section 2 we prove the existence of a trajectory attractor for equation (1). The structure of the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ is described as well.

Section 3 deals with systems (1) for which the uniqueness for the Cauchy problem holds. In this case, the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ is more regular and $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ attracts bounded sets B from $\mathcal{K}_\Sigma^+$ in a stronger topology Θ_+^s.

In Section 4 we study trajectory attractors for some class of ordinary differential equations in finite-dimensional Euclidean space. The corresponding Cauchy problem can have non-unique solution. This class includes the Galerkin approximation system of order m for equation (1). We prove the theorem on the existence and structure of trajectory attractors for such equations. In particular, the Galerkin approximation system with symbol σ_{0m} has a trajectory attractor $\mathcal{A}^{(m)}_{\mathcal{H}_+(\sigma_{0m})}$ in the space $P_m\Theta_+$, where P_m is the orthogonal projection onto the corresponding m-dimensional space.

In Section 1 we present the general scheme of construction of a trajectory attractor for an abstract non-autonomous operator evolution equation.

In Section 5 we study some perturbation problems for the trajectory attractor of system (1). The translation semigroup $\{T(t)\}$ acts also on the hull $\mathcal{H}_+(\sigma_0)$: $T(t)\sigma(s) = \sigma(t+s)$, $t \geq 0$. Consider the attractor $\omega(\mathcal{H}_+(\sigma_0))$ of the translation semigroup on $\mathcal{H}_+(\sigma_0)$: $T(t)\omega(\mathcal{H}_+(\sigma_0)) = \omega(\mathcal{H}_+(\sigma_0))$ for all $t \geq 0$. The following equality holds: $\mathcal{A}_{\mathcal{H}_+(\sigma_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(\sigma_0))}$ (see Section 2). Now, assume for brevity that one perturbs the term g_0 in the following way: $g(x,s) = g_0(x,s) + a(x,s)$, where $a(\,\cdot\,,s) \rightharpoonup 0$ $(s \to \infty)$ in a weak sense. (For example, $a(x,s) = A(x)\sin(s^2)$, $A \in H$). Then the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(f_0,g_0)}$ does not change: $\mathcal{A}_{\mathcal{H}_+(f_0,g_0+a)} = \mathcal{A}_{\mathcal{H}_+(f_0,g_0)}$. We also study the analogous perturbation of the non-linear term f_0 in the corresponding space. Other perturbation problems arise when the symbol $\sigma_0(s)$ depends on a small parameter ε, i.e., for example, when the terms of equation (1) are: $f_0(v,s) + \varepsilon f_1(v,s)$ and $g_0(x,s) + \varepsilon g_1(x,s)$. We describe the limit behaviour of the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(f_0+\varepsilon f_1, g_0+\varepsilon g_1)}$ as $\varepsilon \to 0$. Finally, dealing with the approximation problem, we prove that the trajectory attractor $\mathcal{A}^{(m)}_{\mathcal{H}_+(\sigma_{0m})}$ of the Galerkin approximation system of order m converges as $m \to \infty$ to the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ of the reaction-diffusion system (1) with the original symbol $\sigma_0(s)$.

1. Preliminaries

In this section, we describe the general scheme of constructing a trajectory attractor for an abstract operator evolution equation. In the next sections, this scheme will be applied to the study of reaction-diffusion systems. Consider a non-autonomous equation of the type

$$\partial_t u = A_{\sigma(t)}(u), \quad t \geq 0. \tag{1.1}$$

For every $s \in \mathbb{R}_+$, we are given an operator $A_{\sigma(s)}(\,\cdot\,) : E \to E_0$, where E, E_0 are Banach spaces. The function parameter $\sigma(s), s \in \mathbb{R}_+$, in (1.1) reflects the dependence of the equation on time. The function σ is called the *time symbol* (or *symbol*) of equation (1.1). The values of σ belong to some Banach space Ψ.

For example, in the reaction-diffusion system

$$\partial_t u = a\Delta u - f(u,t) + g(x,t), \quad u|_{\partial\Omega} = 0, \quad t \geq 0,$$

where $x \in \Omega \Subset \mathbb{R}^n, u = (u^1, \dots, u^N)$, $f = (f^1, \dots, f^N)$, $g = (g^1, \dots, g^N)$, the symbol is the pair $\sigma(s) = (f(v,s), g(x,s))$, $s \geq 0$. The component $g(x,s)$ is viewed as a mapping from $\mathbb{R}_+$ into $(L_2(\Omega))^N$, and $f(v,s)$ as a mapping from $\mathbb{R}_+$ into a specially selected function space $\mathcal{M} = \{\psi(v) = (\psi^1(v), \dots, \psi^N(v)),$ $v \in \mathbb{R}^N\} \subset C(\mathbb{R}^N; \mathbb{R}^N)$. The norm of $\mathcal{M}$ takes into account the growth of $f(v,s)$ with respect to v (see (2.30) and (3.48)). In this case $\Psi = \mathcal{M} \times (L_2(\Omega))^N$.

We assume that the symbol $\sigma(s)$ of equation (1.1), as a function of s, belongs to a topological space Ξ_+ of functions $\xi : \mathbb{R}_+ \to \Psi$. Usually, in applications, the topology in Ξ_+ is a local convergence topology on every segment $[t_1, t_2] \subset \mathbb{R}_+$. Consider the translation operator $T(t)$ on Ξ_+:

$$T(t)\xi(s) = \xi(t+s), \quad s \in \mathbb{R}_+, \quad t \geq 0. \tag{1.2}$$

We assume that the mapping $T(t)$ takes Ξ_+ into itself for all $t \geq 0$. Evidently, $\{T(t) \mid t \geq 0\}$ forms a semigroup on Ξ_+.

Now consider a family of equations (1.1) with various symbols σ belonging to a set $\Sigma \subseteq \Xi_+$. The set Σ is called the *symbol space* of the family. It is assumed that Σ, together with any symbol $\sigma \in \Sigma$, contains all positive translations of σ: $\sigma(t + \cdot\,) = T(t)\sigma(\,\cdot\,) \in \Sigma$ for any $t \geq 0$, i.e.

$$T(t)\Sigma \subseteq \Sigma \quad \forall t \geq 0. \tag{1.3}$$

We suppose that the symbol space Σ with the topology inherited from Ξ_+ is a metrizable space and the corresponding metric space is complete.

We shall study the family of equations (1.1) with symbols σ from the complete metric space $\Sigma \subseteq \Xi_+$. We assume that the translation semigroup $\{T(t)\}$ is continuous on Σ and it satisfies (1.3).

Let us describe the typical symbol space in particular problems. We are given some fixed symbol $\sigma_0(s), s \geq 0$, consisting of all time-dependent terms of the equation under consideration. Then one chooses an appropriate enveloping topological space Ξ_+ of functions on $\mathbb{R}_+$ such that $\sigma_0 \in \Xi_+$. Consider the closure in Ξ_+ of the set $\{T(t)\sigma_0(\,\cdot\,) \mid t \geq 0\} = \{\sigma_0(t + \cdot\,) \mid t \geq 0\}$. This set is said to be the *hull* of the function $\sigma_0 \in \Xi_+$ and it is denoted by

$$\mathcal{H}_+(\sigma_0) = [\{T(t)\sigma_0 \mid t \geq 0\}]_{\Xi_+}. \tag{1.4}$$

Here $[\,\cdot\,]_{\Xi_+}$ means the closure in Ξ_+. Evidently, $T(t)\mathcal{H}_+(\sigma_0) \subseteq \mathcal{H}_+(\sigma_0)$ for all $t \geq 0$.

Definition 1.1. The function $\sigma_0 \in \Xi_+$ is *translation-compact* (*tr.-c.*) *in* Ξ_+ if the hull $\mathcal{H}_+(\sigma_0)$ is compact in Ξ_+.

In applications, we consider the symbol spaces $\Sigma = \mathcal{H}_+(\sigma_0)$, where $\sigma_0(s)$ is a tr.-c. function in an appropriate topological space Ξ_+.

The aim of this article is to study solutions $u(s)$ of equation (1.1) which are defined for all $s \in \mathbb{R}_+$. We assume that $u(s) \in E$ for any $s \geq 0$. In all applications below, we shall strictly clarify the meaning of the expression: "a function u is a solution of (1.1)". To each symbol $\sigma \in \Sigma$, we assign a set $\mathcal{K}_\sigma^+$ of solutions of equation (1.1). The set $\mathcal{K}_\sigma^+$ is said to be a *trajectory space* of equation (1.1) with the symbol σ. We shall study the family $\{\mathcal{K}_\sigma^+ \mid \sigma \in \Sigma\}$ of trajectory spaces corresponding to equations (1.1) with symbols $\sigma \in \Sigma$. In this section we shall emphasize the properties of $\mathcal{K}_\sigma^+$ needed to construct a general theory of trajectory attractors.

As before, $T(t)$ means the translation operator acting now on the trajectory spaces $\mathcal{K}_\sigma^+$: $T(t)u(s) = u(t+s)$, $t \geq 0$.

Definition 1.2. The family $\{\mathcal{K}_\sigma^+ \mid \sigma \in \Sigma\}$ of trajectory spaces is *translation-coordinated* (tr.-coord.) if for all $\sigma \in \Sigma$ and all $u \in \mathcal{K}_\sigma^+$,

$$T(t)u \in \mathcal{K}_{T(t)\sigma}^+ \quad \forall t \geq 0. \tag{1.5}$$

Definition 1.3. The set $\mathcal{K}_\Sigma^+ = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma^+$ is called the *united trajectory space* of the family $\{\mathcal{K}_\sigma^+ \mid \sigma \in \Sigma\}$.

Proposition 1.1. *If the family $\{\mathcal{K}_\sigma^+ \mid \sigma \in \Sigma\}$ is tr.-coord. then the translation semigroup $\{T(t)\}$ takes $\mathcal{K}_\Sigma^+$ to itself:*

$$T(t)\mathcal{K}_\Sigma^+ \subseteq \mathcal{K}_\Sigma^+ \quad \forall t \geq 0. \tag{1.6}$$

The proof is evident.

Definition 1.4. A compact set $A \Subset X$ is said to be a *global attractor* of a semigroup $\{S(t)\}$ acting on a complete metric space X if

(i) A attracts every set B, bounded in X: $\operatorname{dist}_X(S(t)B, A) \to 0$ $(t \to \infty)$;
(ii) A is strictly invariant with respect to $\{S(t)\}$: $S(t)A = A$ for all $t \geq 0$.

In the case $X = \Sigma$ and $\{S(t)\} = \{T(t)\}$ we have

Proposition 1.2. *Every continuous translation semigroup $\{T(t)\}$ acting on a compact metric space Σ has a global attractor A which coincides with the ω-limit set of Σ:*

$$A = \omega(\Sigma) = \bigcap_{t\geq 0} \Big[\bigcup_{h\geq t} T(h)\Sigma\Big]_\Sigma, \quad \omega(\Sigma) \subseteq \Sigma. \tag{1.7}$$

The set A is strictly invariant with respect to $\{T(t)\}$: $T(t)A = A$ for $t \geq 0$.

This statement follows from well-known theorems from the theory of attractors of semigroups acting in metric spaces (see, for example, [1], [10], [15]).

To describe the limit behaviour of the translation semigroup $\{T(t)\}$ acting on the united trajectory space $\mathcal{K}_\Sigma^+$ we need a more general notion of global attractor, a trajectory attractor $\mathcal{A}_\Sigma$ of the translation semigroup. Let $\mathcal{K}_\Sigma^+ \subseteq \mathcal{F}_+ \subseteq \Theta_+$, where Θ_+ is a Hausdorff topological space and $\mathcal{F}_+$ is a Banach space. We use $\mathcal{F}_+$ to define bounded sets in $\mathcal{K}_\Sigma^+$. Let the translation semigroup $\{T(t)\}$ be continuous with respect to the topology of Θ_+. The set $\mathcal{A}_\Sigma$ attracts every set $T(t)B$ as $t \to \infty$ in the topological space Θ_+, where $B \subset \mathcal{K}_\Sigma^+$ and B is bounded in the Banach space $\mathcal{F}_+$.

Definition 1.5. A set $P \subseteq \Theta_+$ is said to be *a uniformly* (with respect to $\sigma \in \Sigma$) *attracting set* for the family $\{\mathcal{K}_\sigma \mid \sigma \in \Sigma\}$ in the topology Θ_+ if for every bounded set B in $\mathcal{F}_+$ and $B \subseteq \mathcal{K}_\Sigma^+$, the set P attracts $T(t)B$ as $t \to \infty$ in the topology Θ_+, i.e. for every neighbourhood $\mathcal{O}(P)$ in Θ_+ there exists $t_1 \geq 0$ such that $T(t)B \subseteq \mathcal{O}(P)$ for all $t \geq t_1$.

Definition 1.6. A compact set $\mathcal{A}_\Sigma \Subset \Theta_+$ is said to be a *uniform* (with respect to $\sigma \in \Sigma$) *trajectory attractor* of the translation semigroup $\{T(t)\}$ on $\mathcal{K}_\Sigma^+$ in the topology Θ_+ if

- (i) $\mathcal{A}_\Sigma$ is a minimal compact uniformly attracting set of $\{\mathcal{K}_\sigma \mid \sigma \in \Sigma\}$, i.e. $\mathcal{A}_\Sigma$ is contained in every compact uniformly attracting set P of $\{\mathcal{K}_\sigma \mid \sigma \in \Sigma\}$,
- (ii) $T(t)\mathcal{A}_\Sigma = \mathcal{A}_\Sigma$ for all $t \geq 0$.

Notice that a trajectory attractor $\mathcal{A}_\Sigma$ is contained in $\mathcal{K}_\Sigma^+$ and it depends on the symbol space Σ.

Definition 1.7. The family $\{\mathcal{K}_\sigma \mid \sigma \in \Sigma\}$ is said to be (Θ_+, Σ)-*closed* if the graph set $\bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma \times \{\sigma\}$ is closed in the topological space $\Theta_+ \times \Sigma$ with the usual product topology.

Proposition 1.3. *Let Σ be a compact metric space and $\{\mathcal{K}_\sigma \mid \sigma \in \Sigma\}$ be (Θ_+, Σ)-closed. Then the set $\mathcal{K}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma$ is closed in Θ_+.*

Proof. The reasoning is standard. Let $u \notin \mathcal{K}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma$. Then $(u, \sigma) \notin \bigcup_{\sigma' \in \Sigma} \mathcal{K}_{\sigma'} \times \{\sigma'\}$ for all $\sigma \in \Sigma$. By the assumption, the set $\cup_{\sigma' \in \Sigma} \mathcal{K}_{\sigma'} \times \{\sigma'\}$ is closed in $\Theta_+ \times \Sigma$, so there is a neighbourhood $\mathcal{W}_\sigma \times \mathcal{O}_\sigma$ in $\Theta_+ \times \Sigma$ such that $\mathcal{W}_\sigma \times \mathcal{O}_\sigma \cap (\bigcup_{\sigma' \in \Sigma} \mathcal{K}_{\sigma'} \times \{\sigma'\}) = \emptyset$, $u \in \mathcal{W}_\sigma$, $\sigma \in \mathcal{O}_\sigma$, where $\mathcal{W}_\sigma$ and $\mathcal{O}_\sigma$ are open sets in Θ_+ and Σ respectively. The family $\{\mathcal{O}_\sigma \mid \sigma \in \Sigma\}$ forms an open covering of Σ. Since Σ is compact, there is a finite subcovering $\{\mathcal{O}_{\sigma_i} \mid i = 1, \ldots, N\}$. Put $\mathcal{W}(u) = \bigcap_{i=1}^N \mathcal{W}_{\sigma_i}$. Evidently, $\mathcal{W}(u) \cap \mathcal{K}_\Sigma = \emptyset$. Hence, for every $u \notin \mathcal{K}_\Sigma$ there is a neighbourhood $\mathcal{W}(u)$ with $\mathcal{W}(u) \cap \mathcal{K}_\Sigma = \emptyset$, i.e. $\mathcal{K}_\Sigma$ is closed in Θ_+. □

Together with the family $\{\mathcal{K}_\sigma^+ \mid \sigma \in \Sigma\}$ of trajectory spaces we shall consider a more slender family $\{\mathcal{K}_\sigma^+ \mid \sigma \in \omega(\Sigma)\}$ (see (1.7)), which corresponds to the strictly invariant symbol space $\omega(\Sigma) \subseteq \Sigma$. Now we have the following result about trajectory attractors of families of equations (1.1).

Theorem 1.1. *Let Σ be a compact metric space and suppose that a continuous translation semigroup $\{T(t) \mid t \geq 0\}$ acts on Σ: $T(t)\Sigma \subseteq \Sigma$. Assume the family $\{\mathcal{K}_\sigma^+ \mid \sigma \in \Sigma\}$ corresponding to equation* (1.1) *is tr.-coord. and* (Θ_+, Σ)-*closed, and $\{T(t)\}$ is continuous in Θ_+. Let there exist a uniformly* (*with respect to $\sigma \in \Sigma$*) *attracting set P for $\{\mathcal{K}_\sigma^+ \mid \sigma \in \Sigma\}$ in Θ_+ such that P is compact in Θ_+ and bounded in $\mathcal{F}_+$. Then the translation semigroup $\{T(t) \mid t \geq 0\}$ acting on*

$\mathcal{K}_\Sigma^+$ has a uniform (with respect to $\sigma \in \Sigma$) trajectory attractor $\mathcal{A}_\Sigma$ in the topology Θ_+, in particular $\mathcal{A}_\Sigma \subseteq \mathcal{K}_\Sigma^+ \cap P$, and

$$T(t)\mathcal{A}_\Sigma = \mathcal{A}_\Sigma \quad \forall t \geq 0. \tag{1.8}$$

Moreover,

$$\mathcal{A}_\Sigma = \mathcal{A}_{\omega(\Sigma)},$$

where $\mathcal{A}_{\omega(\Sigma)}$ is the uniform (with respect to $\sigma \in \omega(\Sigma)$) trajectory attractor of the family $\{\mathcal{K}_\sigma^+ \mid \sigma \in \omega(\Sigma)\}$, $\mathcal{A}_{\omega(\Sigma)} \subseteq \mathcal{K}_{\omega(\Sigma)}^+$. The set $\mathcal{A}_\Sigma = \mathcal{A}_{\omega(\Sigma)}$ is compact in Θ_+ and bounded in $\mathcal{F}_+$.

The proof of Theorem 1.1 is given in [3] (see also [8]).

Remark 1.1. The introduced notion of a trajectory attractor depends on the selected trajectory space $\{\mathcal{K}_\sigma^+ \mid \sigma \in \Sigma\}$. Therefore, generally speaking, different trajectory attractors correspond to different families $\{\mathcal{K}_\sigma^+ \mid \sigma \in \Sigma\}$ of trajectory spaces of equation (1.1).

Theorem 1.1 shows that to construct the trajectory attractor one needs a uniformly attracting set P, compact in Θ_+ and bounded in $\mathcal{F}_+$. Usually, in applications, a large ball $B_R = \{\|f\|_{\mathcal{F}_+} \leq R\}$ in $\mathcal{F}_+$ $(R \gg 1)$ serves as such an attracting set (or even an absorbing set). Attraction to B_R follows from the inequality

$$\|T(t)u\|_{\mathcal{F}_+} \leq C(\|u\|_{\mathcal{F}_+})e^{-\gamma t} + R_0 \quad \forall t \geq 0 \tag{1.9}$$

for all $u \in \mathcal{K}_\Sigma^+$, where $C(R)$ depends on R, and γ, R_0 do not depend on u. Usually, inequality (1.9) follows from a priori estimates for a solution of equation (1.1). If, in addition, a ball B_R in $\mathcal{F}_+$ is compact in Θ_+ then B_{2R_0} is the required compact uniformly attracting set.

Corollary. *If $u \in \mathcal{A}_\Sigma$ then u is tr.-c. in Θ_+.*

Indeed, using (1.8), we get $T(t)u \in \mathcal{A}_\Sigma$ for any $t \geq 0$, that is, the set $(T(t)u \mid t \geq 0$ is compact in Θ_+, i.e. u is tr.-c. in Θ_+ (see Definition 1.1). $\square$

Now, we shall describe the structure of the trajectory attractors from Theorem 1.1 in terms of the complete trajectories of equation (1.1), i.e. solutions defined on the whole time axis. Let Σ be a compact symbol space, $\Sigma \Subset \Xi_+$, and suppose that the semigroup $\{T(t)\}$ is continuous on Σ. Consider any symbol $\sigma \in \omega(\Sigma)$. The invariance property (1.7) implies that there is a symbol $\sigma_1 \in \omega(\Sigma)$ such that $T(1)\sigma_1 = \sigma$. Consider the function $\widehat{\sigma}(s) = \sigma_1(s+1)$ defined for $s \geq -1$. Obviously, $\widehat{\sigma}(s) \equiv \sigma(s)$ for $s \geq 0$, hence, $\widehat{\sigma}$ is a prolongation of σ on the semiaxis $[-1, \infty[$. Next, there is $\sigma_2 \in \omega(\Sigma)$ such that $T(1)\sigma_2 = \sigma_1$ and $T(2)\sigma_2 = \sigma$. Put $\widehat{\sigma}(s) = \sigma_2(s+2)$ for $s \geq -2$. Evidently, the function $\widehat{\sigma}$ is well defined, since $\sigma_2(s+2) = \sigma_1(s+1)$ for $s \geq -1$. Continuing this process, we define

$\widehat{\sigma}(s) = \sigma_n(s+n)$ for $s \in [-n, \infty[$, where $\sigma_n \in \omega(\Sigma)$ and $n \in \mathbb{N}$. We have defined a function $\widehat{\sigma}$ on $\mathbb{R}$ which is a prolongation of the initial symbol σ defined on $\mathbb{R}_+$. Moreover, $\widehat{\sigma}$ has the following property: $\Pi_+\widehat{\sigma}_t \in \omega(\Sigma)$ for all $t \in \mathbb{R}$, where $\widehat{\sigma}_t(s) = \widehat{\sigma}(t+s)$. Here $\Pi_+ = \Pi_{0,\infty}$ is the restriction operator to the semiaxis $\mathbb{R}_+$.

Definition 1.8. A function $\zeta(s), s \in \mathbb{R}$, is said to be a *complete symbol in* $\omega(\Sigma)$ if $\Pi_+\zeta_t(\,\cdot\,) = \Pi_+\zeta(t+\,\cdot\,) \in \omega(\Sigma)$ for all $t \in \mathbb{R}$.

As shown above, for every symbol $\sigma \in \omega(\Sigma)$ there exists at least one complete symbol $\zeta = \widehat{\sigma}$ which is the prolongation of σ for negative s. Notice at once that, in general, this prolongation need not be unique.

Now consider some complete symbol $\zeta(s), s \in \mathbb{R}$, in $\omega(\Sigma)$. It is easily seen that to ζ there corresponds the family of operators $A_{\zeta(t)}(\,\cdot\,) : E \to E_0,\ A_{\zeta(t)}(\,\cdot\,) \equiv A_{\Pi_+\zeta_t}(\,\cdot\,),\ t \in \mathbb{R}$. Consider the corresponding evolution equation on the whole axis:

$$\partial_t u = A_{\zeta(t)}(u), \quad t \in \mathbb{R}. \tag{1.10}$$

Definition 1.9. A function $u(s), s \in \mathbb{R}$, is said to be a *complete trajectory* of equation (1.10) with complete symbol $\zeta(s), s \in \mathbb{R}$, (with respect to the family $\{\mathcal{K}_\sigma^+ \mid \sigma \in \Sigma\}$) if

$$\Pi_+u_t \in \mathcal{K}^+_{\Pi_+\zeta_t}, \quad \text{i.e.} \quad \Pi_+u(t+\,\cdot\,) \in \mathcal{K}^+_{\Pi_+\zeta(t+\,\cdot\,)} \quad \forall t \in \mathbb{R}.$$

Definition 1.10. The *kernel* $\mathcal{K}_\zeta$ of equation (1.10) in the space $\mathcal{F}_+$ with complete symbol $\zeta(s), s \in \mathbb{R}$, in $\omega(\Sigma)$ is the union of all complete trajectories $u(s), s \in \mathbb{R}$, of equation (1.10) (with respect to the family $\{\mathcal{K}_\sigma^+ \mid \sigma \in \Sigma\}$) that are bounded in $\mathcal{F}_+$:

$$\|\Pi_+u(t+\,\cdot\,)\|_{\mathcal{F}_+} \le C_u \quad \forall t \in \mathbb{R}. \tag{1.11}$$

Denote by $Z = Z(\Sigma)$ the set of all complete symbols in $\omega(\Sigma)$, $Z = \{\zeta(s), s \in \mathbb{R} \mid \Pi_+\zeta_t \in \omega(\Sigma)\ \forall t \in \mathbb{R}\}$. Evidently, $\Pi_+Z(\Sigma) = \omega(\Sigma)$. Let $\mathcal{K}_{Z(\Sigma)}$ denote the union of all kernels $\mathcal{K}_\zeta$ of all complete symbols $\zeta \in Z(\Sigma) : \mathcal{K}_{Z(\Sigma)} = \bigcup_{\zeta\in Z(\Sigma)} \mathcal{K}_\zeta$.

Theorem 1.2. *Let the conditions of Theorem* 1.1 *be valid. Then*

$$\mathcal{A}_\Sigma = \mathcal{A}_{\omega(\Sigma)} = \Pi_+\biggl(\bigcup_{\zeta\in Z(\Sigma)} \mathcal{K}_\zeta\biggr) = \Pi_+\mathcal{K}_{Z(\Sigma)}, \tag{1.12}$$

the set $\Pi_+\mathcal{K}_{Z(\Sigma)}$ *is compact in* Θ_+ *and bounded in* $\mathcal{F}_+$. *If the family* $\{\mathcal{K}_\sigma^+ \mid \sigma \in \Sigma\}$ *satisfies the condition: for some ball* B_R *in* $\mathcal{F}_+$ *the set* $B_R \cap \mathcal{K}_\sigma^+ \neq \emptyset$ *for all* $\sigma \in \Sigma$, *then* $\mathcal{K}_\zeta \neq \emptyset$ *for all* $\zeta \in Z(\Sigma)$.

The proof of Theorem 1.2 is given in [3] and it uses the invariance property (1.8) of the trajectory attractor $\mathcal{A}_\Sigma$.

2. Trajectory attractors of reaction-diffusion systems

In this section we study the reaction-diffusion system

$$\partial_t u = a\Delta u - f(u,t) + g(x,t), \quad u|_{\partial\Omega} = 0, \quad x \in \Omega \Subset \mathbb{R}^n,\ t \geq 0. \tag{2.13}$$

(Similarly one can study a problem with the Neumann boundary conditions.) In system (2.13), $u = (u^1, \dots, u^N)$ is an unknown vector-function, $g = g(x,s) = (g^1, \dots, g^N)$ are external forces, and $f = f(v,s) = (f^1, \dots, f^N)$ are interaction functions. The real $N \times N$ matrix a has a positive symmetric part $\frac{1}{2}(a + a^*) \geq \beta I,\ \beta > 0$. Set $H = (L_2(\Omega))^N$ and $V = (H_0^1(\Omega))^N$. The norms in these spaces are $|v|^2 = \int_\Omega \sum_{i=1}^N |v^i(x)|^2\, dx$ and $\|v\|^2 = \int_\Omega \sum_{i=1}^N |\nabla v^i(x)|^2\, dx$ respectively. Let also $V' = (H^{-1}(\Omega))^N$. We suppose that $g \in L_2^{\mathrm{loc}}(\mathbb{R}_+; V')$ and g is translation-bounded in $L_2^{\mathrm{loc}}(\mathbb{R}_+; V')$:

$$\|g\|_{\mathrm{a}}^2 = \sup_{t\geq 0} \int_t^{t+1} \|g(s)\|_{V'}^2\, ds < \infty. \tag{2.14}$$

We also suppose that $f \in C(\mathbb{R}^N \times \mathbb{R}_+; \mathbb{R}^N)$ and

$$\gamma |v|^p - C_1 \leq (f(v,s), v), \quad p \geq 2,\ \gamma > 0, \tag{2.15}$$

$$|f(v,s)| \leq C_2(|v|^{p-1} + 1) \quad \forall v \in \mathbb{R}^N,\ s \in \mathbb{R}_+. \tag{2.16}$$

Let q be the conjugate exponent for p, $1/p + 1/q = 1,\ 1 < q \leq 2$. Let $u(x,\,\cdot\,) \in L_p(t_1, t_2; (L_p(\Omega))^N)$, $[t_1, t_2] \subseteq \mathbb{R}_+$. It follows from (2.16) that $f(u(x,\,\cdot\,),\,\cdot\,) \in L_q(t_1, t_2; (L_q(\Omega))^N)$ and

$$\|f(u(x,\,\cdot\,),\,\cdot\,)\|^q_{L_q(t_1,t_2;(L_q(\Omega))^N)} \leq C_3(\|u(x,\,\cdot\,)\|^p_{L_p(t_1,t_2;(L_p(\Omega))^N)} + t_2 - t_1), \tag{2.17}$$

where $C_3 = C_3(p, |\Omega|, C_2)$. At the same time, if $u(x,\,\cdot\,) \in L_2(t_1, t_2; V)$ then $\Delta u(x,\,\cdot\,) + g(x,\,\cdot\,) \in L_2(t_1, t_2; V')$. The Sobolev embedding theorem implies, by passing to the conjugate spaces, that $L_q(\Omega) \subset H^{-r}(\Omega)$, where $r \geq n(1/q - 1/2)$. If, in addition, $r \geq 1$, then the right-hand side of (2.13) belongs to $L_q(t_1, t_2; (H^{-r}(\Omega))^N)$. Put $r \equiv \max\{1, n(1/q - 1/2)\}$ for $p \geq 2$. Let $X = (H^{-r}(\Omega))^N$. We conclude that if $u(x,\,\cdot\,) \in L_p(t_1, t_2; (L_p(\Omega))^N) \cap L_2(t_1, t_2; V)$ then equation (2.13) can be considered in the distribution sense of the space $D'(t_1, t_2; X)$ and $\partial_t u \in L_q(t_1, t_2; (H^{-r}(\Omega))^N)$.

Definition 2.1. A function $u(x,s)$, $x \in \Omega$, $s \geq 0$, is said to be a *weak solution* of equation (2.13) if $u \in L_p(t_1, t_2; (L_p(\Omega))^N) \cap L_2(t_1, t_2; V)$ and u satisfies equation (2.13) in the distribution sense of the space $D'(t_1, t_2; X)$, where $X = (H^{-r}(\Omega))^N$ and $r \equiv \max\{1, n(1/q - 1/2)\}$.

Lemma 2.1 (Lions–Magenes [12]). *Let X and Y be Banach spaces such that $Y \subset X$ with a continuous injection. If $f \in C([t_1,t_2];X)$ and $f \in L_\infty([t_1,t_2];Y)$ then f is weakly continuous on $[t_1,t_2]$ with values in Y, i.e. for every $\psi \in Y^*$ the function $\langle \psi, f(\,\cdot\,)\rangle \in C([t_1,t_2])$.*

If u is a weak solution of (2.13) then, evidently, $u \in C([t_1,t_2];X)$ for $X = (H^{-r}(\Omega))^N$. If, in addition, $u \in L_\infty(t_1,t_2;H)$ then, by Lemma 2.1 we have $u \in C_w([t_1,t_2];H)$. Therefore, the initial-value problem

$$u|_{t=t_1} = u_0 \tag{2.18}$$

for equation (2.13) is meaningful when $u_0 \in H$. Let us formulate the weak solution existence theorem.

Theorem 2.1. *Let $g \in L_2(t_1,t_2;V')$, let f satisfy conditions* (2.15), (2.16), *and $u_0 \in H$. Then there exists a weak solution u of equation* (2.13) *satisfying $u \in L_p(t_1,t_2;(L_p(\Omega))^N) \cap L_2(t_1,t_2;V) \cap L_\infty(t_1,t_2;H)$ and $u(t_1) = u_0$.*

Proof. We implement the Galerkin approximation method, with a complete system $\{w_j\}$ of functions in $V \cap (L_p(\Omega))^N$. We outline the main points of the method. Let $u_m(x,s) = \sum_{i=1}^m a_{j,m}(s) w_j(x)$ be a solution of the ordinary differential system

$$\frac{du_m}{dt} = P_m a\Delta u_m - P_m f(u_m,t) + P_m g(t), \quad u_m(t_1) = P_m u_0, \tag{2.19}$$

where P_m is the orthogonal projection in H onto $H_m = [w_1,\dots,w_m]$. Evidently, $P_m u_0 \to u_0$ $(m \to \infty)$ strongly in H and $P_m g \rightharpoonup g$ $(m \to \infty)$ weakly in $L_2(t_1,t_2;V')$. Equation (2.19) implies that

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + (a\nabla u_m(t), \nabla u_m(t)) + (f(u_m(t),t), u_m(t)) = \langle g(t), u_m(t)\rangle. \tag{2.20}$$

Using condition (2.15) and integrating in s from τ to t, we obtain

$$\begin{aligned} |u_m(t)|^2 - |u_m(\tau)|^2 + \beta\int_\tau^t \|u_m(s)\|^2\,ds + 2\gamma\int_\tau^t \|u_m(s)\|_{L_p}^p\,ds \\ \le \frac{1}{\beta}\int_\tau^t \|g(s)\|_{V'}^2\,ds + 2(t-\tau)C_2, \end{aligned} \tag{2.21}$$

for all $t,\tau \in [t_1,t_2]$, $t \ge \tau$, where $\|v\|_{L_p}^p = \|v\|_{(L_p(\Omega))^N}^p$. It follows from (2.21) that the sequence $\{u_m\}$ remains in a bounded subset of $L_2(t_1,t_2;V) \cap L_p(t_1,t_2;(L_p(\Omega))^N) \cap L_\infty(t_1,t_2;H)$, since $|u_m(t_1)|^2$ is bounded. By (2.17), $\{f(u_m(\,\cdot\,),\,\cdot\,)\}$ is bounded in $L_q(t_1,t_2;(L_q(\Omega))^N)$. So, by refining, we may assume that there exists a function $u \in L_2(t_1,t_2;V) \cap L_p(t_1,t_2;(L_p(\Omega))^N) \cap L_\infty(t_1,t_2;H)$ such that $u_m \rightharpoonup u$ $(m \to \infty)$ weakly in $L_2(t_1,t_2;V)$, weakly in $L_p(t_1,t_2;(L_p(\Omega))^N)$, and $*$-weakly in $L_\infty(t_1,t_2;H)$. In particular, $\Delta u_m \rightharpoonup \Delta u$ $(m \to \infty)$ weakly in $L_2(t_1,t_2;V')$, $\partial_t u_m \rightharpoonup \partial_t u$ $(m \to \infty)$ weakly in $L_q(t_1,t_2;X)$ and $f(u_m(\,\cdot\,),\,\cdot\,) \rightharpoonup$

w $(m \to \infty)$ weakly in $L_q(t_1, t_2; (L_q(\Omega))^N)$ for some $w \in L_q(t_1, t_2; (L_q(\Omega))^N)$. Passing to the limit in equation (2.19) we obtain the equality

$$\partial_t u = a\Delta u - w + g$$

in the distribution sense of the space $D'(t_1, t_2; X)$. Notice that $u_m \to u$ $(m \to \infty)$ in $C_w([t_1, t_2]; H)$, so that $u(t_1) = u_0$.

To prove that u is a weak solution of (2.13) we have to check that $f(u(s), s) = w(s)$. From (2.19) it follows that $\{\partial_t u_m\}$ is bounded in $L_q(t_1, t_2; X)$. Due to the compactness theorem (see [9], [11]), we extract a subsequence of $\{u_m\}$ (which we label the same) strongly convergent to u in $L_2(t_1, t_2; H)$ and $u_m(x, s) \to u(x, s)$ $(m \to \infty)$ for almost all $(x, s) \in \Omega \times [t_1, t_2]$. Therefore $f(u_m(x, s), s) \to f(u(x, s), s)$ $(m \to \infty)$ for almost all $(x, s) \in \Omega \times [t_1, t_2]$, since f is continuous. On the other hand, the sequence $\{f(u_m(x, \cdot), \cdot)\}$ is bounded in $L_q(t_1, t_2; L_q(\Omega))$. From the Lions lemma (see [11, Chapter 1, Lemma 1.3]), we conclude that $f(u_m(x, \cdot), \cdot) \rightharpoonup f(u(x, \cdot), \cdot)$ $(m \to \infty)$ weakly in $L_q(t_1, t_2; L_q(\Omega))$, hence, $f(u(s), s) = w(s)$. □

REMARK 2.1. It is easily seen that conditions (2.15) and (2.16) do not ensure the uniqueness for the Cauchy problem (2.13), (2.18).

PROPOSITION 2.1. *Let* $u \in L_p(t_1, t_2; (L_p(\Omega))^N) \cap L_2(t_1, t_2; V)$ *be a weak solution of* (2.13). *Then*

(i) $u \in C([t_1, t_2]; H)$;

(ii) *the function* $|u(\cdot)|^2$ *is absolutely continuous on* $[t_1, t_2]$, *and*

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + (a\nabla u(t), \nabla u(t)) + (f(u(t), t), u(t)) = \langle g(t), u(t)\rangle, \tag{2.22}$$

for almost all $t \in [t_1, t_2]$.

This proposition follows directly from

LEMMA 2.2. *Let* H *be a Hilbert space and let* V, E, X *be Banach spaces with* $V \subseteq H \subseteq V' \subseteq X$ *and* $E \subseteq H \subseteq E' \subseteq X$, V' *and* E' *being the duals of* V *and* E *respectively. Here* H' *is identified with* H. *Assume that* $u \in L_2(t_1, t_2; V) \cap L_p(t_1, t_2; E)$ $(p > 1)$ *and the distribution* $\partial_t u$ *from* $D'(t_1, t_2; X)$ *is representable as* $\partial_t u(s) = w(s) + h(s)$, *where* $w \in L_2(t_1, t_2; V')$ *and* $h \in L_q(t_1, t_2; E')$ $(1/p + 1/q = 1)$. *Then*

(i) $u \in C([t_1, t_2]; H)$;

(ii) *the function* $|u(\cdot)|^2$ *is absolutely continuous on* $[t_1, t_2]$, *and*

$$\frac{d}{dt}|u(t)|^2 = 2\langle \partial_t u(t), u(t)\rangle = 2\langle w(t), u(t)\rangle + 2\langle h(t), u(t)\rangle \tag{2.23}$$

for almost all $t \in [t_1, t_2]$.

This statement is a generalization of the known interpolation result. (See [12] and [16, Chapter 3, Lemma 1.2], where the author considered the case $V = E$, $p = 2$, V being a Hilbert space.) The proof uses the regularization technique.

To prove Proposition 2.1 we use Lemma 2.2 for $E = (L_p(\Omega))^N$. If $u \in L_p(t_1, t_2; (L_p(\Omega))^N) \cap L_2(t_1, t_2; V)$ then, by equation (2.13), $\partial_t u = w + h$, where $w(s) = a\Delta u(s) + g(s)$ and $h(s) = f(u(s), s)$. Evidently, $w \in L_2(t_1, t_2; V')$ and $h \in L_q(t_1, t_2; E')$ (see (2.17)). So $u \in C_w([t_1, t_2]; H)$ and $|u(\cdot)|$ is continuous on $[t_1, t_2]$, consequently, $u \in C([t_1, t_2]; H)$.

Corollary 2.1. *Let $u \in L_p^{\mathrm{loc}}(\mathbb{R}_+; (L_p(\Omega))^N) \cap L_2^{\mathrm{loc}}(\mathbb{R}_+; V)$ be a weak solution of* (2.13). *Then for all $t \geq 0$,*

$$|u(t)|^2 \leq |u(0)|^2 e^{-\alpha t} + R_1^2, \quad R_1^2 = \frac{2C_1}{\alpha} + \frac{1}{\beta(1 - e^{-\alpha})}\|g\|_{\mathrm{a}}^2, \ \alpha = \beta\lambda_1, \tag{2.24}$$

$$\beta \int_t^{t+1} \|u(s)\|^2 \, ds + 2\gamma \int_t^{t+1} \|u(s)\|_{L_p}^p \, ds \leq |u(t)|^2 + R_2^2, \qquad R_2^2 = 2C_1 + \frac{1}{\beta}\|g\|_{\mathrm{a}}^2, \tag{2.25}$$

$$\beta \int_0^t \|u(s)\|^2 e^{\alpha s} \, ds \leq (1 + \alpha t)|u(0)|^2 + 2R_1^2 e^{\alpha t}. \tag{2.26}$$

Proof. Every weak solution u satisfies (2.22). Using condition (2.15) and the Bouniakovsky–Schwarz inequality, we get

$$\frac{d}{dt}|u(t)|^2 + \beta\|u(t)\|^2 + 2\gamma\|u(t)\|_{L_p}^p \leq 2C_1 + \frac{1}{\beta}\|g(t)\|_{V'}^2. \tag{2.27}$$

In particular,

$$\frac{d}{dt}|u(t)|^2 + \alpha|u(t)|^2 \leq 2C_1 + \frac{1}{\beta}\|g(t)\|_{V'}^2,$$

and hence

$$\frac{d}{dt}(|u(t)|^2 \exp(\alpha t)) \leq 2C_1 \exp(\alpha t) + \frac{1}{\beta}\|g(t)\|_{V'}^2 \exp(\alpha t).$$

Integrating from 0 to t, we get

$$|u(t)|^2 e^{\alpha t} - |u(0)|^2 \leq \frac{2C_1}{\alpha}(e^{\alpha t} - 1) + \frac{1}{\beta}\int_0^t \|g(s)\|_{V'}^2 e^{\alpha s} \, ds.$$

Estimating the last integral,

$$\begin{aligned}(2.28)\quad \int_0^t \|g(s)\|_{V'}^2 e^{\alpha s} ds &\le \int_{t-1}^t \|g(s)\|_{V'}^2 e^{\alpha s}\, ds + \int_{t-2}^{t-1} \|g(s)\|_{V'}^2 e^{\alpha s}\, ds + \dots \\ &\le e^{\alpha t}\int_{t-1}^t \|g(s)\|_{V'}^2\, ds + e^{\alpha(t-1)}\int_{t-2}^{t-1} \|g(s)\|_{V'}^2\, ds \\ &\quad + e^{\alpha(t-2)}\int_{t-3}^{t-2} \|g(s)\|_{V'}^2\, ds + \dots \\ &\le \|g\|_{\mathrm{a}}^2 e^{\alpha t}(1 + e^{-\alpha} + e^{-2\alpha} + \dots) \\ &= \|g\|_{\mathrm{a}}^2 e^{\alpha t}(1 - e^{-\alpha})^{-1},\end{aligned}$$

we get

$$|u(t)|^2 e^{\alpha t} - |u(0)|^2 \le \frac{2C_1}{\alpha} e^{\alpha t} + \frac{1}{\beta}\|g\|_{\mathrm{a}}^2 e^{\alpha t}(1 - e^{-\alpha})^{-1} = R_1^2 e^{\alpha t}$$

and (2.24) is proved. Inequality (2.25) follows directly from (2.27) by integrating over $[t, t+1]$.

Let us check inequality (2.26). We shall use this inequality in the next section. Multiplying (2.24) by $\alpha e^{\alpha t}$ and integrating, we obtain

$$(2.29)\qquad \alpha\int_0^t |u(s)|^2 e^{\alpha s}\, ds \le \alpha t|u(0)|^2 + R_1^2 e^{\alpha t}.$$

Inequality (2.27) implies that

$$\begin{aligned}&\frac{d}{dt}(|u(t)|^2 \exp(\alpha t)) + \beta\|u(t)\|^2 \exp(\alpha t) \\ &\qquad \le 2C_1 \exp(\alpha t) + \frac{1}{\beta}\|g(t)\|_{V'}^2 \exp(\alpha t) + \alpha|u(t)|^2 \exp(\alpha t).\end{aligned}$$

Therefore, using the above reasoning and (2.29), we get

$$\begin{aligned}|u(t)|^2 e^{\alpha t} + \beta\int_0^t \|u(s)\|^2 e^{\alpha s}\, ds &\le |u(0)|^2 + R_1^2 e^{\alpha t} + \alpha t|u(0)|^2 + R_1^2 e^{\alpha t} \\ &= (1 + \alpha t)|u(0)|^2 + 2R_1^2 e^{\alpha t}.\end{aligned}$$

The proof is complete. □

The pair $\sigma(s) = (f(v,s), g(x,s))$ is the symbol of equation (2.13). With each symbol σ satisfying (2.14)–(2.16), we associate the trajectory space $\mathcal{K}_\sigma^+$. By definition, $\mathcal{K}_\sigma^+$ is the union of all weak solutions $u \in L_p^{\mathrm{loc}}(\mathbb{R}_+; (L_p(\Omega))^N) \cap L_2^{\mathrm{loc}}(\mathbb{R}_+; V)$ of equation (2.13).

Corollary 2.2.

(i) *For every* $u_0 \in H$ *there exists* $u \in \mathcal{K}_\sigma^+$ *such that* $u(0) = u_0$.
(ii) $u \in L_p^{\mathrm{loc}}(\mathbb{R}_+; (L_p(\Omega))^N) \cap L_2^{\mathrm{loc}}(\mathbb{R}_+; V) \cap L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H)$ *for all* $u \in \mathcal{K}_\sigma^+$.

According to the general scheme of Section 1, we introduce spaces Θ_+ and $\mathcal{F}_+$. Put

$$\begin{aligned}\mathcal{F}_+^{\mathrm{loc}} &= L_\infty^{\mathrm{loc}}(\mathbb{R}_+;H)\cap L_2^{\mathrm{loc}}(\mathbb{R}_+;V)\cap L_p^{\mathrm{loc}}(\mathbb{R}_+;(L_p(\Omega))^N)\\ &\qquad \cap\{v\mid \partial_t v\in\ L_q^{\mathrm{loc}}(\mathbb{R}_+;X)\},\\ X &= (H^{-r}(\Omega))^N,\quad r=\max\{1,\ n(1/2-1/p)\}.\end{aligned}$$

Denote by Θ_+ the space $\mathcal{F}_+^{\mathrm{loc}}$ with the following convergence topology: a sequence $\{v_m\}\subset\mathcal{F}_+^{\mathrm{loc}}$ converges to $v\in\mathcal{F}_+^{\mathrm{loc}}$ as $m\to\infty$ in Θ_+ if $v_m\rightharpoonup v$ $(m\to\infty)$ $*$-weakly in $L_\infty(t_1,t_2;H)$, weakly in $L_2(t_1,t_2;V)$, weakly in $L_p(t_1,t_2;(L_p(\Omega))^N)$, and $\partial_t v_m\rightharpoonup\partial_t v$ $(m\to\infty)$ weakly in $L_q(t_1,t_2;X)$ for all $[t_1,t_2]\subseteq\mathbb{R}_+$. It is easy to prove that Θ_+ is a Hausdorff space with a countable base. The translation semigroup $\{T(t)\}$ is continuous on Θ_+.

Now, define $\mathcal{F}_+$ by putting

$$\mathcal{F}_+=\{v\in\mathcal{F}_+^{\mathrm{loc}}\mid \|v\|_{\mathcal{F}_+}<\infty\},$$

where

$$\begin{aligned}\|v\|_{\mathcal{F}_+} = \sup_{t\ge0}\{&\|T(t)v\|_{L_\infty(0,1;H)}+\|T(t)v\|_{L_2(0,1;V)}\\ &+\|T(t)v\|_{L_p(0,1;(L_p(\Omega))^N)}+\|T(t)\partial_t v\|_{L_q(0,1;X)}\}.\end{aligned}$$

Evidently, $\mathcal{F}_+$ is a Banach space and $\mathcal{F}_+\subset\Theta_+$.

To describe a symbol space Σ suppose we are given a fixed symbol $\sigma_0(s)=(f_0(v,s),g_0(x,s))$.

The function g_0 is translation-bounded in $L_2^{\mathrm{loc}}(\mathbb{R}_+;V')$ (see (2.14)). Therefore, it is tr.-c. in the space $L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+;V')$ with the following local weak convergence topology: $h_m\to h$ $(m\to\infty)$ in $L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+;V')$ if $h_m\rightharpoonup h$ $(m\to\infty)$ weakly in $L_2(t_1,t_2;V')$ for all $[t_1,t_2]\subseteq\mathbb{R}_+$. The hull $\mathcal{H}_+(g_0)$ (see (1.14)) is a compact set in $L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+;V')$.

The function f_0 satisfies (2.15) and (2.16). Consider the space $\mathcal{M}_1=\{\psi\in C(\mathbb{R}^N;\mathbb{R}^N)\mid |\psi(v)|\le C_2(|v|^{p-1}+1)\ \forall v\in\mathbb{R}^N\}$ endowed with the following local uniform convergence topology: $\psi^{(m)}\to\psi$ $(m\to\infty)$ in $\mathcal{M}_1$ if

$$\max_{|v|\le R}|\psi^{(m)}(v)-\psi(v)|\to0\quad(m\to\infty)$$

for each $R>0$. This topology is generated by the norm

$$\|\psi\|_{\mathcal{M}_1}=\sum_{m=1}^\infty\frac{1}{m^{p+1}}\max_{|v|\le m}|\psi(v)|. \tag{2.30}$$

$\mathcal{M}_1$ is a Banach space. Consider the space $C(\mathbb{R}_+;\mathcal{M}_1)$ of continuous functions with values in $\mathcal{M}_1$. We assume that f_0 is a tr.-c. function in $C(\mathbb{R}_+;\mathcal{M}_1)$. By (2.16), f_0 is bounded in every semicylinder $Q_+(R)=\{(v,s)\mid |v|\le R,\ s\ge0\}$.

By [3], f_0 is tr.-c. in $C(\mathbb{R}_+;\mathcal{M}_1)$ if and only if for each $R>0$ it is bounded and uniformly continuous in each $Q_+(R)$, i.e.

$$(2.31)\quad |f_0(v_1,s_1)-f_0(v_2,s_2)|\le \alpha_1(|v_1-v_2|+|s_1-s_2|,R)\\ \forall(v_1,s_1),(v_2,s_2)\in Q_+(R);\ \alpha_1(s,R)\to 0+\quad (s\to 0+).$$

The hull $\mathcal{H}_+(f_0)$ is a compact set in $C(\mathbb{R}_+;\mathcal{M}_1)$. Evidently, the symbol $\sigma_0(s)=(f_0(v,s),g_0(x,s))$ is a tr.-c. function in $\Xi_+=C(\mathbb{R}_+,\mathcal{M}_1)\times L^{\mathrm{loc}}_{2,w}(\mathbb{R}_+;V')$.

Now define the symbol space Σ of equation (2.13) to be $\mathcal{H}_+(\sigma_0)$, where $\mathcal{H}_+(\sigma_0)$ is the hull of the function σ_0 in Ξ_+. It can be shown that the space $\mathcal{H}_+(\sigma_0)$ is metrizable. Therefore, $\Sigma=\mathcal{H}_+(\sigma_0)$ is a compact metric space. The translation semigroup $\{T(t)\}$ acts continuously on Σ.

PROPOSITION 2.2. *For all symbols $\sigma(s)=(f(v,s),g(x,s))\in\Sigma=\mathcal{H}_+(\sigma_0)$,*

(i) $\|g\|_{\mathrm{a}}^2=\sup_{t\ge 0}\int_t^{t+1}\|g(s)\|_{V'}^2\,ds\le\|g_0\|_{\mathrm{a}}^2$;

(ii) $f(v,s)$ *satisfies conditions* (2.15), (2.16), *and* (2.31) *with the same constants and with the same function* $\alpha_1(s,R)$.

The proof is straightforward.

To each symbol $\sigma\in\mathcal{H}_+(\sigma_0)$ there corresponds the trajectory space $\mathcal{K}_\sigma^+$. Evidently, the family $\{\mathcal{K}_\sigma^+\mid\sigma\in\Sigma\}$ is tr.-coord. Define $\mathcal{K}_\Sigma^+=\bigcup_{\sigma\in\Sigma}\mathcal{K}_\sigma^+$.

Let $\omega(\Sigma)$ be the global attractor of the semigroup $\{T(t)\}$ on $\Sigma=\mathcal{H}_+(\sigma_0)$ (see Proposition 1.2). Let $Z(\sigma_0):=Z(\Sigma)$ be the set of all complete symbols in Σ, i.e. the set of functions $\xi\in\Xi=C(\mathbb{R},\mathcal{M}_1)\times L^{\mathrm{loc}}_{2,w}(\mathbb{R};V')$ such that $\zeta_t\in\omega(\Sigma)$ for any $t\in\mathbb{R}$, where $\zeta_t(s)=\Pi_+\zeta(s+t), s\ge 0$. To each complete symbol $\zeta(s)=(h(v,s),r(x,s))\in Z(\sigma_0)$ there corresponds, by Definition 1.9, the kernel $\mathcal{K}_\zeta$ of equation (2.13). $\mathcal{K}_\zeta$ consists of all weak solutions $u(s), s\in\mathbb{R}$, of the equation

$$\partial_t u=\Delta u-h(u,t)+r(x,t),\quad t\in\mathbb{R},$$

that are bounded in the space $\mathcal{F}$ with the norm

$$\|v\|_{\mathcal{F}}=\sup_{t\in\mathbb{R}}\{\|T(t)v\|_{L_\infty(0,1;H)}+\|T(t)v\|_{L_2(0,1;V)}\\ +\|T(t)v\|_{L_p(0,1;(L_p(\Omega))^N)}+\|T(t)\partial_t v\|_{L_q(0,1;X)}\}.$$

Here $T(t)$ is the translation operator for $t\in\mathbb{R}$. Let us formulate the main theorem on trajectory attractors of equation (2.13).

THEOREM 2.2. *Let $\sigma_0(s)=(f_0(v,s),g_0(x,s))$, $s\in\mathbb{R}_+$, where g_0 is translation-bounded in $L_2^{\mathrm{loc}}(\mathbb{R}_+;V')$ and f_0 satisfies conditions* (2.14)–(2.16) *and* (2.31). *Let $\Sigma=\mathcal{H}_+(\sigma_0)$ be the symbol space of equation* (2.13). *Then the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}_\Sigma^+=\bigcup_{\sigma\in\Sigma}\mathcal{K}_\sigma^+$ has a uniform* (*with respect to $\sigma\in\mathcal{H}_+(\sigma_0)$*) *trajectory attractor $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$* (*in the topology Θ_+; bounded sets are* *taken in the Banach space $\mathcal{F}_+$). The set $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ is bounded in $\mathcal{F}_+$, compact in Θ_+, and*

$$\mathcal{A}_{\mathcal{H}_+(\sigma_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(\sigma_0))} = \Pi_+\Big(\bigcup_{\zeta\in Z(\sigma_0)} \mathcal{K}_\zeta\Big) = \Pi_+\mathcal{K}_{Z(\sigma_0)}.$$

The kernel $\mathcal{K}_\zeta$ is not empty for any $\zeta \in Z(\sigma_0)$. The set $\mathcal{K}_{Z(\sigma_0)}$ is bounded in $\mathcal{F}$.

Notice that the following embedding is continuous: $\Theta_+ \subset L_2^{\mathrm{loc}}(\mathbb{R}_+; H_{1-\delta})$, $0 < \delta \le 1$, so we get

Corollary 2.3. *For each set $B \subset \mathcal{K}_\Sigma^+$ bounded in $\mathcal{F}_+$,*

$$\operatorname{dist}_{L_2(0,R;H_{1-\delta})}(\Pi_{0,R}T(t)B, \Pi_{0,R}\mathcal{K}_{Z(\sigma_0)}) \to 0 \quad (t\to\infty)$$

for all R, where $\Pi_{0,R}$ is the restriction operator to the segment $[0,R]$.

Proof of Theorem 2.2. To apply Theorems 1.1 and 1.2 to the family $\{\mathcal{K}_\sigma^+ \mid \sigma \in \Sigma\}$ of trajectory spaces, we have to prove Propositions 2.3 and 2.4 below. By (2.32), the translation semigroup $\{T(t)\}$, acting on $\mathcal{K}_\Sigma^+$, has a uniformly (with respect to $\sigma \in \Sigma$) absorbing set $P = B_R = \{v \in \mathcal{F}_+ \mid \|v\|_{\mathcal{F}_+} \le R\}$ for the family $\{\mathcal{K}_g^+ \mid g \in \mathcal{H}_+(g_0)\}$ whenever $R > R_0$. The ball B_R is compact in Θ_+ and bounded in $\mathcal{F}_+$. This completes the proof. $\square$

Proposition 2.3. *If σ_0 satisfies conditions* (2.14)–(2.16) *and* (2.31) *then*

(i) *$\mathcal{K}_\sigma^+ \subseteq \mathcal{F}_+$ for all $\sigma \in \mathcal{H}_+(\sigma_0)$;*
(ii) *for all $u \in \mathcal{K}_\sigma^+$,*

$$\|T(t)u\|_{\mathcal{F}_+}^2 \le C_4|u(0)|^2\exp(-\alpha t) + R_0^2 \quad \forall t \ge 0, \tag{2.32}$$

where $\alpha = \beta\lambda_1$, C_4 depends on a, γ, C_3, and R_0 depends on α, C_1, and $\|g_0\|_{\mathrm{a}}^2$.

Proof. (i) follows from Corollary 2.2 and (2.32). It follows from (2.24) that

$$\|T(t)u\|_{L_\infty(\mathbb{R}_+;H)}^2 \le |u(0)|^2\exp(-\alpha t) + R_{01}^2 \quad \forall t \ge 0, \tag{2.33}$$

where $R_{01}^2 = 2C_1/\alpha + \|g_0\|_{\mathrm{a}}^2/(\beta(1-e^{-\alpha}))$. Using (2.25) and (2.33), we obtain

$$\beta\|T(t)u\|_{L_2(0,1;V)}^2 + 2\gamma\|T(t)u\|_{L_p(0,1;(L_p(\Omega))^N)}^p \le |u(0)|^2\exp(-\alpha t) + R_{02}^2, \tag{2.34}$$

where $R_{02}^2 = R_{01}^2 + 2C_1 + \beta^{-1}\|g_0\|_{\mathrm{a}}^2$. Taking into account (2.17), from equation (2.13) we get

$$\begin{aligned}
\bigg(\int_t^{t+1} |\partial_t u(s)|_X^q \, ds\bigg)^{1/q} \\
&\le \bigg(\int_t^{t+1} |a\Delta u(s)|_X^q \, ds\bigg)^{1/q} + \bigg(\int_t^{t+1} |f(u(s),s)|_X^q \, ds\bigg)^{1/q} \\
&\quad + \bigg(\int_t^{t+1} |g(s)|_X^q \, ds\bigg)^{1/q} \\
&\le C_5\bigg(\int_t^{t+1} \|u(s)\|^2 \, ds\bigg)^{1/2} + \bigg(C_3\int_t^{t+1} \|u(s)\|_{L_p}^p \, ds + 1\bigg)^{1/q} \\
&\quad + \bigg(\int_t^{t+1} |g(s)|_{V'}^2 \, ds\bigg)^{1/2} \\
&\le C_5(\beta^{-1}|u(0)|^2 e^{-\alpha t} + \beta^{-1}R_{02}^2)^{1/2} \\
&\quad + (C_3(2\gamma)^{-1}|u(0)|^2 e^{-\alpha t} + C_3(2\gamma)^{-1}R_{02}^2 + 1)^{1/q} \\
&\le C_6|u(0)|^2 e^{-\alpha t} + R_3^2,
\end{aligned}$$

since $1 < q \le 2$. Thus,

$$\|T(t)\partial_t u\|_{L_q(\mathbb{R}_+;X)} \le C_6|u(0)|^2 e^{-\alpha t} + R_3^2. \tag{2.35}$$

Finally, inequalities (2.33)–(2.35) imply (2.32). □

Proposition 2.4. *The family $\{\mathcal{K}_\sigma^+ \mid \sigma \in \Sigma\}$ of trajectory spaces is (Θ_+, Σ)-closed.*

Proof. Let $\sigma_m(s) = (f_m(v,s), g_m(x,s)) \in \Sigma$, $u_m \in \mathcal{K}_{\sigma_m}^+$, and $u_m \to u$ $(m \to \infty)$ in Θ_+, $f_m \to f$ $(m \to \infty)$ in $C(\mathbb{R}_+;\mathcal{M}_1)$, and $g_m \to g$ $(m \to \infty)$ in $L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+;V')$. The functions u_m satisfy the equations

$$\partial_t u_m = a\Delta u_m - f_m(u_m,t) + g_m. \tag{2.36}$$

Fix any interval $[t_1,t_2] \subseteq \mathbb{R}_+$. Passing to a subsequence, we may assume that $u_m \to u$ $(m \to \infty)$ strongly in $L_2(t_1,t_2;H)$, $u_m(x,s) \to u(x,s)$ $(m \to \infty)$ for almost all $(x,s) \in \Omega \times [t_1,t_2]$, and $f_m(u_m(\,\cdot\,),\,\cdot\,) \rightharpoonup w$ $(m \to \infty)$ weakly in $L_q(t_1,t_2;L_q(\Omega))$ (see the proof of Theorem 2.1). Passing to the limit in equation (2.36) we find that

$$\partial_t u = a\Delta u - w + g. \tag{2.37}$$

It remains to show that $w(s) = f(u(x,s),s)$. To do this, similarly to the proof of Theorem 2.1, we have to check that $f_m(u_m(x,s),s) \to f(u(x,s),s)$ $(m \to \infty)$ for almost all $(x,s) \in \Omega \times [t_1,t_2]$. Let (x,s) be a point such that $u_m(x,s) \to$

$u(x,s)$ $(m\to\infty)$ in $\mathbb{R}^N$. The set $\{u_m(x,s)\}$ is bounded in $\mathbb{R}^N$, i.e. $|u_m(x,s)|\le R$ for all $m\in\mathbb{N}$ and for some R. But

$$\begin{aligned}&|f_m(u_m(x,s),s)-f(u(x,s),s)|\\&\qquad\le\max_{|v|\le\mathbb{R}}|f_m(v,s)-f(v,s)|+\alpha_1(|u_m(x,s)-u(x,s)|,R)\to 0\quad(m\to\infty)\end{aligned}$$

(see (2.31)). Hence, $f_m(u_m(x,s),s)\to f(u(x,s),s)$ $(m\to\infty)$ for almost all $(x,s)\in\Omega\times[t_1,t_2]$. Therefore $u\in\mathcal{K}_\sigma^{t_1,t_2}$ for all $[t_1,t_2]\subseteq\mathbb{R}_+$, so that $u\in\mathcal{K}_\sigma^+$. $\square$

3. Reaction-diffusion systems with uniqueness. More regular trajectory attractors

In this section we study the reaction-diffusion systems for which the Cauchy problem (2.13), (2.18) has a unique solution under some regularity conditions. The results of Section 1 are also applicable in this case. The trajectory attractor $\mathcal{A}_\Sigma$ has a more regular structure and it attracts bounded sets in a stronger topology Θ_+^s than Θ_+ described in Section 2.

Consider once more the system (2.13). Now, we assume that the function g is translation-bounded in $L_2^{\rm loc}(\mathbb{R}_+;H)$:

$$|g|_{\rm a}^2=\sup_{t\ge0}\int_t^{t+1}|g(s)|^2\,ds<\infty. \tag{3.38}$$

For the vector-function f, we suppose, besides (2.15) and (2.16), that $f'_{v^j}\in C(\mathbb{R}^N\times\mathbb{R}_+;\mathbb{R}^N)$ for $j=1,\dots,N$ and

$$-C_7|w|^2\le(f_v(v,s)w,w); \tag{3.39}$$

$$|f_v(v,s)|\le C_8(|v|^{p-2}+1)\quad\forall v,w\in\mathbb{R}^N,s\in\mathbb{R}_+. \tag{3.40}$$

Theorem 3.1. *Let $g\in L_2(t_1,t_2;V')$ and f satisfy conditions* (2.15), (2.16), *and* (3.39). *Then a weak solution $u\in L_p(t_1,t_2;(L_p(\Omega))^N)\cap L_2(t_1,t_2;V)$ of the Cauchy problem* (2.13), (2.18) *is unique.*

Proof. We follow the proof from [1] where the autonomous case of system (2.13) was studied. Suppose there are two weak solutions $u_1,u_2\in L_p(t_1,t_2;(L_p(\Omega))^N)\cap L_2(t_1,t_2;V)$ such that $u_1(t_1)=u_2(t_1)=u_0$. The difference $w(s)=u_1(s)-u_2(s)$ satisfies

$$\partial_t w=a\Delta w-(f(u_1,t)-f(u_2,t)),\quad w(t_1)=0.$$

Since $w\in L_p(t_1,t_2;(L_p(\Omega))^N)\cap L_2(t_1,t_2;V)$ and $\partial_t w\in L_q(t_1,t_2;(L_q(\Omega))^N)+L_2(t_1,t_2;V')$, Lemma 2.2 is applicable and we have

$$\frac12\frac{d}{dt}|w(t)|^2+(a\nabla w(t),\nabla w(t))=-(f(u_1,t)-f(u_2,t),w(t))$$

for almost all $t \in [t_1, t_2]$, and the function $|w(\,\cdot\,)|^2$ is absolutely continuous on $[t_1, t_2]$. From (3.39) it follows that $(f(v_1,t)-f(v_2,t), v_1-v_2) \geq -C_7|v_1-v_2|^2$ for all $v_1, v_2 \in \mathbb{R}^N$ and $t \geq 0$. Consequently, $-(f(u_1,t)-f(u_2,t), w(t)) \leq C_7|w(t)|^2$ and, hence,

$$\frac{d}{dt}|w(t)|^2 \leq 2C_7|w(t)|^2, \quad \frac{d}{dt}(|w(t)|^2 \exp(-C_7 t)) \leq 0,$$

because $(a\nabla w(t), \nabla w(t)) \geq 0$. Finally, we get $|w(t)|^2 \leq \exp(C_7 t)|w(0)|^2 = 0$, and so $w(t) = u_1(t) - u_2(t) \equiv 0$ for all $t \in [t_1, t_2]$. $\square$

Proposition 3.1. *Suppose that $g \in L_2^{\mathrm{loc}}(\mathbb{R}_+; H)$ satisfies* (3.38) *and f satisfies* (2.15), (2.16), *and* (3.39). *Then for every weak solution $u \in L_p^{\mathrm{loc}}(\mathbb{R}_+; (L_p(\Omega))^N) \cap L_2^{\mathrm{loc}}(\mathbb{R}_+; V)$ of* (2.13) *the following inequalities hold for $t > 0$:*

$$\|u(t)\|^2 \leq (t^{-1} + 1 + t)C_9|u(0)|^2 e^{-\alpha t} + (t^{-1}+1)R_4^2, \tag{3.41}$$

$$\int_t^{t+1} |\Delta u(s)|^2\, ds \leq (t^{-1} + 1 + t)C_{10}|u(0)|^2 e^{-\alpha t} + (t^{-1}+1)R_5^2. \tag{3.42}$$

Proof. It is sufficient to prove (3.41) and (3.42) for every solution $u_m = u$ of the Galerkin approximation system (2.19), since the corresponding exact solution is unique and we can pass to the limit as $m \to \infty$ in the resulting inequality. So, we multiply (2.19) by $-\Delta u_m(s)$ and integrate over $x \in \Omega$. After standard transformations we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \beta|\Delta u(t)|^2 \leq (f(u(t),t), \Delta u(t)) - (g(t), \Delta u(t)). \tag{3.43}$$

Here we omit the index m for brevity.

Without loss of generality, we may assume that $f(0,t) = 0$ for all $t \geq 0$. Otherwise we can replace $f(v,s)$ and $g(x,s)$ by $\widetilde{f}(v,s) = f(v,s) - f(0,s)$ and $\widetilde{g}(v,s) = g(v,s) - f(0,s)$ respectively. The functions $\widetilde{f}$ and $\widetilde{g}$ satisfy the same conditions with modified constants C_i, because $|f(0,s)| \leq C_2$ for all $s \geq 0$ (see (2.16)). Using this assumption, we can integrate by parts in the term $(f(u(t),t), \Delta u(t))$ since $f(u(t),t)|_{\partial\Omega} = 0$:

$$\begin{aligned}(f(u,t),\Delta u) &= \sum_{k=1}^{n}\sum_{i=1}^{N}\int_\Omega f^i(u,t)\frac{\partial^2 u^i}{\partial x_k^2}\,dx \\ &= \sum_{k=1}^{n}\sum_{i=1}^{N}\sum_{j=1}^{N} -\int_\Omega \frac{\partial f^i(u,t)}{\partial u_j}\frac{\partial u^j}{\partial x_k}\frac{\partial u^i}{\partial x_k}\,dx \\ &= \sum_{k=1}^{n} -\int_\Omega \left(f_v(u,t)\frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial x_k}\right)dx \\ &\leq C_7\sum_{k=1}^{n}\left|\frac{\partial u}{\partial x_k}\right|^2 = C_7\|u\|^2.\end{aligned} \tag{3.44}$$

We have used condition (3.39). For the second term $-(g(t),\Delta u(t))$ we have

$$-(g(t),\Delta u(t)) \leq \frac{\beta}{2}|\Delta u(t)|^2+\frac{1}{2\beta}|g(t)|^2.$$

Therefore, by (3.43) and (3.44), we obtain

$$\frac{d}{dt}\|u(t)\|^2+\beta|\Delta u(t)|^2 \leq 2C_7\|u(t)\|^2+\beta^{-1}|g(t)|^2 \tag{3.45}$$

and

$$\frac{d}{dt}\|u(t)\|^2+\alpha\|u(t)\|^2 \leq 2C_7\|u(t)\|^2+\beta^{-1}|g(t)|^2.$$

Multiplying by $te^{\alpha t}$, we get

$$\frac{d}{dt}(t\|u(t)\|^2e^{\alpha t}) \leq (2C_7t+1)\|u(t)\|^2e^{\alpha t}+\beta^{-1}t|g(t)|^2e^{\alpha t}. \tag{3.46}$$

Integrating from 0 to t, we obtain

$$t\|u(t)\|^2e^{\alpha t} \leq (2C_7t+1)\int_0^t \|u(s)\|^2e^{\alpha s}\,ds+\beta^{-1}t\int_0^t |g(s)|^2e^{\alpha s}\,ds.$$

Now, using estimates (2.26) and (2.28), we have

$$\begin{aligned} t\|u(t)\|^2e^{\alpha t} &\leq (2C_7t+1)\beta^{-1}[(1+\alpha t)|u(0)|^2+2R_1^2e^{\alpha t}] \\ &\qquad +\beta^{-1}t|g|_{\mathrm{a}}^2(1-e^{-\alpha})^{-1}e^{\alpha t} \\ &\leq (1+t+t^2)C_9|u(0)|^2+(1+t)R_4^2e^{\alpha t}. \end{aligned} \tag{3.47}$$

Therefore, estimate (3.41) is proved.

Finally, integrating (3.45) over $[t,t+1]$, we deduce

$$\beta\int_t^{t+1} |\Delta u(s)|^2\,ds \leq \|u(t)\|^2+2C_7\int_t^{t+1}\|u(s)\|^2\,ds+\beta^{-1}\int_t^{t+1}|g(s)|^2\,ds.$$

Combining (2.25), (2.24), and (3.47) we get

$$\begin{aligned} \beta\int_t^{t+1}|\Delta u(s)|^2\,ds &\leq \beta((t^{-1}+1+t)C_9|u(0)|^2e^{-\alpha t}+(t^{-1}+1)R_4^2) \\ &\qquad +2C_7\beta^{-1}(|u(0)|^2e^{-\alpha t}+R_1^2+R_2^2)+\beta^{-1}|g|_{\mathrm{a}}^2 \\ &\leq (t^{-1}+1+t)C_{10}|u(0)|^2e^{-\alpha t}+(t^{-1}+1)R_5^2. \end{aligned}$$

Thus, estimate (3.42) is valid. □

In this section we assume that the symbol $\sigma(s)=(f(v,s),g(s))$ of equation (2.13) satisfies more regularity conditions: (2.15), (2.16) and (3.38)–(3.40). As before, $\mathcal{K}_\sigma^+$ denotes the set of all weak solutions $u\in L_2^{\mathrm{loc}}(\mathbb{R}_+;V)\cap L_p^{\mathrm{loc}}(\mathbb{R}_+;(L_p(\Omega))^N)$ of equation (2.13).

Let $H_2=(H^2(\Omega))^N$. It is well known that the norm in H_2 on the subspace $H_2\cap V$ is equivalent to the norm $\|u\|_2=|\Delta u|$.

Corollary 3.1. *If the symbol $\sigma(s) = (f(v,s), g(s))$ satisfies the above conditions of regularity then $T(t)u \in L_\infty^{\mathrm{loc}}(\mathbb{R}_+; V) \cap L_2^{\mathrm{loc}}(\mathbb{R}_+; H_2)$ and $T(t)\partial_t u \in L_q^{\mathrm{loc}}(\mathbb{R}_+; (L_q(\Omega))^N)$ for all $u \in \mathcal{K}_\sigma^+$ when $t > 0$.*

Indeed, by Proposition 3.1, $T(t)u \in L_\infty^{\mathrm{loc}}(\mathbb{R}_+; V) \cap L_2^{\mathrm{loc}}(\mathbb{R}_+; H_2)$, i.e. $T(t)(a\Delta u + g) \in L_2^{\mathrm{loc}}(\mathbb{R}_+; (L_2(\Omega))^N)$ for $t > 0$. By (2.17), $f(u,s) \in L_q^{\mathrm{loc}}(\mathbb{R}_+; (L_q(\Omega))^N)$. Hence, $T(t)\partial_t u \in L_q^{\mathrm{loc}}(\mathbb{R}_+; (L_q(\Omega))^N)$ for $t > 0$.

Therefore we can study the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}_\Sigma^+$ in a space with a stronger topology. Denote by Θ_+^s the space $L_\infty^{\mathrm{loc}}(\mathbb{R}_+; V) \cap L_2^{\mathrm{loc}}(\mathbb{R}_+; H_2) \cap L_p^{\mathrm{loc}}(\mathbb{R}_+; (L_p(\Omega))^N) \cap \{v \mid \partial_t v \in L_q^{\mathrm{loc}}(\mathbb{R}_+; (L_q(\Omega))^N)\}$ with the following convergence topology: a sequence $\{v_m\}$ converges to v in Θ_+^s as $m \to \infty$ if $v_m \rightharpoonup v$ $(m \to \infty)$ $*$-weakly in $L_\infty(t_1, t_2; V)$, weakly in $L_2(t_1, t_2; H_2)$, weakly in $L_p(t_1, t_2; (L_p(\Omega))^N)$, and $\partial_t v_m \rightharpoonup \partial_t v$ $(m \to \infty)$ weakly in $L_q(t_1, t_2; (L_q(\Omega))^N)$ for every $[t_1, t_2] \subseteq \mathbb{R}_+$. It is easy to prove that Θ_+^s is a Hausdorff space with a countable base. We also define $\mathcal{F}_+^s$ by setting

$$\mathcal{F}_+^s = \{v \in \Theta_+^s \mid \|v\|_{\mathcal{F}_+^s} < \infty\},$$

where

$$\begin{aligned}\|v\|_{\mathcal{F}_+^s} = \sup_{t \ge 0} \{ & \|T(t)v\|_{L_\infty(0,1;V)} + \|T(t)v\|_{L_2(0,1;H_2)} \\ & + \|T(t)v\|_{L_p(0,1;(L_p(\Omega))^N)} + \|T(t)\partial_t v\|_{L_q(0,1;(L_q(\Omega))^N)}\}.\end{aligned}$$

Evidently, $\mathcal{F}_+^s$ is a Banach space.

As before, we are given a fixed symbol $\sigma_0(s) = (f_0(v,s), g_0(x,s))$.

The function g_0 is translation-bounded in $L_2^{\mathrm{loc}}(\mathbb{R}_+; H)$ (see (3.38)). Therefore, it is tr.-c. in $L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+; H)$.

The function f_0 satisfies (2.15), (2.16), (3.39), and (3.40). Consider the space $\mathcal{M}_2 = \{\psi \in C^1(\mathbb{R}^N; \mathbb{R}^N) \mid |\psi(v)| \le C_2(|v|^{p-1} + 1), |\psi_v(v)| \le C_8(|v|^{p-2} + 1)\ \forall v \in \mathbb{R}^N\}$ endowed with the following local uniform convergence topology: $\psi^{(m)} \to \psi$ $(m \to \infty)$ in $\mathcal{M}_2$ if

$$\max_{|v| \le R} (|\psi^{(m)}(v) - \psi(v)| + |\psi_v^{(m)}(v) - \psi_v(v)|) \to 0 \quad (m \to \infty)$$

for each $R > 0$. This topology is generated by the norm

$$\|\psi\|_{\mathcal{M}_2} = \sum_{m=1}^\infty \left(\frac{1}{m^{p+1}} \max_{|v| \le m} |\psi(v)| + \frac{1}{m^p} \max_{|v| \le m} |\psi_v(v)| \right). \tag{3.48}$$

Thus $\mathcal{M}_2$ is a Banach space. Consider the space $C(\mathbb{R}_+; \mathcal{M}_2)$ of continuous functions with values in $\mathcal{M}_2$. We assume that f_0 is a tr.-c. function in $C(\mathbb{R}_+; \mathcal{M}_2)$. By (3.40), the function f_{0v} is bounded in each semicylinder $Q_+(R) = \{(v,s) \mid |v| \le R,\ s \ge 0\}$. By [3], f_0 is tr.-c. in $C(\mathbb{R}_+; \mathcal{M}_2)$ if and only if for any $R > 0$

the functions f_0 and f_{0v} are bounded and uniformly continuous in each $Q_+(R)$, i.e.

$$
\begin{aligned}
(3.49)\qquad &|f_0(v_1,s_1)-f_0(v_2,s_2)|+|f_{0v}(v_1,s_1)-f_{0v}(v_2,s_2)| \\
&\qquad\qquad \le \alpha_2(|v_1-v_2|+|s_1-s_2|,R) \\
&\forall (v_1,s_1),(v_2,s_2)\in Q_+(R);\ \alpha_2(s,R)\to 0+\ (s\to 0+).
\end{aligned}
$$

Evidently, the symbol $\sigma_0(s)=(f_0(v,s),g_0(x,s))$ is a tr.-c. function in $\Xi_+^s = C(\mathbb{R}_+,\mathcal{M}_2)\times L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+;H)$.

Now define the symbol space Σ of equation (2.13) to be $\mathcal{H}_+(\sigma_0)$, the hull of σ_0 in Ξ_+^s. It is easily shown that the hull of a function in Ξ_+^s coincides with its hull in $\Xi_+ = C(\mathbb{R}_+;\mathcal{M}_1)\times L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+;V')$. Moreover, the topology on $\mathcal{H}_+(\sigma_0)$ is the same in Ξ_+ and in Ξ_+^s.

Proposition 3.2. *For all symbols $\sigma(s)=(f(v,s),g(x,s))\in\Sigma=\mathcal{H}_+(\sigma_0)$,*

(i) $|g|_{\mathrm{a}}^2=\sup_{t\ge 0}\int_t^{t+1}|g(s)|^2\,ds\le |g_0|_{\mathrm{a}}^2$;
(ii) *f satisfies conditions* (3.39), (3.40), *and* (3.49) *with the same constants and with the same function $\alpha_2(s,R)$.*

Now consider the family $\{\mathcal{K}_\sigma^+\mid\sigma\in\Sigma\}$ of trajectory spaces corresponding to equation (2.13). Thanks to Proposition 3.1, the translation semigroup $\{T(t)\}$, acting on $\mathcal{K}_\Sigma^+$, has a uniformly (with respect to $\sigma\in\Sigma$) absorbing set $P'=B_R=\{v\in\mathcal{F}_+^s\mid \|v(\cdot)\|_{\mathcal{F}_+^s}\le R\}$ for the family $\{\mathcal{K}_\sigma^+\mid\sigma\in\mathcal{H}_+(\sigma_0)\}$, whenever $R\gg 1$. The ball B_R is compact in Θ_+^s and bounded in $\mathcal{F}_+^s$. Hence we can formulate the following result.

Theorem 3.2. *Let the function $\sigma_0(s)=(f_0(v,s),g_0(x,s))$ satisfy conditions* (2.15), (2.16), (3.38)–(3.40), *and* (3.49). *Then the trajectory attractor $\mathcal{A}_\Sigma$ of the family of equations* (2.13) *for $\sigma\in\Sigma=\mathcal{H}_+(\sigma_0)$ from Theorem* 2.2 *is also a uniform* (*with respect to $\sigma\in\mathcal{H}_+(\sigma_0)$*) *trajectory attractor in the topology Θ_+^s. The set $\mathcal{A}_\Sigma$ is compact in Θ_+^s and bounded in $\mathcal{F}_+^s$.*

The following embedding is continuous: $\Theta_+^s\subset L_2^{\mathrm{loc}}(\mathbb{R}_+;H_{2-\delta})$, $0<\delta\le 2$, so we get

Corollary 3.2. *For every set $B\subset\mathcal{K}_\Sigma^+$ bounded in $\mathcal{F}_+$,*

$$
\operatorname{dist}_{L_2(0,R;H_{2-\delta})}(\Pi_{0,R}T(t)B,\Pi_{0,R}\mathcal{K}_{Z(\sigma_0)})\to 0\quad (t\to\infty)
$$

for all $R>0$.

4. Trajectory attractors for ordinary differential equations

In this section, we briefly describe how the methods of Section 1 can be applied to the study of non-autonomous ordinary differential equations. We do not suppose the unique solvability of the corresponding Cauchy problem. As an example of such equations, we consider the Galerkin approximation system (2.19) of order m for the reaction-diffusion system (2.13). It contains $M = mN$ ordinary differential equations. This is why we shall study the following type of ordinary differential equations in $\mathbb{R}^M$:

$$\frac{dy}{dt} = -F(y,t) + \varphi(t), \quad t \geq 0. \tag{4.50}$$

Here $y = (y^1, \dots, y^M),\ F = (F^1, \dots, F^M),\ \varphi = (\varphi^1, \dots, \varphi^M)$. We assume that $\varphi \in L_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^M)$ and $F \in C(\mathbb{R}^M \times \mathbb{R}_+; \mathbb{R}^M)$. Suppose also that

$$(F(z,s), z) = \sum_{i=1}^{M} F^i(z,s) z^i \geq \delta |z|^2 - C \quad \forall z \in \mathbb{R}^M,\ s \geq 0, \tag{4.51}$$

where $\delta > 0$. The pair $\sigma(s) = (F(z,s), \varphi(s))$ is the symbol of equation (4.50). We are looking for solutions y of (4.50) such that $y \in H^{1,\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^M) = \{z \mid z' \in L_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^M)\}$. It is classical that for any $y_0 \in \mathbb{R}^M$ equation (4.50) has a solution $y \in H^{1,\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^M)$ such that $y(0) = y_0$. (Condition (4.51) provides prolongation of every local solution $y \in H^1([0, t_0[; \mathbb{R}^M)$ on the whole $\mathbb{R}_+$.) Evidently, this solution need not be unique (we do not suppose any Lipschitz conditions for F with respect to z).

Denote by $\mathcal{K}_\sigma^+$ the set of all solutions $y \in H^{1,\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^M)$ of (4.50). We shall study the trajectory attractor of the translation semigroup $\{T(t)\}$ acting on the union $\mathcal{K}_\Sigma^+$ of trajectory spaces, where the symbol space Σ is described below.

Let $\sigma_0 = (F_0, \varphi_0) \in C(\mathbb{R}^M \times \mathbb{R}_+; \mathbb{R}^M) \times L_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^M)$ be some initial symbol. Assume that φ_0 is tr.-c. in $L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^M)$, i.e.

$$|\varphi_0|_{\mathrm{a}}^2 = \sup_{t \geq 0} \int_t^{t+1} |\varphi_0(s)|_{\mathbb{R}^M}^2 \, ds < \infty. \tag{4.52}$$

Suppose also that F_0 satisfies (4.51) and it is tr.-c. in $C(\mathbb{R}_+; \mathcal{M}_0)$, i.e. F_0 is bounded and uniformly continuous on every semicylinder $Q_+(R) = \{(z,s) \mid |z|_{\mathbb{R}^M} \leq R,\ s \geq 0\}$:

$$\begin{aligned} &|F_0(z,s)| \leq C_1(R) \quad \forall (z,s) \in Q_+(R); \\ &|F_0(z_1,s_1) - F_0(z_2,s_2)| \leq \alpha_0(|z_1 - z_2| + |s_1 - s_2|, R) \\ &\qquad\qquad \forall (z_1,s_1),(z_2,s_2) \in Q_+(R);\ \alpha_0(s,R) \to 0+\ (s \to 0+). \end{aligned} \tag{4.53}$$

Here $\mathcal{M}_0 = C(\mathbb{R}^M; \mathbb{R}^M)$ is endowed with the following local uniform convergence topology: $\Psi^{(m)} \to \Psi$ $(m \to \infty)$ in $\mathcal{M}_0$ if $\max_{|v| \leq R} |\Psi^{(m)}(v) - \Psi(v)| \to 0$ $(m \to \infty)$ for all $R > 0$.

Evidently $\sigma_0 = (F_0, \varphi_0)$ is tr.-c. in $\Xi_+ = C(\mathbb{R}_+; \mathcal{M}_0) \times L_{2,w}^{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^M)$. Put, as usual, $\Sigma = \mathcal{H}_+(\sigma_0) = [\{\sigma_0(\,\cdot\, + h) \mid h \geq 0\}]_{\Xi_+}$. The space Σ is metrizable and compact in Ξ_+.

Proposition 4.1. *For all symbols* $\sigma(s) = (F(z,s), \varphi(s)) \in \Sigma = \mathcal{H}_+(\sigma_0)$,

(i) $|\varphi|_{\mathrm{a}}^2 = \sup_{t\ge 0} \int_t^{t+1} |\varphi(s)|^2\,ds \le |\varphi_0|_{\mathrm{a}}^2$;

(ii) F *satisfies conditions* (4.51) *and* (4.53).

Consider the family $\{\mathcal{K}_\sigma^+ \mid \sigma \in \Sigma\}$ of trajectory spaces corresponding to equation (4.50). Evidently, this family is tr.-coord. Define $\mathcal{K}_\Sigma^+ = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma^+$.

As an enveloping space Θ_+, we consider the space $H_w^{1,\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^M)$ with the local weak convergence topology: $z_m \to z$ $(m \to \infty)$ in Θ_+ if $z_m \rightharpoonup z$ $(m\to\infty)$ weakly in $H^1([t_1,t_2];\mathbb{R}^M)$ for every $[t_1,t_2] \subset \mathbb{R}_+$. Let also

$$\mathcal{F}_+ = \{z \in H^{1,\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^M) \mid \|z\|_{\mathcal{F}_+} = \sup_{t\ge 0}(|T(t)z|_{L_2(0,1)} + |T(t)z'|_{L_2(0,1)}) < \infty\}.$$

Proposition 4.2. *If* σ_0 *satisfies the above conditions then*

(i) $\mathcal{K}_\sigma^+ \subset \mathcal{F}_+$ *for all* $\sigma \in \mathcal{H}_+(\sigma_0)$;

(ii) *for every* $y \in \mathcal{K}_\sigma^+$,

$$|y(t)|^2 \le |y(0)|^2 e^{-\delta t} + R_1^2, \quad R_1^2 = \frac{2C}{\delta} + \frac{1}{\delta(1-e^{-\delta})}|g_0|_{\mathrm{a}}^2;$$

$$\bigg(\int_t^{t+1} |y'(s)|^2\,ds\bigg)^{1/2} \le C_1((|y(0)|^2 e^{-\delta t} + R_1^2)^{1/2}) + |g_0|_{\mathrm{a}} \quad \forall t \ge 0.$$

Consequently,

$$\|T(t)y(\,\cdot\,)\|_{\mathcal{F}_+} \le C_2(|y(0)|^2 e^{-\delta t}) + R_0 \quad \forall t\ge 0.$$

Proposition 4.3. *The family* $\{\mathcal{K}_\sigma^+ \mid \sigma\in\Sigma\}$ *of trajectory spaces is* (Θ_+,Σ)-*closed.*

We omit the proofs of Propositions 4.2 and 4.3 as they are similar to the proofs of Propositions 2.3 and 2.4.

Let $\omega(\Sigma)$ be the global attractor of the semigroup $\{T(t)\}$ on $\Sigma = \mathcal{H}_+(\sigma_0)$. Let $Z(\sigma_0) := Z(\Sigma)$ be the set of all complete symbols in Σ, i.e. the set of functions $\zeta \in \Xi = C(\mathbb{R};\mathcal{M}_0) \times L_{2,w}^{\mathrm{loc}}(\mathbb{R};\mathbb{R}^M)$ such that $\zeta_t \in \omega(\Sigma)$ for all $t \in \mathbb{R}$, where $\zeta_t(s) = \Pi_+\zeta(s+t), s \ge 0$. To any complete symbol $\zeta(s) = (H(z,s),\Phi(s)) \in Z(\sigma_0)$, there corresponds, by Definition 1.10, the kernel $\mathcal{K}_\zeta$ of equation (2.13). It consists of all solutions $y(s), s\in\mathbb{R}$, of the equation

$$\frac{dy}{dt} = -H(y,t) + \Phi(t), \quad t \in \mathbb{R},$$

that are bounded in the space $\mathcal{F}$ with the norm

$$\|z\|_{\mathcal{F}} = \sup_{t\in\mathbb{R}}(|T(t)z|_{L_2(0,1;\mathbb{R}^M)} + |T(t)z'|_{L_2(0,1;\mathbb{R}^M)}).$$

Let us formulate the main theorem on the trajectory attractor of equation (4.50).

Theorem 4.1. *Let $\sigma_0(s) = (F_0(z,s), \varphi_0(s)), s \in \mathbb{R}_+$, where φ_0 is translation-bounded in $L_2^{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^M)$ and satisfies* (4.51), *and F_0 satisfies* (4.53). *Then the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}_\Sigma^+ = \bigcup_{\sigma\in\Sigma}\mathcal{K}_\sigma^+$ has a uniform (with respect to $\sigma \in \mathcal{H}_+(\sigma_0)$) trajectory attractor $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ (in the topology Θ_+; bounded sets are taken in the Banach space $\mathcal{F}_+$). The set $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ is bounded in $\mathcal{F}_+$, compact in Θ_+, and*

$$\mathcal{A}_{\mathcal{H}_+(\sigma_0)} = \mathcal{A}_{\omega(\mathcal{H}_+(\sigma_0))} = \Pi_+\Bigg(\bigcup_{\zeta\in Z(\sigma_0)}\mathcal{K}_\zeta\Bigg) = \Pi_+\mathcal{K}_{Z(\sigma_0)}.$$

The kernel $\mathcal{K}_\zeta$ is not empty for any $\zeta \in Z(\sigma_0)$. The set $\mathcal{K}_{Z(\sigma_0)}$ is bounded in $\mathcal{F}$.

Notice that the following embedding is continuous: $\Theta_+ \subset C(\mathbb{R}_+;\mathbb{R}^M)$, so we get

Corollary 4.1. *For every set $B \subset \mathcal{K}_\Sigma^+$ bounded in $\mathcal{F}_+$,*

$$\mathrm{dist}_{C(0,R;\mathbb{R}^M)}(\Pi_{0,R}T(t)B, \Pi_{0,R}\mathcal{K}_{Z(\sigma_0)}) \to 0 \quad (t\to\infty),$$

for all $R > 0$.

Theorem 4.1 follows directly from Propositions 4.1–4.3 (see Theorems 1.1 and 1.3).

Remark 4.1. The Galerkin approximation system (2.19) of m equations satisfies all the conditions of Theorem 4.1.

Remark 4.2. Consider a more regular case: $\varphi \equiv 0,\ \Sigma = \mathcal{H}_+(F_0)$. Then every solution y of equation (4.50) belongs to $C^1(\mathbb{R}_+;\mathbb{R}^M)$, i.e. $\mathcal{K}_\sigma^+ \subset C^1(\mathbb{R}_+;\mathbb{R}^M)$. Consider the topological space $\Theta_+^s = C^1(\mathbb{R}_+;\mathbb{R}^M)$ with the local uniform convergence topology: $z_m \to z$ $(m\to\infty)$ in Θ_+^s if $\max_{s\in[t_1,t_2]}\{|z_m(s)-z(s)| + |z_m'(s)-z'(s)|\} \to 0$ $(m\to\infty)$ for every segment $[t_1,t_2] \subset \mathbb{R}_+$. It can be proved that for every set $B \subset \mathcal{K}_\Sigma^+$ bounded in $\mathcal{F}_+$,

$$\mathrm{dist}_{C^1(0,R;\mathbb{R}^M)}(\Pi_{0,R}T(t)B, \Pi_{0,R}\mathcal{K}_{Z(F_0)}) \to 0 \quad (t\to\infty),$$

and the set $\Pi_+\mathcal{K}_{Z(F_0)}$ is compact in $\Theta_+^s = C^1(\mathbb{R}_+;\mathbb{R}^M)$ and bounded in $\mathcal{F}_+^s$. The norm in $\mathcal{F}_+^s$ is $\|z\|_{\mathcal{F}_+^s} = \sup_{t\geq 0}|T(t)z|_{C^1(0,1;\mathbb{R}^M)}$.

5. Some applications to the perturbation theory of trajectory attractors

Below we study some perturbation and approximation problems for trajectory attractors of reaction-diffusion systems considered in Sections 2 and 3. We prove that trajectory attractors are stable with respect to small perturbations of symbols of equations. In some cases, it is shown that perturbations do not affect the trajectory attractors. We investigate the convergence of the trajectory attractors of Galerkin approximation systems to the trajectory attractor of the original reaction-diffusion system.

1. Consider system (2.13) with a perturbed external force g_0+g_1 and with a perturbed interaction function f_0+f_1. We assume that both g_0 and g_1 are tr.-c. in $L^{\rm loc}_{2,w}(\mathbb{R}_+;V')$, i.e. inequalities (2.14) hold. The functions f_0 and f_1 satisfy conditions (2.15), (2.16), and they are tr.-c. in $C(\mathbb{R}_+;\mathcal{M}_1)$, i.e. conditions (2.31) are valid. Moreover, we suppose that

$$T(h)g_1(x,s)=g_1(x,s+h)\to 0 \quad (h\to\infty) \quad \text{in } L^{\rm loc}_{2,w}(\mathbb{R}_+;V'); \tag{5.54}$$

$$T(h)f_1(v,s)=f_1(v,s+h)\to 0 \quad (h\to\infty) \quad \text{in } C(\mathbb{R}_+;\mathcal{M}_1). \tag{5.55}$$

(See Section 2.) Put $\sigma_0=(f_0,g_0)$, $\sigma_1=(f_1,g_1)$.

Theorem 5.1. *Under the above conditions, the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(\sigma_0+\sigma_1)}$ of the perturbed reaction-diffusion system* (2.13) *coincides with the trajectory attractor $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ of the unperturbed system:*

$$\mathcal{A}_{\mathcal{H}_+(\sigma_0+\sigma_1)}=\mathcal{A}_{\mathcal{H}_+(\sigma_0)}. \tag{5.56}$$

The proof follows from formula (1.12) because $\omega(\mathcal{H}_+(\sigma_0+\sigma_1))=\omega(\mathcal{H}_+(\sigma_0))$ by (5.54) and (5.55):

$$\mathcal{A}_{\mathcal{H}_+(\sigma_0+\sigma_1)}=\mathcal{A}_{\omega(\mathcal{H}_+(\sigma_0+\sigma_1))}=\mathcal{A}_{\omega(\mathcal{H}_+(\sigma_0))}=\mathcal{A}_{\mathcal{H}_+(\sigma_0)}.$$

As an example, consider the disturbing external force $g_1(x,t)=G\sin t^2$, where $G(x)\in H$. Evidently, $G(x)\sin(t+h)^2\rightharpoonup 0$ $(h\to\infty)$ weakly in $L^{\rm loc}_{2,w}(t_1,t_2;H)$ for every $[t_1,t_2]\subseteq\mathbb{R}_+$ and (5.54) holds. Let also $f_1(v,s)=\alpha(s)F_1(v)$, where $\alpha(s)\to 0$ $(s\to\infty)$ and $F_1(v)$ satisfies conditions (2.15) and (2.16).

2. Consider system (2.13) with a symbol $\sigma^0(s,\varepsilon)=(f_0(v,s)+\varepsilon f_1(v,s), g_0(x,s)+\varepsilon g_1(x,s))$, where f_i and g_i $(i=1,2)$ satisfy (2.14)–(2.16) and (2.31). To construct the trajectory attractor for (2.13) with symbol $\sigma^0(s,\varepsilon)$, we study the family of equations (2.13) with symbols $\sigma(s,\varepsilon)=\sigma(\varepsilon)\in\Sigma(\varepsilon)=\mathcal{H}_+(\sigma_0)+\varepsilon\mathcal{H}_+(\sigma_1)$, where $\sigma_0(s)=(f_0(v,s),g_0(x,s))$, $\sigma_1(s)=(f_1(v,s),g_1(x,s))$. The hulls $\mathcal{H}_+(\sigma_i)$ are taken in the space $\Xi_+=C(\mathbb{R}_+;\mathcal{M}_1)\times L^{\rm loc}_{2,w}(\mathbb{R}_+;V')$. According to Theorem 2.2, the translation semigroup $\{T(t)\}$, acting on the united trajectory space $\mathcal{K}^+(\varepsilon)=\mathcal{K}^+_{\Sigma(\varepsilon)}=\bigcup_{\sigma(\varepsilon)\in\Sigma(\varepsilon)}\mathcal{K}^+_{\sigma(\varepsilon)}$, has the trajectory attractor $\mathcal{A}_{\Sigma(\varepsilon)}$ in the topology Θ_+ which was described in Section 2. The following statement generalizes Theorem 2.2.

Theorem 5.2. *Assume that the symbol $\sigma^0(s,\varepsilon)=(f_0(v,s)+\varepsilon f_1(v,s), g_0(x,s)+\varepsilon g_1(x,s))$ satisfies the above conditions. Then the semigroup $\{S(t)\mid t\ge 0\}$ acting on $\bigcup_{|\varepsilon|\le\varepsilon_0}\mathcal{K}^+(\varepsilon)\times\{\varepsilon\}$ by the formula*

$$S(t)(u_{\sigma(\varepsilon)}(s),\varepsilon)=(u_{\sigma(\varepsilon)}(s+t),\varepsilon),$$

where $u_{\sigma(\varepsilon)}(s) \in \mathcal{K}^+_{\sigma(\varepsilon)}$, *has a global attractor* $\mathcal{A}$ *with the following properties:*

(i) $\mathcal{A}$ *is compact in* $\Theta_+ \times \{|\varepsilon| \le \varepsilon_0\}$;

(ii) $\mathcal{A} = \bigcup_{|\varepsilon| \le \varepsilon_0} \mathcal{A}_{\Sigma(\varepsilon)} \times \{\varepsilon\}$, *where* $\mathcal{A}_{\Sigma(\varepsilon)}$ *is the trajectory attractor of the family of equations* (2.13) *with symbols* $\sigma \in \Sigma(\varepsilon)$;

(iii) *the trajectory attractors* $\mathcal{A}_{\Sigma(\varepsilon)}$ *converge to the trajectory attractor* $\mathcal{A}_{\Sigma(0)} = \mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ *as* $\varepsilon \to 0$ *in the topology* Θ_+. *In particular,*

$$\operatorname{dist}_{L_2(0,R;H_{1-\delta})}(\Pi_{0,R}\mathcal{A}_{\Sigma(\varepsilon)}, \Pi_{0,R}\mathcal{A}_{\mathcal{H}_+(\sigma_0)}) \to 0 \quad (\varepsilon \to 0) \quad \forall R > 0,\ 0 < \delta \le 1.$$

The proof is similar to one given in [1].

Using the results of Section 3 one can formulate the analog of Theorem 5.2 when the initial symbol $\sigma^0(s,\varepsilon) = (f_0(v,s) + \varepsilon f_1(v,s), g_0(x,s) + \varepsilon g_1(x,s))$ satisfies more regularity conditions: (2.15), (2.16), (3.38)–(3.40), (3.49), and the corresponding Cauchy problem is uniquely solvable. In this case, everywhere in Theorem 5.2, one can replace the topological space Θ_+ by a space Θ_+^s with a stronger topology. One can also combine these statements in the following way. Suppose that a regular symbol $\sigma_0(s) = (f_0(v,s), g_0(x,s))$ is perturbed by a non-regular one $\sigma_1(s) = (f_1(v,s), g_1(x,s))$. Then the $\mathcal{A}_{\Sigma(\varepsilon)}$ converge to $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ in Θ_+, and $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ is compact in Θ_+^s and it has a more regular structure.

3. Now fix some symbol $\sigma_0(s) = (f_0(v,s), g_0(x,s))$ that satisfies the general conditions (2.14)–(2.16), (2.31), and consider the corresponding trajectory attractor $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ in Θ_+. Suppose we are given some complete system $\{w_j\}$ of functions in $V \cap (L_p(\Omega))^N$. Let P_m be the orthogonal projection from H onto the space $H_m = [w_1, \dots, w_m]$. Consider the Galerkin approximation system (2.19) of order m. It follows easily that this system of ordinary differential equations satisfies conditions (4.51)–(4.53). Therefore, the results of Section 4 apply and the Galerkin system has a trajectory attractor $\mathcal{A}^{(m)}$ in the space $L^{\text{loc}}_{\infty,w*}(\mathbb{R}_+; H) \cap L^{\text{loc}}_{2,w}(\mathbb{R}_+; V) \cap L^{\text{loc}}_{p,w}(\mathbb{R}_+; (L_p(\Omega))^N) \cap \{v \mid \partial_t v \in L^{\text{loc}}_{q,w}(\mathbb{R}_+; X)\} = \Theta_+$, $X = (H^{-r}(\Omega))^N$.

Theorem 5.3. *The trajectory attractors* $\mathcal{A}^{(m)}$ *of the Galerkin approximation system* (2.19) *converge as* $m \to \infty$ (*in* Θ_+) *to the trajectory attractor* $\mathcal{A}_{\mathcal{H}_+(\sigma_0)}$ *of the system* (2.13). *In particular,*

$$\operatorname{dist}_{L_2(0,R;H_{1-\delta})}(\Pi_{0,R}\mathcal{A}^{(m)}, \Pi_{0,R}\mathcal{A}_{\mathcal{H}_+(\sigma_0)}) \to 0 \quad (\varepsilon \to 0) \quad \forall R > 0,\ 0 < \delta \le 1.$$

The proof is standard.

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