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MULTIPLICITY OF SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS

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Dedicated to Louis Nirenberg on the occasion of his 70th birthday

Introduction

The semilinear elliptic equation

$$-\Delta u = g(x,u)$$

has been the object of several studies in the last twenty years. For instance, let us mention the well-known result proved by Ambrosetti and Rabinowitz (cf. [1]): if g is superlinear and odd with respect to the second variable, then the above equation has a sequence of solutions $u_h \in H_0^1(\Omega)$ with $\|u_h\|_{H_0^1} \to \infty$.

In order to get such a result, a variational technique has been employed. In fact, the above equation is the Euler equation associated with the functional

$$f(u) = \frac{1}{2}\int_\Omega |Du|^2\,dx - \int_\Omega G(x,u)\,dx,$$

where $G(x,s) = \int_0^s g(x,t)\,dt$.

Now, let us deal with the quasilinear elliptic equation

$$(1)\qquad -\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_iu) + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_ju = g(x,u),$$

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where $a_{ij}(x,s) = a_{ji}(x,s)$. Classical critical point theory fails in this case. In fact, consider the associated functional $f : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$f(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j u \, dx - \int_\Omega G(x,u)\, dx.$$

Under reasonable assumptions on a_{ij} and g, it is possible to prove that f is continuous and that for every $u \in H_0^1(\Omega)$ and $v \in C_0^\infty(\Omega)$,

$$\begin{aligned} f'(u)(v) &:= \lim_{t\to 0} \frac{f(u+tv)-f(u)}{t} \\ &= \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j v \, dx \\ &\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u v\, dx - \int_\Omega g(x,u) v\, dx. \end{aligned}$$

Under natural hypotheses, it is also possible to have $a_{ij}(x,u) \in L^\infty(\Omega)$, $(\partial a_{ij}/\partial s)(x,u) \in L^\infty(\Omega)$, and $g(x,u) \in L^1_{\mathrm{loc}}(\Omega) \cap H^{-1}(\Omega)$ for every $u \in H_0^1(\Omega)$.

Now, if f is locally lipschitzian, we must have, for every $u \in H_0^1(\Omega)$,

$$\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u \in L^1_{\mathrm{loc}}(\Omega) \cap H^{-1}(\Omega).$$

This seems to be possible only if a_{ij} is independent of s or $n = 1$.

From now on, let us concentrate our attention on the case $n \geq 2$.

Quasilinear elliptic equations like (1) have been studied, by means of different techniques, in [4], [5], [7]–[9], [13], [17]. However, a hypothesis is assumed, which implies an a priori bound on the solutions with respect to the H_0^1-norm. Therefore, the superlinear case seems not to be treatable by that approach. On the contrary, there are few applications of techniques of critical point theory to (1). We are only aware of [18], where a nonlinear eigenvalue problem is treated[1].

The aim of our paper is to prove the existence of infinitely many solutions for (1) under suitable symmetry assumptions. The main tool for the proof is the nonsmooth critical point theory as developed in [11] and [12]. In fact, we reduce the problem to finding "critical" (in a suitable sense) points of the functional f and then we apply a symmetric mountain-pass theorem for continuous functionals.

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[1]After the completion of this paper, we learned about [2] where the existence of one nontrivial solution for certain quasilinear equations is proved.

1. Nonsmooth critical point theory

Let us begin by recalling from [11] and [12] some notions and results of nonsmooth critical point theory.

In the following, X will denote a metric space endowed with the metric d.

DEFINITION 1.1. Let $f : X \to \mathbb{R}$ be a continuous function and let $u \in X$. We denote by $|df|(u)$ the supremum of the σ's in $[0, \infty[$ such that there exist $\delta > 0$ and a continuous map

$$H : B(u, \delta) \times [0, \delta] \to X$$

such that

$$\forall \nu \in B(u, \delta),\ \forall t \in [0, \delta]: \quad d(H(\nu, t), \nu) \le t,$$
$$\forall \nu \in B(u, \delta),\ \forall t \in [0, \delta]: \quad f(H(\nu, t)) \le f(\nu) - \sigma t.$$

The extended real number $|df|(u)$ is called the *weak slope* of f at u.

Let us point out that the above notion has been independently introduced also in [15], while a similar notion can be found in [14].

The following two definitions are related to the previous notion.

DEFINITION 1.2. Let $f : X \to \mathbb{R}$ be a continuous function. A point $u \in X$ is said to be (*lower*) *critical* for f if $|df|(u) = 0$. A real number c is said to be a (*lower*) *critical value* for f if there exists $u \in X$ such that $|df|(u) = 0$ and $f(u) = c$.

DEFINITION 1.3. Let $f : X \to \mathbb{R}$ be a continuous function and let $c \in \mathbb{R}$. We say that f satisfies $(\text{P-S})_c$, i.e. the Palais–Smale condition at level c, if from every sequence (u_h) in X with $|df|(u_h) \to 0$ and $f(u_h) \to c$ it is possible to extract a subsequence (u_{h_k}) converging in X.

Now, let us mention a fundamental theorem in critical point theory. By now, it is a classical result for C^1-functionals. Owing to the results in [11], we can state a generalized version for the case of continuous functionals.

THEOREM 1.4. *Let E be an infinite-dimensional Banach space and let $f : E \to \mathbb{R}$ be continuous, even and satisfying* $(\text{P-S})_c$ *for every $c \in \mathbb{R}$. Assume that*

(a) *there exist $\varrho > 0$, $\alpha > f(0)$ and a subspace $V \subset E$ of finite codimension such that*

$$\forall u \in V: \quad \|u\| = \varrho \Rightarrow f(u) \ge \alpha, \tag{1.4.1}$$

(b) *for every finite-dimensional subspace $W \subset E$, there exists $R > 0$ such that*

$$\forall u \in W: \quad \|u\| > R \Rightarrow f(u) \le f(0). \tag{1.4.2}$$

Then there exists a sequence (c_h) of critical values of f with $c_h \to \infty$.

Proof. If $f \in C^1(E)$, the result can be found in [16, Th. 9.12]. On the other hand, the symmetric deformation lemma has been extended to the continuous case in [11, Th. 2.16, 2.17]. Then the argument of [16, Th. 9.12] can be easily adapted to our situation. $\square$

In view of the application we shall consider in the next section, let us focus our attention on the functional

$$f(u) = \int_\Omega L(x, u, Du)\, dx.$$

Let Ω be an open bounded subset of $\mathbb{R}^n$. For simplicity, assume $n \geq 3$. Consider an integrand

$$L : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

such that:

- for all $(s, z) \in \mathbb{R} \times \mathbb{R}$, $L(x, s, z)$ is measurable with respect to x,
- for a.e. $x \in \Omega$, $L(x, s, z)$ is of class C^1 with respect to (s, z).

Let us impose the following growth conditions: there exist $a_0 \in L^1(\Omega)$, $b_0 \in \mathbb{R}$, $a_1 \in L^1_{\mathrm{loc}}(\Omega)$ and $b_1 \in L^\infty_{\mathrm{loc}}(\Omega)$ such that

$$|L(x, s, z)| \leq a_0(x) + b_0(|s|^{2n/(n-2)} + |z|^2), \tag{1.1}$$

$$\left|\frac{\partial L}{\partial s}(x, s, z)\right| \leq a_1(x) + b_1(x)(|s|^{2n/(n-2)} + |z|^2), \tag{1.2}$$

$$\left|\frac{\partial L}{\partial z}(x, s, z)\right| \leq a_1(x) + b_1(x)(|s|^{2n/(n-2)} + |z|^2). \tag{1.3}$$

We want to provide us with some tools which allow handling the abstract notions we have recalled.

Theorem 1.5. *Let $f : H^1_0(\Omega) \to \mathbb{R}$ be defined by*

$$f(u) = \int_\Omega L(x, u, Du)\, dx.$$

Then f is continuous and for every $u \in H^1_0(\Omega)$ we have

$$|df|(u) \geq \sup\left\{ \int_\Omega \left[\frac{\partial L}{\partial z}(x, u, Du)Dv + \frac{\partial L}{\partial s}(x, u, Du)v\right] dx : v \in C^\infty_0(\Omega), \|v\|_{H^1_0} \leq 1\right\}.$$

Proof. It is easy to verify that f is continuous and that for every $u \in H^1_0(\Omega)$ and for every $v \in C^\infty_0(\Omega)$,

$$\begin{aligned} f'(u)(v) &:= \lim_{t \to 0} \frac{f(u + tv) - f(u)}{t} \\ &= \int_\Omega \left[\frac{\partial L}{\partial z}(x, u, Du)Dv + \frac{\partial L}{\partial s}(x, u, Du)v\right] dx. \end{aligned}$$

Moreover, for every $v \in C_0^\infty(\Omega)$ the function $\{u \to f'(u)(v)\}$ is continuous. Let u be fixed. If

$$\sup\left\{\int_\Omega \left[\frac{\partial L}{\partial z}(x,u,Du)Dv + \frac{\partial L}{\partial s}(x,u,Du)v\right] dx : v \in C_0^\infty(\Omega), \|v\|_{H_0^1} \le 1\right\} = 0,$$

the assertion is true.

Otherwise, consider $\sigma > 0$ such that

$$\sigma < \sup\left\{\int_\Omega \left[\frac{\partial L}{\partial z}(x,u,Du)Dv + \frac{\partial L}{\partial s}(x,u,Du)v\right] dx : v \in C_0^\infty(\Omega), \|v\|_{H_0^1} \le 1\right\};$$

then there exists $v \in C_0^\infty(\Omega)$ with $\|v\|_{H_0^1} \le 1$ and

$$\sigma < \int_\Omega \left[\frac{\partial L}{\partial z}(x,u,Du)Dv + \frac{\partial L}{\partial s}(x,u,Du)v\right] dx.$$

Let $\widetilde{\delta} > 0$ be such that for every $w \in B(u, \widetilde{\delta})$,

$$\sigma < \int_\Omega \left[\frac{\partial L}{\partial z}(x,w,Dw)Dv + \frac{\partial L}{\partial s}(x,w,Dw)v\right] dx. \tag{1.5.1}$$

Define a continuous map

$$H : B(u,\delta) \times [0,\delta] \to H_0^1(\Omega) \quad (\delta = \widetilde{\delta}/2)$$

by $H(w,t) = w - tv$. It is trivial that $\|H(w,t) - w\|_{H_0^1} \le t$. On the other hand, by (1.5.1), it is easy to see that

$$f(H(w,t)) \le f(w) - \sigma t.$$

It follows that $|df|(u) \ge \sigma$, whence the assertion follows by the arbitrariness of σ. □

We immediately draw the obvious conclusion:

Corollary 1.6. *If $u \in H_0^1(\Omega)$ is a critical point of f, we have*

$$\int_\Omega \left[\frac{\partial L}{\partial z}(x,u,Du)Dv + \frac{\partial L}{\partial s}(x,u,Du)v\right] dx = 0 \quad \forall v \in C_0^\infty(\Omega).$$

In order to treat the Palais–Smale condition, let us introduce an auxiliary notion.

Definition 1.7. Let c be a real number. We say that f satisfies the *concrete Palais–Smale condition at level c* (denoted by $(\text{C-P-S})_c$) if from every sequence $(u_h) \subset H_0^1(\Omega)$ satisfying $\lim_h f(u_h) = c$ and

$$\int_\Omega \left[\frac{\partial L}{\partial z}(x,u_h,Du_h)Dv + \frac{\partial L}{\partial s}(x,u_h,Du_h)v\right] dx = \langle \alpha_h, v\rangle \quad \forall v \in C_0^\infty(\Omega) \tag{1.7.1}$$

with $\lim_h \alpha_h = 0$ in $H^{-1}(\Omega)$, it is possible to extract a subsequence strongly convergent in $H_0^1(\Omega)$.

COROLLARY 1.8. *Let c be a real number. If f satisfies* (C-P-S)$_c$, *then f satisfies* (P-S)$_c$.

PROOF. Let $(u_h) \subset H_0^1(\Omega)$ be such that $\lim_h |df|(u_h) = 0$ and $\lim_h f(u_h) = c$. Of course we can assume $|df|(u_h) < \infty$. By Theorem 1.5, there exists $\alpha_h \in H^{-1}(\Omega)$ such that $\|\alpha_h\|_{H^{-1}} \leq |df|(u_h)$ and

$$\int_\Omega \left[\frac{\partial L}{\partial z}(x, u_h, Du_h)Dv + \frac{\partial L}{\partial s}(x, u_h, Du_h)v\right] dx = \langle \alpha_h, v\rangle \quad \forall v \in C_0^\infty(\Omega).$$

Then the conclusion follows. □

2. The main result

Let Ω be an open bounded subset of $\mathbb{R}^n$. For simplicity, suppose $n \geq 3$.

Let $a_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ $(1 \leq i, j \leq n)$ be such that

- for each $s \in \mathbb{R}$, $a_{ij}(x, s)$ is measurable with respect to x,
- for a.e. $x \in \Omega$, $a_{ij}(x, s)$ is of class C^1 with respect to s.

Let us make the following assumptions:

- there exists $\nu > 0$ such that for a.e. $x \in \Omega$, and all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$,

$$\sum_{i,j=1}^n a_{ij}(x, s)\xi_i\xi_j \geq \nu|\xi|^2; \tag{2.1}$$

- there exists $c > 0$ such that for a.e. $x \in \Omega$, and all $s \in \mathbb{R}$ and $1 \leq i, j \leq n$,

$$|a_{ij}(x, s)| \leq c, \tag{2.2}$$

$$\left|\frac{\partial a_{ij}}{\partial s}(x, s)\right| \leq c; \tag{2.3}$$

- for a.e. $x \in \Omega$, and all $s \in \mathbb{R}$ and $1 \leq i, j \leq n$,

$$a_{ij}(x, s) = a_{ji}(x, s). \tag{2.4}$$

Now, consider a Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

- for a.e. $x \in \Omega$, and all $s \in \mathbb{R}$,

$$|g(x, s)| \leq a(x) + b|s|^p \tag{2.5}$$

  with $a \in L^{2n/(n+2)}(\Omega)$, $b \in \mathbb{R}$, and $1 < p < (n + 2)/(n - 2)$.

Set

$$G(x, s) = \int_0^s g(x, t)\, dt$$

and suppose that:

- there exist $q > 2$, $R > 0$ such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$ with $|s| \geq R$,

$$0 < qG(x,s) \leq sg(x,s); \tag{2.6}$$

- there exists $\alpha < q-2$ such that for a.e. $x \in \Omega$, and all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$,

$$0 \leq s \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x,s)\xi_i\xi_j \leq \alpha \sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j. \tag{2.7}$$

Assumptions (2.1–2.6) seem to be very natural. On the contrary, hypothesis (2.7) looks rather technical. However, conditions of this kind have already been considered in the literature.

For instance, the sign condition

$$s \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x,s)\xi_i\xi_j \geq 0$$

has been assumed in [4], [7], while a condition like

$$s \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x,s)\xi_i\xi_j \leq \alpha \sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j$$

is typical in superlinear problems (see e.g. [3], where a problem in one dimension is treated).

We are interested in weak solutions $u \in H_0^1(\Omega)$ of the quasilinear elliptic equation

$$(P) \qquad -\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_iu) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_ju = g(x,u) \quad \text{in } D'(\Omega),$$

namely functions $u \in H_0^1(\Omega)$ such that

$$(P') \qquad \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u)D_iuD_jv\,dx + \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_juv\,dx = \int_\Omega g(x,u)v\,dx \quad \forall v \in C_0^\infty(\Omega).$$

To this end, define the functional $f : H_0^1(\Omega) \to \mathbb{R}$ by setting

$$f(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u)D_iuD_ju\,dx - \int_\Omega G(x,u)\,dx.$$

Theorem 2.1. *Let $u \in H_0^1(\Omega)$ and $\alpha \in H^{-1}(\Omega)$ be such that*

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j v \, dx + \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u v \, dx - \int_\Omega g(x,u) v \, dx = \langle \alpha, v \rangle \quad \forall v \in C_0^\infty(\Omega).$$

Then

$$u \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u \in L^1(\Omega)$$

and

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j u \, dx + \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u u \, dx - \int_\Omega g(x,u) u \, dx = \langle \alpha, u \rangle.$$

Proof. We have

$$\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u - g(x,u) \in L^1_{\mathrm{loc}}(\Omega) \cap H^{-1}(\Omega)$$

and

$$\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u u - g(x,u) u \geq -g(x,u) u$$

with $g(x,u)u \in L^1(\Omega)$. Then the conclusion follows by the result of [10]. $\square$

Theorem 2.2. *For every real number c the functional f satisfies* (C-P-S)$_c$.

To prove this theorem, we need some lemmas.

Lemma 2.3. *Let (u_h) be a bounded sequence in $H_0^1(\Omega)$ satisfying*

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h) D_i u_h D_j v \, dx + \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_h) D_i u_h D_j u_h v \, dx = \langle \beta_h, v \rangle \quad \forall v \in C_0^\infty(\Omega) \tag{2.3.1}$$

with (β_h) strongly convergent to β in $H^{-1}(\Omega)$. Then $Du_h \to Du$ a.e. and there exists a subsequence (u_{h_k}) weakly convergent to some u in H_0^1 with

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j v \, dx + \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u v \, dx = \langle \beta, v \rangle \quad \forall v \in C_0^\infty(\Omega). \tag{2.3.2}$$

The proof follows the argument of [7] and it will be given in the Appendix for the reader's convenience.

Lemma 2.4. *Let (u_h) be a bounded sequence in $H_0^1(\Omega)$ satisfying*

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h) D_i u_h D_j v \, dx + \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_h) D_i u_h D_j u_h v \, dx = \langle \beta_h, v \rangle \quad \forall v \in C_0^\infty(\Omega) \tag{2.3.1}$$

with (β_h) strongly convergent in $H^{-1}(\Omega)$. Then it is possible to extract a subsequence (u_{h_k}) strongly convergent in $H_0^1(\Omega)$.

Proof. Denote still by (u_h) a subsequence as in Lemma 2.3; without loss of generality, we can assume that $u_h \to u$ a.e. By Theorem 2.1, we have, in particular,

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j u \, dx + \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u u \, dx = \langle \beta, u \rangle,$$

where $\beta \in H^{-1}(\Omega)$ is the limit of (β_h).

Now, let us prove that

$$\limsup_h \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h) D_i u_h D_j u_h \le \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j u. \tag{2.4.1}$$

By Fatou's lemma, we have

$$\int_\Omega \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u u \, dx \le \liminf_h \int_\Omega \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_h) D_i u_h D_j u_h u_h \, dx.$$

Therefore

$$\begin{aligned}
\limsup_h & \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h) D_i u_h D_j u_h \, dx \\
&= \limsup_h \left[-\frac{1}{2} \int_\Omega \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_h) D_i u_h D_j u_h u_h \, dx + \langle \beta_h, u_h \rangle \right] \\
&\le -\frac{1}{2} \int_\Omega \sum_{i,j} \frac{\partial a_{ij}}{\partial s}(x,u) D_i u D_j u u \, dx + \langle \beta, u \rangle \\
&= \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j u \, dx.
\end{aligned}$$

Finally, let us show that u_h converges to u in the strong topology of $H^1_0(\Omega)$. Observe that

$$\begin{aligned}(2.4.2)\qquad &\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)D_i(u_h-u)D_j(u_h-u)\,dx\\ &=\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)D_iu_hD_ju_h\,dx-2\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)D_iuD_ju_h\,dx\\ &\quad+\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)D_iuD_ju\,dx.\end{aligned}$$

For every $j=1,\dots,n$, we have

$$\lim_h \sum_{i=1}^n a_{ij}(x,u_h)D_iu=\sum_{i=1}^n a_{ij}(x,u)D_iu$$

in the strong topology of $L^2(\Omega)$. Then, passing to the lim sup in (2.4.2), we have, by (2.4.1),

$$\begin{aligned}(2.4.3)\qquad &\limsup_h \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)D_i(u_h-u)D_j(u_h-u)\,dx\\ &=\limsup_h \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)D_iu_hD_ju_h\,dx\\ &\quad-\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u)D_iuD_ju\,dx\le 0.\end{aligned}$$

Using (2.4.3) and hypothesis (2.1), we conclude that

$$\begin{aligned}&\nu\limsup_h \|Du_h-Du\|^2_{L^2}\\ &\qquad\le \limsup_h \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)D_i(u_h-u)D_j(u_h-u)\,dx\le 0.\end{aligned}$$

Thus the assertion is proved. □

Lemma 2.5. *Let c be a real number. Let $(u_h)\subset H^1_0(\Omega)$ be such that*

$$(2.5.1)\qquad \lim_h f(u_h)=c$$

and

$$\begin{aligned}(2.5.2)\qquad &\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)D_iu_hD_jv\,dx+\frac12\int_\Omega \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_h)D_iu_hD_ju_hv\,dx\\ &\quad-\int_\Omega g(x,u_h)v\,dx=\langle\alpha_h,v\rangle\quad\forall v\in C^\infty_0(\Omega)\end{aligned}$$

with $\alpha_h\to 0$ in $H^{-1}(\Omega)$. Then (u_h) is bounded in $H^1_0(\Omega)$.

Proof. Since $f(u_h)$ is bounded, we deduce from hypothesis (2.6) that there exists $k \in \mathbb{R}$ such that

$$k \geq \frac{1}{2} \int_\Omega \sum_{i,j=1}^n a_{ij}(x, u_h) D_i u_h D_j u_h \, dx - \frac{1}{q} \int_\Omega g(x, u_h) u_h \, dx. \tag{2.5.3}$$

Moreover, from (2.5.2) and Theorem 2.1, we have

$$\begin{aligned} \bigg| \int_\Omega \sum_{i,j=1}^n & a_{ij}(x, u_h) D_i u_h D_j u_h \, dx \\ &+ \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x, u_h) D_i u_h D_j u_h u_h \, dx - \int_\Omega g(x, u_h) u_h \, dx \bigg| \\ &\leq \varepsilon_h \|u_h\|_{H_0^1} \quad (\lim_h \varepsilon_h = 0). \end{aligned} \tag{2.5.4}$$

Now, from (2.5.3) and (2.5.4), we deduce that

$$\begin{aligned} \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega \sum_{i,j=1}^n a_{ij}(x, u_h) D_i u_h D_j u_h \, dx & \\ - \frac{1}{2q} \int_\Omega \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x, u_h) D_i u_h D_j u_h u_h \, dx & \\ \leq \frac{\varepsilon_h}{q} \|u_h\|_{H_0^1} + k. & \end{aligned}$$

By hypotheses (2.1) and (2.7), the assertion follows. □

We now easily prove the previous theorem.

Proof of Theorem 2.2. Let $(u_h) \subset H_0^1(\Omega)$ satisfy (2.5.1)–(2.5.2). By Lemma 2.5, (u_h) is bounded in $H_0^1(\Omega)$. By hypotheses on g, it follows that, up to a subsequence, $(g(x, u_h))$ is strongly convergent in $H^{-1}(\Omega)$. Therefore we can apply Lemma 2.4 with $\beta_h = g(x, u_h) + \alpha_h$. □

Finally, we can state the main result of the paper.

Theorem 2.6. *Let a_{ij} and g satisfy hypotheses* (2.1)–(2.7). *Moreover, let*

$$a_{ij}(x, -s) = a_{ij}(x, s), \quad g(x, -s) = -g(x, s).$$

Then there exists a sequence $(u_h) \subset H_0^1(\Omega)$ of solutions of (P') with $f(u_h) \to \infty$.

Proof. The functional $f : H_0^1(\Omega) \to \mathbb{R}$ is evidently continuous and even. Moreover, by Theorem 2.2 and Corollary 1.8, the functional f satisfies $(\text{P-S})_c$ for every $c \in \mathbb{R}$.

For some constant $k > 0$, we also have

$$\frac{\nu}{2} \int_\Omega |Du|^2 \, dx - \int_\Omega G(x, u) \, dx \leq f(u) \leq k \int_\Omega |Du|^2 \, dx - \int_\Omega G(x, u) \, dx.$$

Then the arguments of [16], [19] show that f satisfies conditions (a) and (b) of Theorem 1.4. By Theorem 1.4 and Corollary 1.6, the conclusion follows. □

Appendix

Proof of Lemma 2.3. Up to a subsequence, u_h is convergent to u weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and a.e. in Ω. Moreover, since (u_h) satisfies (2.3.1), by Theorem 2.1 of [6] we have, up to a further subsequence, $Du_h \to Du$ a.e. in Ω.

We will use the device of [7]. We consider the test functions

$$v_h = \varphi \exp\{-Mu_h^+\}, \tag{2.3.3}$$

where $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\varphi \geq 0$, u_h^+ is the positive part of u_h, and $M > 0$, according to (2.1) and (2.3), is such that

$$\frac{1}{2}\bigg|\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,s)\xi_i\xi_j\bigg| \leq M \sum_{i,j=1}^n a_{ij}(x,s)\xi_i\xi_j.$$

Since (2.3.1) holds by density for every $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we can put $v = v_h$ in (2.3.1), obtaining

$$\begin{aligned}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u_h)D_iu_hD_j\varphi \exp\{-Mu_h^+\}\,dx \\ + \int_\Omega \bigg[\frac{1}{2}\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_h)D_iu_hD_ju_h \\ - M\sum_{i,j=1}^n a_{ij}(x,u_h)D_iu_hD_ju_h^+\bigg]\varphi\exp\{-Mu_h^+\}\,dx \\ - \langle \beta_h, \varphi\exp\{-Mu_h^+\}\rangle = 0.\end{aligned} \tag{2.3.4}$$

Since

$$\begin{aligned}\bigg[\frac{1}{2}\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u_h)D_iu_hD_ju_h \\ - M\sum_{i,j=1}^n a_{ij}(x,u_h)D_iu_hD_ju_h^+\bigg]\varphi\exp\{-Mu_h^+\} \leq 0,\end{aligned}$$

by Fatou's lemma, we have

$$\begin{aligned}(2.3.5)\qquad \int_\Omega \sum_{i,j=1}^n & a_{ij}(x,u)D_iuD_j\varphi \exp\{-Mu^+\}\,dx \\ &+ \int_\Omega \Big[\frac{1}{2}\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_ju \\ &- M\sum_{i,j=1}^n a_{ij}(x,u)D_iuD_ju^+\Big]\varphi\exp\{-Mu^+\}\,dx \\ &\geq \langle \beta, \varphi\exp\{-Mu^+\}\rangle \quad \forall \varphi\in H_0^1(\Omega)\cap L^\infty(\Omega),\ \varphi\geq 0.\end{aligned}$$

Now, we consider the test functions

$$(2.3.6)\qquad \varphi_h = \varphi H(u/h)\exp\{Mu^+\}$$

with $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ and

$$H:\mathbb{R}\to\mathbb{R},\quad H\in C^1(\mathbb{R}),\quad 0\leq H\leq 1,$$
$$H=1 \quad \text{on } [-1/2,1/2],\quad H=0\quad \text{on }]-\infty,-1]\cup[1,\infty[.$$

Putting them in (2.3.5), we obtain

$$\begin{aligned}(2.3.7)\qquad \int_\Omega \sum_{i,j=1}^n & a_{ij}(x,u)D_iuD_j(\varphi H(u/h))\,dx \\ &+ \int_\Omega \frac{1}{2}\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_ju\varphi H(u/h)\,dx \\ &\geq \langle \beta, \varphi H(u/h)\rangle \quad \forall\varphi\in C_0^\infty(\Omega),\ \varphi\geq 0.\end{aligned}$$

Passing to the limit as $h\to\infty$ in (2.3.7), we obtain

$$\begin{aligned}(2.3.8)\qquad \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u)D_iuD_j\varphi\,dx + \int_\Omega \frac{1}{2}\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_ju\varphi\,dx \\ \geq \langle\beta,\varphi\rangle \quad \forall\varphi\in C_0^\infty(\Omega),\ \varphi\geq 0.\end{aligned}$$

In a similar way, by considering the test functions $v_h = \varphi\exp\{-Mu_h^-\}$, it is possible to prove the opposite inequality. It follows that

$$\begin{aligned}(2.3.9)\qquad \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u)D_iuD_j\varphi\,dx + \int_\Omega \frac{1}{2}\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial s}(x,u)D_iuD_ju\varphi\,dx \\ = \langle\beta,\varphi\rangle \quad \forall\varphi\in C_0^\infty(\Omega),\ \varphi\geq 0.\end{aligned}$$

By (2.3.9), we deduce (2.3.2). □

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