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THE NIRENBERG PROBLEM IN A DOMAIN WITH BOUNDARY

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Dedicated to L. Nirenberg with admiration

0. Introduction

There has been much work on the Nirenberg problem: which function $K(x)$ on $\mathbb{S}^n$ is the scalar curvature of a metric g on $\mathbb{S}^n$ pointwise conformal to the standard metric g_0? It is quite natural to ask the following question on the half sphere $\mathbb{S}^n_-$: which function $K(x)$ on $\mathbb{S}^n_-$ is the scalar curvature of a metric g on $\mathbb{S}^n_-$ which is pointwise conformal to the standard metric g_0 with $\partial\mathbb{S}^n_-$ being minimal with respect to g? For $n = 2$, this has been studied by J. Q. Liu and P. L. Li in [LL]. In this note we study the higher dimensional cases along the lines of [L1-2]. For much work on the Nirenberg problem see, for example, [L1-2] and the references therein. See also some more recent work in [CL1], [HL], [Bi1-2], [SZ], [B], [ChL] and [CL2].

For $n \geq 3$, by writing $g = u^{4/(n-2)}g_0$, the problem is equivalent to solving the following Neumann problem on $\mathbb{S}^n_- = \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \mid x_{n+1} < 0\}$:

$$(0.1)\qquad \begin{cases} -\Delta_{g_0} u + c(n)R_0 u = c(n)Ku^{(n+2)/(n-2)}, \quad u > 0, & \text{on } \mathbb{S}^n_-,\\ \partial u/\partial\nu = 0 & \text{on } \partial\mathbb{S}^n_-, \end{cases}$$

where $c(n) = (n-2)/(4(n-1))$, $R_0 = n(n-1)$, and ν denotes the unit outer normal at points of $\partial\mathbb{S}^n_-$.

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We introduce

$$\mathcal{A} = \{K \in C^1(\overline{\mathbb{S}^3_-}) \mid K > 0 \text{ on } \overline{\mathbb{S}^3_-},\ \nabla K \neq 0 \text{ on } \partial\mathbb{S}^3_-\},$$
$$\mathcal{K}^- = \left\{ q \in \partial\mathbb{S}^3_- \,\middle|\, \nabla_{\tan} K(q) = 0,\ \frac{\partial K}{\partial \nu}(q) > 0 \right\},$$
$$\mathcal{M}_K = \{u \in H^1(\mathbb{S}^3_-) \mid u \text{ satisfies } (0.1)\},$$

where $\nabla_{\tan} K(q)$ denotes the tangential derivatives of K at $q \in \partial\mathbb{S}^3_-$. Clearly $\mathcal{A}$ is open and dense in $C^1(\overline{\mathbb{S}^3_-})^+$, which consists of positive functions in $C^1(\overline{\mathbb{S}^3_-})$. We will introduce an integer-valued continuous function Index : $\mathcal{A} \to \mathbb{Z}$, with an explicit formula for $K \in \mathcal{A} \cap C^2(\overline{\mathbb{S}^3_-})$ with $K|_{\partial\mathbb{S}^3_-}$ being a Morse function. In fact, for any such K, let $i(\overline{P})$ denote the Morse index of $K|_{\partial\mathbb{S}^3_-}$ at $\overline{P} \in \mathcal{K}^-$. Then

$$\text{Index}(K) = -1 + \sum_{\overline{P} \in \mathcal{K}^-} (-1)^{i(\overline{P})}. \tag{0.2}$$

It is proved in Section 3 that Index can be extended from (0.2) as a *continuous* function on $\mathcal{A}$ with respect to the $C^1(\mathbb{S}^3)$ topology.

THEOREM 0.1. (a) *For any $K \in \mathcal{A}$, there exists some positive constant $C = C(K)$ such that for any $K_i \to K$ in $C^1(\overline{\mathbb{S}^3_-})$, and any $u_i \in \mathcal{M}_{K_i}$,*

$$C^{-1} \leq \liminf_{i\to\infty}(\min_{\overline{\mathbb{S}^3_-}} u_i) \leq \limsup_{i\to\infty}(\max_{\overline{\mathbb{S}^3_-}} u_i) \leq C.$$

Furthermore, for all $0 < \alpha < 1$, there exists $R_0 = R_0(K, \alpha) \gg 1$ such that for all $R > R_0$,

$$\deg\left(u - \tfrac{1}{8}\left(-\Delta_{g_0} + \tfrac{3}{4}\right)^{-1}(Ku^5), \mathcal{O}_R, 0\right) = \text{Index}(K), \tag{0.3}$$

where $\mathcal{O}_R = \{u \in C^{2,\alpha}(\overline{\mathbb{S}^3_-}) \mid 1/R < u < R$ on $\mathbb{S}^3_-$, $\|u\|_{C^{2,\alpha}(\mathbb{S}^3_-)} < R\}$, and deg *denotes the Leray–Schauder degree.*

(b) *For any $K \in C^1(\overline{\mathbb{S}^3_-})^+ \setminus \mathcal{A} \equiv \partial\mathcal{A}$, there exist $K_i \to K$ in $C^1(\overline{\mathbb{S}^3_-})$ and $u_i \in \mathcal{M}_{K_i}$ such that*

$$\lim_{i\to\infty}(\max_{\overline{\mathbb{S}^3_-}} u_i) = \infty, \quad \lim_{i\to\infty}(\min_{\overline{\mathbb{S}^3_-}} u_i) = 0.$$

COROLLARY 0.1. *For any $K \in \mathcal{A}$ with* Index$(K) \neq 0$*, (0.1) has at least one solution.*

REMARK 0.1. For $K \in \mathcal{A} \cap C^2(\overline{\mathbb{S}^3_-})$, $K|_{\partial\mathbb{S}^3_-}$ being a Morse function, we can use Theorem 3.1 to easily establish a strong Morse inequality as in [SZ], which gives more general existence results than Corollary 0.1.

In deriving Theorem 0.1, we have obtained some detailed information on blow up behavior of solutions which is of independent interest. See Proposition 2.4, Theorem 2.1 and Theorem 3.1.

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1. A Pokhozhaev type identity

For $\sigma > 0$ and $\overline{x} \in \mathbb{R}^n$, we set $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$, $B_\sigma(\overline{x}) = \{x \in \mathbb{R}^n \mid |x| < \sigma\}$, $B_\sigma = B_\sigma(0)$, $B^+_\sigma(\overline{x}) = B_\sigma(\overline{x}) \cap \mathbb{R}^n_+$, and $B^+_\sigma = B^+_\sigma(0)$.

The following is a Pokhozhaev type identity. The proof is standard by now (see e.g. [L1]).

LEMMA 1.1. *Let $p \geq 1$, $\sigma > 0$, $n \geq 3$, $B^+_\sigma \subset \mathbb{R}^n_+$, and $u \in C^2(B^+_\sigma) \cap C^1(\overline{B^+_\sigma})$ be a solution of*

$$\begin{cases} -\Delta u = c(n)K(x)|u|^{p-1}u, & x \in B^+_\sigma, \\ \partial u/\partial x_n = 0, & x \in \partial B^+_\sigma \cap \partial \mathbb{R}^n_+. \end{cases}$$

We have

$$\frac{c(n)}{p+1}\sum_i \int_{B^+_\sigma} x_i \frac{\partial K}{\partial x_i}|u|^{p+1} + \left(\frac{n}{p+1} - \frac{n-2}{2}\right)c(n)\int_{B^+_\sigma} K|u|^{p+1} \\ - \frac{\sigma c(n)}{p+1}\int_{\partial B_\sigma \cap \mathbb{R}^n_+} K|u|^{p+1} = \int_{\partial B_\sigma \cap \mathbb{R}^n_+} B(\sigma, x, u, \nabla u),$$

where

$$B(\sigma, x, u, \nabla u) = \frac{n-2}{2}u\frac{\partial u}{\partial \nu} - \frac{\sigma}{2}|\nabla u|^2 + \sigma\left(\frac{\partial u}{\partial \nu}\right)^2$$

with ν denoting the unit outer normal of ∂B_σ.

2. Analysis of blow ups

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded domain containing the origin, $\Omega^+ = \Omega \cap \mathbb{R}^n_+$, $\tau_i \geq 0$ satisfy $\lim_{i\to\infty} \tau_i = 0$, $p_i = \frac{n+2}{n-2} - \tau_i$, and $\{K_i\} \in L^\infty(\Omega^+)$ satisfy, for some constant $A_1 > 0$,

$$1/A_1 \leq K_i(x) \leq A_1 \quad \text{for all } x \in \Omega^+. \tag{2.1}$$

Consider

$$\begin{cases} -\Delta u_i = c(n)K_i(x)u^{p_i}, \quad u_i > 0, & \text{in } \Omega^+, \\ \partial u_i/\partial x_n = 0 & \text{on } \partial\Omega^+ \cap \partial\mathbb{R}^n_+. \end{cases} \tag{2.2}$$

DEFINITION 2.1. A point $\overline{y} \in \Omega \cap \overline{\mathbb{R}^n_+}$ is called a *blow up point* of $\{u_i\}$ if there exists a sequence $y_i \in \overline{\Omega^+}$ tending to $\overline{y}$ such that $u_i(y_i) \to \infty$.

Definition 2.2. A point $\overline{y} \in \Omega \cap \overline{\mathbb{R}^n_+}$ is called an *isolated blow up point* of $\{u_i\}$ if there exist $0 < \overline{r} < \operatorname{dist}(\overline{y}, \partial\Omega \cap \mathbb{R}^n_+)$, $\overline{C} > 0$, and a sequence y_i tending to $\overline{y}$ such that y_i is a local maximum of u_i in $\overline{\Omega^+}$, $u_i(y_i) \to \infty$ and

$$u_i(y) \le \overline{C}|y - y_i|^{-2/(p_i-1)} \quad \text{for all } y \in B_{\overline{r}}(y_i) \cap \Omega^+.$$

We point out that the $\{y_i\}$ in Definition 2.2 are uniquely determined for large i provided $\{K_i\}$ is bounded in $C^\alpha(\overline{\Omega^+})$ for some $0 < \alpha < 1$. Let $y_i \to \overline{y}$ be an isolated blow up point of $\{u_i\}$. We define

$$\overline{u}_i(r) = \frac{1}{|\partial B_r(y_i) \cap \Omega^+|} \int_{\partial B_r(y_i) \cap \Omega^+} u_i, \quad r > 0,$$

and

$$\overline{w}_i(r) = r^{2/(p_i-1)} \overline{u}_i(r), \quad r > 0.$$

Definition 2.3. $\overline{y} \in \Omega \cap \overline{\mathbb{R}^n_+}$ is called an *isolated simple blow up point* of $\{u_i\}$ if $y_i \to \overline{y}$ is an isolated blow up point such that, for some $\varrho > 0$ (independent of i),

$$\overline{w}_i \text{ has precisely one critical point in } (0, \varrho), \tag{2.3}$$

for large i. In addition,

$$y_i \in \Omega \cap \partial\mathbb{R}^n_+ \tag{2.4}$$

for large i if $\overline{y} \in \Omega \cap \partial\mathbb{R}^n_+$.

If $\overline{y} \in \Omega \cap \partial\mathbb{R}^n_+$ in the above, we call it a *boundary isolated simple blow up point.*

Lemma 2.1. *Let $\{K_i\} \in L^\infty(\Omega^+)$, $\{u_i\}$ satisfy* (2.2) *and $y_i \to \overline{y} \in \Omega$ be an isolated blow up point. Then for any $0 < r < \frac{1}{3}\overline{r}$, we have the following Harnack inequality:*

$$\sup_{y \in B^+_{2r}(y_i) \setminus B^+_{r/2}(y_i)} u_i(y) \le C \inf_{y \in B^+_{2r}(y_i) \setminus B^+_{r/2}(y_i)} u_i(y),$$

where C is a positive constant depending only on n, $\overline{C}$ and $\sup_i \|K_i\|_{L^\infty(B^+_{\overline{r}}(y_i))}$.

Proof. Reflect u_i evenly to $\mathbb{R}^n_-$, and apply Lemma 2.1 of [L1].

Proposition 2.1. *Suppose $\{K_i\} \in C^{0,1}(\overline{\Omega} \cap \overline{\mathbb{R}^n_+})$ satisfies* (2.1) *for some $A_1 > 0$, and*

$$\|\nabla K_i\|_{L^\infty(\Omega^+)} \le A_2 \tag{2.5}$$

for some $A_2 > 0$. *Let* $\{u_i\}$ *satisfy* (2.2), $\overline{y} \in \Omega \cap \overline{\mathbb{R}^n_+}$ *be an isolated blow up point of* $\{u_i\}$ *and* $\{y_i\}$ *be the sequence of points as in Definition* 2.2. *Then for any* $R_i \to \infty$ *and* $\varepsilon_i \to 0^+$, *after passing to a subsequence, we have either*

$$\begin{cases} r_i := R_i u_i(y_i)^{-(p_i-1)/2} \to 0 \quad \textit{as } i \to \infty, \quad B_{2r_i}(y_i) \subset \Omega^+, \\ \|u_i(y_i)^{-1} u_i(u_i(y_i)^{-(p_i-1)/2} \cdot + y_i) - (1 + k_i|\cdot|^2)^{(2-n)/2}\|_{C^2(B_{2R_i})} \le \varepsilon_i, \end{cases}$$

or

$$\begin{cases} r_i \to 0 \quad \textit{as } i \to \infty, \quad y_i \in \Omega \cap \partial\mathbb{R}^n_+, \\ \|u_i(y_i)^{-1} u_i(u_i(y_i)^{-(p_i-1)/2} \cdot + y_i) - (1 + k_i|\cdot|^2)^{(2-n)/2}\|_{C^2(\overline{B^+_{2R_i}})} \le \varepsilon_i, \end{cases}$$

where $k_i = c(n)(n(n-2))^{-1} K_i(y_i)$.

Proof. We will only prove this for $\overline{y} \in \Omega \cap \partial\mathbb{R}^n_+$. Without loss of generality, we take $\overline{y} = 0$.

Writing $y_i = (y_{i1}, y_{i2}, y_{i3})$, we consider

$$w_i(z) = u_i(y_i)^{-1} u_i(u_i(y_i)^{(1-p_i)/2} z + y_i), \quad z_3 \ge -u_i(y_i)^{(p_i-1)/2} y_{i3} \equiv -T_i.$$

It is easy to see that $w_i(0) = 1$, $z = 0$ is a local maximum point of w_i in $z_3 \ge -T_i$, and w_i satisfies

$$\begin{cases} -\Delta w_i(z) = c(n) K_i(u_i(y_i)^{(1-P_i)/2} z + y_i) w_i(z)^{p_i}, \quad w_i(z) > 0, & z_3 > -T_i, \\ \partial w_i / \partial z_3 = 0, & z_3 = -T_i. \end{cases}$$

After passing to a subsequence, there are three cases.

Case 1: $T_i \to \infty$.

Case 2: $T_i \to 0$.

Case 3: $T_i \to T \in (0, \infty)$.

It is not difficult to see that Case 1 and Case 2 lead to the conclusion of Proposition 2.1. Case 3 cannot occur since if it occurred, the limit function w of $\{w_i\}$ would satisfy

$$\begin{cases} -\Delta w = w^{(n+2)/(n-2)}, \quad w > 0, & z_3 > -T, \\ \partial w / \partial z_3 = 0, & z_3 = -T < 0, \\ \nabla w(0) = 0. \end{cases}$$

Making an even extension across $z_3 = -T$ produces a positive solution of $-\Delta w = w^{(n+2)/(n-2)}$ in $\mathbb{R}^n$ with two critical points, which violates the uniqueness result of [CGS].

PROPOSITION 2.2. *Suppose* $\{K_i\} \in C^1(\overline{B_2^+})$ *satisfies* (2.1) *and* (2.5) *for some constants* $A_1, A_2 > 0$ *with* $\Omega = B_2$. *Suppose also that* u_i *satisfies* (2.2) *with* $\Omega = B_2$, *and* $y_i \to \overline{y} \in \overline{B_{1/4}^+}$ *is an isolated blow up point with, for some positive constant* A_3,

$$|y - y_i|^{2/(p_i-1)} u_i(y) \le A_3 \quad \text{for all } y \in B_2^+. \tag{2.6}$$

Then there exists some positive constant $C = C(n, A_1, A_2, A_3)$ *such that, for* i *large enough,*

$$u_i(y) \ge C^{-1} u_i(y_i)(1 + k_i u_i(y_i)^{p_i-1} |y - y_i|^2)^{(2-n)/2} \quad \text{for all } y \in B_1^+(y_i).$$

In particular, for i *large enough, we have*

$$u_i(y_i + e) \ge C^{-1} u_i(y_i)^{-1+(n-2)\tau_i/2}$$

for all $e \in \mathbb{R}^n$ *with* $|e| = 1$ *and* $y_i + e \in B_2^+$.

PROOF. Set $r_i = R_i u_i(y_i)^{-(p_i-1)/2}$. It follows from Proposition 2.1 that

$$u_i(y) \ge C^{-1} u_i(y_i) R_i^{2-n} \quad \text{for all } y \in \partial B_{r_i}(y_i) \cap B_2^+.$$

Set

$$\varphi_i(y) = C^{-1} R_i^{2-n} r_i^{n-2} u_i(y_i)(|y - y_i|^{2-n} - (3/2)^{2-n}), \qquad y \in B_{3/2}(y_i) \setminus B_{r_i}(y_i) \cap B_2^+.$$

Clearly φ_i satisfies

$$\begin{cases} \Delta\varphi_i(y) = 0 \ge \Delta u_i(y), & y \in B_{3/2}(y_i) \setminus B_{r_i}(y_i) \cap B_2^+, \\ \varphi_i(y) = 0 \le u_i(y), & y \in \partial B_{3/2}(y_i) \cap B_2^+, \\ \varphi_i(y) \le u_i(y), & y \in \partial B_{r_i}(y_i) \cap B_2^+, \\ \dfrac{\partial \varphi_i}{\partial y_n}(y) \ge 0 = \dfrac{\partial u_i}{\partial y_n}(y), & y \in \partial(B_{3/2}(y_i) \setminus B_{r_i}(y_i)) \cap \partial\mathbb{R}_+^n. \end{cases}$$

It follows from the maximum principle that

$$u_i(y) \ge \varphi_i(y) \quad \text{for all } y \in (B_{3/2}(y_i) \setminus B_{r_i}(y_i)) \cap \mathbb{R}_+^n.$$

Proposition 2.2 follows immediately from the above and Proposition 2.1.

PROPOSITION 2.3. *Suppose* $\{K_i\} \subset C^{0,1}(\overline{B}_2 \cap \overline{\mathbb{R}_+^n})$ *satisfies* (2.1), (2.5) *with* $\Omega = B_2$ *for some* $A_1, A_2 > 0$. *Suppose also that* u_i *satisfies* (2.2) *with* $\Omega = B_2$, *and* $y_i \to 0$ *is a boundary isolated simple blow up point with* (2.3), (2.4) *and* (2.6) *for some positive constants* ϱ *and* A_3. *Then there exists some positive constant* $C = C(n, A_1, A_2, A_3, \varrho)$ *such that*

$$u_i(y) \le C u_i(y_i)^{-1} |y - y_i|^{2-n} \quad \text{for all } y \in B_1(y_i)^+,$$
$$\tau_i = O(u_i(y_i)^{-2/(n-2)+o(1)}), \quad u_i(y_i)^{\tau_i} = 1 + o(1),$$

where ϱ is the constant in Definition 2.3, and $o(1)$ denotes some quantity tending to 0 as i tends to ∞. Furthermore, for some harmonic function $b(y)$ in B_1^+ with $\partial b/\partial y_n = 0$ on $\partial B_1^+ \cap \partial \mathbb{R}_+^n$, we have, after passing to a subsequence,

$$u_i(y_i)u_i(y) \to h(y) = a|y|^{2-n} + b(y) \quad \text{in } C^2_{\text{loc}}(\overline{B_1^+} \setminus \{0\}),$$

where

$$a = \lim_{i\to\infty} k_i^{(2-n)/2} = c(n)^{(2-n)/2}[n(n-2)]^{(n-2)/2}(\lim_{i\to\infty} K_i(0))^{(2-n)/2}.$$

Remark 2.1. When $y_i \to \overline{y} \in B_2^+$ is an interior isolated simple blow up point, similar results have been given in Proposition 2.3 of [L1]. It is clear that the hypothesis $\{K_i\} \subset C^1_{\text{loc}}(B_2)$ there can be relaxed to $\{K_i\} \subset C^{0,1}_{\text{loc}}(B_2)$, and the same proof works.

Proof of Proposition 2.3. The assertion follows from Proposition 2.3 and Lemma 2.3 of [L1] after extending u_i evenly to $\mathbb{R}^n_-$.

Lemma 2.3. *Under the hypotheses of Proposition 2.3, for any $f \in L^1(\mathbb{S}^{n-1}_+)$ with $\int_{\mathbb{S}^{n-1}_+} f = 0$, we have*

$$\begin{aligned}
&\int_{B^+_{r_i}(y_i)} f\left(\frac{y-y_i}{|y-y_i|}\right)|y-y_i|^s u_i(y)^{p_i+1} \\
&= \begin{cases}
u_i(y_i)^{-2s/(n-2)}\left\{\left[|\mathbb{S}^{n-1}_+|^{-1}\int_{\mathbb{S}^{n-1}_+} f\right]\int_{\mathbb{R}^n_+}|z|^s(1+k_i|z|^2)^{-n}dz + o(1)\right\}, & -n<s<n, \\
O\left(\left|\int_{\mathbb{S}^{n-1}_+} f\right| u_i(y_i)^{-2n/(n-2)}\log u_i(y_i)\right) + o(u_i(y_i)^{-2n/(n-2)}\log u_i(y_i)), & s=n, \\
o(u_i(y_i)^{-2n/(n-2)}), & s>n,
\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B_1^+(y_i)\setminus B^+_{r_i}(y_i)} |y-y_i|^s u_i(y)^{p_i+1} \\
&\qquad \le \begin{cases}
o(u_i(y_i)^{-2s/(n-2)}), & -n<s<n, \\
O(u_i(y_i)^{-2n/(n-2)}\log u_i(y_i)), & s=n, \\
O(u_i(y_i)^{-2n/(n-2)}), & s>n,
\end{cases}
\end{aligned}$$

where $k_i = [n(n-2)]^{-1}c(n)K_i(y_i)$.

Proof. This follows from Proposition 2.1, Proposition 2.3 and some elementary calculations.

LEMMA 2.4. *Suppose* $\{K_i\} \in C^{0,1}(\overline{B_2^+})$ *satisfies* (2.1), (2.5) *with* $\Omega = B_2$, $n = 3$, *and some positive constants* A_1, A_2. *Suppose also that* u_i *satisfies* (2.2), *and* $y_i \to 0$ *is an isolated simple blow up point. Then*

$$\tau_i = O(u_i(y_i)^{-2}).$$

If we further assume that $\{\nabla K_i\} \in C^0(\overline{B_2^+})$ *has a uniform modulus of continuity, then*

$$\lim_{i\to\infty} u_i(y_i)^2 \int_{B_\sigma^+(y_i)} (y - y_i) \cdot \nabla K_i u_i^{p_i+1} = 0.$$

PROOF. It follows from Lemma 1.1 (with $\sigma = 1$), Proposition 2.3, Lemma 2.3 and some standard elliptic estimates that

$$\tau_i \le C \int_{B_1^+(y_i)} |y - y_i| u_i^{p_i+1} + C \int_{\partial B_1(y_i)\cap \mathbb{R}_+^3} \left(u_i^{p_i+1} + u_i^2 + |\nabla u_i|^2\right) \le C u_i(y_i)^{-2}.$$

Using the additional property of $\{\nabla K_i\}$ and Lemma 2.3, we have

$$\begin{aligned}
&\int_{B_\sigma^+(y_i)} (y - y_i) \cdot \nabla K_i u_i^{p_i+1} \\
&\qquad = \nabla K_i(y_i) \cdot \int_{B_\sigma^+(y_i)} (y - y_i) u_i^{p_i+1} \\
&\qquad\quad + O\left(\int_{B_\sigma^+(y_i)} |y - y_i| \cdot |\nabla K_i(y) - \nabla K_i(y_i)| u_i^{p_i+1}\right) \\
&\qquad = o(u_i(y_i)^{-2}).
\end{aligned}$$

PROPOSITION 2.4. *Suppose* $\{K_i\} \in C^1(\overline{B_2^+})$ *satisfies* (2.1) *and* (2.5) *with* $\Omega = B_2$, $n = 3$, *and some positive constants* A_1, A_2. *Suppose also that* u_i *satisfies* (2.2), *and* $y_i \to 0$ *is an isolated blow up point with* (2.6) *for some positive constant* A_3. *Then it is an isolated simple blow up point.*

PROOF. We first show that

$$y_i \in \partial B_1^+ \cap \partial \mathbb{R}_+^3 \quad \text{for } i \text{ large enough.} \tag{2.7}$$

Let $y_i = (y_{i1}, y_{i2}, y_{i3})$. Supposing the contrary of (2.7), we can assume, after passing to a subsequence, that $y_{i3} > 0$ for all i and (using Proposition 2.1) that for some $R_i \to \infty$, $y_{i3} u_i(y_i)^{(p_i-1)/2} > R_i$.

Consider

$$\xi_i(z) = y_{i3}^{2/(p_i-1)} u_i(y_i + y_{i3} z), \quad z \in B_{1/y_{i3}} \cap \{z \mid z_3 > -1\}.$$

Clearly ξ_i satisfies

$$\begin{cases} -\Delta\xi_i(z) = c(n)\widetilde{K}_i(z)\xi_i(z)^{p_i}, & z \in B_{1/y_{i3}} \cap \{z \mid z_3 > -1\} \\ \partial\xi_i/\partial z_3 = 0, & z \in \{z \mid z_3 = -1,\ |z| < 1/y_{i3}\}, \\ |z|^{2/(p_i-1)}\xi_i(z) \le A_3, & z \in B_{1/y_{i3}} \cap \{z \mid z_3 > -1\} \\ \lim_{i\to\infty} \xi_i(0) = \infty, \end{cases}$$

where $\widetilde{K}_i(z) = K_i(y_i + y_{i3}z)$.

It follows from Proposition 3.1 of [L1] that $z = 0$ is an isolated simple blow up point of $\{\xi_i\}$. Extend ξ_i to $\{z_3 < -1\}$ by setting $\xi_i(z_1, z_2, z_3) = \xi_i(z_1, z_2, -2-z_3)$. It follows from Proposition 2.3 of [L1] and the maximum principle that

$$\xi_i(0)\xi_i(z) \to h(z) = a(|z|^{2-n} + |z - (0,0,-2)|^{2-n}) + b \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\}),$$

for some constants $a > 0$ and $b \ge 0$. Applying Corollary 1.1 of [L1], for all $0 < \sigma < 1$ we have

$$\int_{\partial B_\sigma} B(\sigma, z, \xi_i, \nabla\xi_i) \ge \frac{c(n)}{p_i + 1}\int_{B_\sigma} z \cdot \nabla\widetilde{K}_i\xi_i^{p_i+1} + O(\xi_i(0)^{-p_i-1}).$$

Multiplying the above by $\xi_i(0)^2$ and sending i to ∞, we obtain (using Lemma 2.4 of [L1])

$$\int_{\partial B_\sigma} B(\sigma, z, h, \nabla h) \ge 0, \quad \forall 0 < \sigma < 1.$$

However, a direct calculation contradicts the above (using $b \ge 0$) for $\sigma > 0$ small. This establishes (2.7).

It follows from Proposition 2.1 that $r^{2/(p_i-1)}\overline{u}_i(r)$ has precisely one critical point in the interval $0 < r < r_i$. Suppose it is not an isolated simple blow up point and let μ_i be the second critical point of $r^{2/(p_i-1)}\overline{u}_i(r)$. We know that

$$\mu_i \ge r_i, \quad \lim_{i\to\infty} \mu_i = 0. \tag{2.8}$$

Without loss of generality (using (2.7)), we assume that $y_i = 0$. Set

$$\xi_i(y) = \mu_i^{2/(p_i-1)}u_i(\mu_i y), \quad y \in B^+_{1/\mu_i}.$$

It follows from (2.2) and (2.8) that ξ_i satisfies

$$\begin{cases} -\Delta\xi_i(y) = c(n)\widetilde{K}_i(y)\xi_i(y)^{p_i}, & y \in B^+_{1/\mu_i}, \\ \dfrac{\partial\xi_i}{\partial y_3}(y) = 0, & y \in \partial B^+_{1/\mu_i} \cap \partial\mathbb{R}^3_+, \\ |y|^{2/(p_i-1)}\xi_i(y) \le A_3, & y \in B^+_{1/\mu_i}, \\ \lim\limits_{i\to\infty} \xi_i(0) = \infty, \\ r^{2/(p_i-1)}\overline{\xi}_i(r) \text{ has precisely one critical point in } 0 < r < 1, \\ \dfrac{d}{dr}\{r^{2/(p_i-1)}\overline{\xi}_i(r)\}|_{r=1} = 0, \end{cases} \tag{2.9}$$

where $\widetilde{K}_i(y) = K_i(\mu_i y)$ and $\overline{\xi}_i(r) = |\partial B^+_r \cap \mathbb{R}^3_+|^{-1} \int_{\partial B^+_r \cap \mathbb{R}^3_+} \xi_i$.

It follows that $\{0\}$ is an isolated simple blow up point of $\{\xi_i\}$. Using Proposition 2.3 and the maximum principle, for some constants $a > 0$ and $b \geq 0$ we have

$$\xi_i(0)\xi_i(y) \to h(y) = a|y|^{2-n} + b \quad \text{in } C^2_{\text{loc}}(\overline{\mathbb{R}^3_+} \setminus \{0\}). \tag{2.10}$$

Using the last property in (2.9) and (2.10), we obtain $b = a > 0$. Applying Lemma 1.1, for all $0 < \sigma < 1$ we have

$$\int_{\partial B_\sigma \cap \mathbb{R}^3_+} B(\sigma, y, \xi_i, \nabla \xi_i) \geq \frac{c(n)}{p_i + 1} \int_{B^+_\sigma} z \cdot \nabla \widetilde{K}_i \xi_i^{p_i+1} + O(\xi_i(0)^{-p_i-1}).$$

Multiplying the above by $\xi_i(0)^2$ and sending i to ∞, we obtain (by Lemma 2.4)

$$\int_{\partial B_\sigma \cap \mathbb{R}^3_+} B(\sigma, y, h, \nabla h) \geq 0, \quad \forall 0 < \sigma < 1.$$

However, a direct calculation contradicts the above (using $b > 0$) for $\sigma > 0$ small.

PROPOSITION 2.5. *For $n = 3$, let $\{K_i\}$ be a convergent sequence of functions in $C^1(\overline{B^+_2})$. Suppose $\{u_i\}$ satisfies* (2.2) *and $y_i \to 0$ is a boundary isolated simple blow up point. Then*

$$|\nabla_{\tan} K_i(y_i)| = o(1).$$

If we further assume that $K_i \in C^{1,1}(B^+_2)$ with $\|\partial^2 K_i\|_{L^\infty(B^+_2)}$ uniformly bounded, then

$$|\nabla_{\tan} K_i(y_i)| = O(u_i(y_i)^{-2}).$$

PROOF. Define a smooth cutoff function $\eta \in C^\infty(\overline{\mathbb{R}^3_+})$ satisfying

$$\eta(x) = 1, \quad x \in \overline{B^+_{1/4}},$$
$$\eta(x) = 0, \quad x \in \overline{\mathbb{R}^3_+} \setminus B_{1/2}.$$

By multiplying (2.2) by $\eta \partial u_i / \partial x_j$ $(1 \leq j \leq 2)$ and integrating by parts on B^+_1, it follows from Proposition 2.3 and some standard elliptic estimates that

$$\left| \int_{B^+_1} \frac{\partial K_i}{\partial x_j} u_i^{p_i+1} \right| \leq C u_i(y_i)^{-2} + C\tau_i.$$

By a suitable Taylor expansion of $\partial K_i / \partial x_j$ at y_i, Proposition 2.5 follows from Lemmas 2.3 and 2.4.

For $2 \leq p \leq 5$, consider

$$\begin{cases} -\Delta_{g_0} u + \frac{3}{4} u = \frac{1}{8} K u^p, \quad u > 0, & \text{on } \mathbb{S}^3_-, \\ \partial u / \partial \nu = 0 & \text{on } \partial \mathbb{S}^3_-, \end{cases} \tag{2.11}$$

where ν denotes the unit outer normal at points of $\partial \mathbb{S}^3_-$.

THEOREM 2.1. *Let $\{K_i\}$ converge in $C^1(\overline{\mathbb{S}^3_-})$ norm to some positive function. Suppose $\{u_i\}$ satisfies (2.11) with $K = K_i$. Then after passing to a subsequence, $\{u_i\}$ is either uniformly bounded in $\overline{\mathbb{S}^3_-}$, or has precisely one blow up point which is a boundary isolated simple blow up point. Moreover, if we let $q_i \to \overline{q}$ denote the boundary isolated simple blow up point as in Definition 2.2, then for some constant $b_1 > 0$,*

$$\nabla_{\tan} K_i(q_i) = o(1), \quad \tau_i = b_1 \frac{\partial K_i}{\partial \nu}(q_i) u_i(q_i)^{-2} + o(u_i(q_i)^{-2}). \tag{2.12}$$

A consequence of Theorem 2.1 is the following a priori estimate on solutions of (0.1).

COROLLARY 2.1. *Let $K \in C^1(\overline{\mathbb{S}^3_-})$ be some positive function with no critical point on $\partial\mathbb{S}^3_-$. Then for any solution u of (0.1) and any $0 < \alpha < 1$, we have*

$$1/C \le \min_{\overline{\mathbb{S}^3_-}} u \le \max_{\overline{\mathbb{S}^3_-}} u \le C, \quad \|u\|_{C^{2,\alpha}(\mathbb{S}^3_-)} \le C,$$

where $C > 0$ depends continuously on $\min_{\overline{\mathbb{S}^3_-}} K > 0, \|\nabla K\|_{L^\infty(\mathbb{S}^3_-)}, \|\nabla K\|_{L^\infty(\partial\mathbb{S}^3_-)} > 0$, and $0 < \alpha < 1$.

PROPOSITION 2.5. *Suppose that $K \in C^{0,1}(\overline{\mathbb{S}^3_-})$ satisfies, for some positive constants A_1, A_2,*

$$K(q) \ge 1/A_1 \quad \text{for all } q \in \mathbb{S}^3_-, \quad \|\nabla K\|_{L^\infty(\mathbb{S}^3_-)} \le A_2. \tag{2.13}$$

Then for any $0 < \varepsilon < 1$ and $R > 1$, there exist some positive constants $C_0^* = C_0^*(\varepsilon, R, A_1, A_2)$ and $C_1^* = C_1^*(\varepsilon, R,, A_1, A_2) > 1$ such that if u is a solution of (2.11) with

$$\max_{\overline{\mathbb{S}^3_-}} u > C_0^*,$$

then there exists $1 \le k = k(u) < \infty$ and a set $\mathcal{S}(u) = \{q_1, \dots, q_k\} \subset \overline{\mathbb{S}^3_-}$ $(q_j = q_j(u))$ such that

(1) $0 \le \tau \equiv 5 - p < \varepsilon$,

(2) *$q_1, \dots, q_k$ are local maxima of u and if, for each $1 \le j \le k$, we let y be some geodesic normal coordinates centered at q_j, then*

$$\begin{cases} \|u(0)^{-1}u(u(0)^{-(p-1)/2}y) - \delta_j(y)\|_{C^2(B_{2R})} < \varepsilon, & \\ B_{Ru(q_j)^{-(p-1)/2}}(q_j) \subset \mathbb{S}^3_- & \text{if } q_j \in \mathbb{S}^3_-, \\ \|u(0)^{-1}u(u(0)^{-(p-1)/2}y) - \delta_j(y)\|_{C^2(B^+_{2R})} < \varepsilon & \text{if } q_j \in \partial\mathbb{S}^3_-, \end{cases}$$

and $\{B_{Ru(q_j)^{-(p-1)/2}}(q_j)\}_{1\le j\le k}$ are disjoint, where

$$\delta_j(y) = (1 + k_j|y|^2)^{(2-n)/2} \quad \text{and} \quad k_j = c(n)[n(n-2)]^{-1}K(q_j),$$

(3) *$u(q) \le C_1^*\{\mathrm{dist}(q, \mathcal{S}(u))\}^{-2/(p-1)}$ for all $q \in \mathbb{S}^3_-$.*

Proof. The proof follows from the uniqueness results of Caffarelli–Gidas–Spruck, and some blow up argument. We omit the details. The argument is similar to that in [Z], taking into account the proof of Proposition 2.1 here.

Proposition 2.6. *Suppose that $K \in C^{0,1}(\overline{\mathbb{S}^3_-})$ satisfies* (2.13) *for some positive constants A_1, A_2. Then for any $\varepsilon > 0$ and $R > 1$, there exists some positive constant $\delta^* = \delta^*(\varepsilon, R, A_1, A_2)$ such that for any solution u of* (2.11) *with $\max_{\overline{\mathbb{S}^3_-}} u > C_0^*$ we have*

$$|q_j - q_l| \ge \delta^* \quad \textit{for all } 1 \le j \ne l \le k,$$

where $q_j = q_j(u)$, $q_l = q_l(u)$, and $k = k(u)$ are defined in Proposition 2.5.

Proof. The proof is similar to the proof of Proposition 4.2 of [L1]. As always we often pass to subsequences when necessary. Suppose the contrary: for some constants $A_1, A_2, \varepsilon > 0$ and $R > 1$, there exists a sequence $\{K_i\} \in C^{0,1}(\overline{\mathbb{S}^3_-})$ satisfying (2.13), and a sequence $\{u_i\}$ of solutions of (2.11) corresponding to $\{K_i\}$ satisfying

$$\max_{\overline{\mathbb{S}^3_-}} u_i > C_0^*, \qquad \min_{1 \le j \ne l \le k} |q_j - q_l| \to 0^+.$$

Without loss of generality,

$$|q_1 - q_2| = \min_{1 \le j \ne l \le k} |q_j - q_l|, \quad q_1, q_2 \to \overline{q} \in \overline{\mathbb{S}^3_-}.$$

Using (2) of Proposition 2.5, we know that $Ru_i(q_1)^{-(p_i-1)/2}, Ru_i(q_2)^{-(p_i-1)/2} \le |q_1 - q_2|$. It follows that $u_i(q_1), u_i(q_2) \to \infty$. By making a suitable stereographic projection to transform $\mathbb{S}^3_-$ to $\mathbb{R}^3_+$, u_i is transformed to v_i which satisfies

$$\begin{cases} -\Delta v_i = \frac{1}{8} K_i H_i^{\tau_i} v_i^{p_i}, \quad v_i > 0, & \text{in } \mathbb{R}^3_+, \\ \partial v_i / \partial y_3 = 0 & \text{on } \partial \mathbb{R}^3_+, \end{cases} \tag{2.14}$$

where $H_i(y) = (2/(1+|y|^2))^{(n-2)/2}$. We can assume without loss of generality that

$$\begin{aligned} & v_i(q) \le C_1^* \min_{1 \le j \le k} |q - q_j|, \quad \forall q \in \mathbb{R}^3_+, \\ & q_1, q_2 \text{ are local maxima of } v_i, \\ & \sigma_i \equiv |q_1 - q_2| = \min_{1 \le j \ne l \le k} |q_j - q_l| \to 0^+, \quad q_1, q_2 \to \overline{q} \in \overline{\mathbb{R}^3_+}. \end{aligned} \tag{2.15}$$

Notice that we have abused notation slightly by not distinguishing points in $\mathbb{S}^3_-$ from points in $\mathbb{R}^3_+$. Also we need to reselect points q_1, q_2 in order to satisfy (2.15) since this property is not preserved by stereographic projection. With the help of Proposition 2.1, this can be easily achieved by going to a subsequence.

Set

$$w_i(y) = \sigma_i^{2/(p_i-1)} v_i(q_1 + \sigma_i y), \quad |y| < 2/\sigma_i, \ y_3 \ge T_i,$$

where $-\sigma_i T_i$ is the third coordinate component of q_1. It is clear that w_i satisfies

$$\begin{cases} -\Delta w_i(y) = \frac{1}{8}\widetilde{K}_i(y)\widetilde{H}_i(y)^{\tau_i} w_i(y)^{p_i}, & |y| \le 2/\sigma_i,\ y_3 \ge T_i, \\ w_i(y) > 0, & |y| \le 2/\sigma_i,\ y_3 \ge T_i, \end{cases}$$

where $\widetilde{K}_i(y) = K_i(\sigma_i y)$ and $\widetilde{H}_i(y) = H_i(\sigma_i y)$.

The following properties can be deduced from properties of v_i and Proposition 2.1:

$$\begin{cases} w_i(0), w_i(\widetilde{y}_i) \ge 1/C, & \nabla w_i(0) = \nabla w_i(\tilde{y}_i) = 0, \\ |y|^{2/(p_i-1)} w_i(y) \le C_1, & |y| \le 1/2,\ y_3 \ge T_i, \\ |y-\widetilde{y}_i|^{2/(p_i-1)} w_i(y) \le C_1, & |y-\widetilde{y}_i| \le 1/2,\ y_3 \ge T_i, \end{cases}$$

where $\widetilde{y}_i = (\widetilde{y}_{i1}, \widetilde{y}_{i2}, \widetilde{y}_{i3}) = (q_2 - q_1)/\sigma_i$. It is not difficult to see that $w_i(0), w_i(\tilde{y}_i) \to \infty$, since otherwise they both have to have finite limits and after passing to the limit, lead to a positive solution of $-\Delta w = w^{(n+2)/(n-2)}$ in the upper half plane with two critical points. This violates the uniqueness theorem of [CGS]. Therefore $\{0\}$ and $\widetilde{y}_i \to \widetilde{y}$ are both isolated blow up points of $\{w_i\}$, hence isolated simple blow up points due to Proposition 3.1 of [L1] and Proposition 2.4. By multiplying the equation by $w_i(0)$, it follows from Proposition 2.3 and the maximum principle (see the proof of Proposition 4.2 in [L1]) that there exists a closed set $\mathcal{S}_2 \subset \mathbb{R}^3_+$ containing neither $\{0\}$ nor $\{\widetilde{y}\}$, and some function $b^* \in C^\infty(\overline{\mathbb{R}^3_+} \setminus \mathcal{S}_2)$ satisfying $\Delta b^* = 0$ and $b^* \ge 0$ on $\mathbb{R}^3_+ \setminus \mathcal{S}_2$ such that

$$\lim_{i\to\infty} w_i(0) w_i(y) = h^*(y) \equiv a_1 |y|^{2-n} + a_2 |y - \widetilde{y}|^{2-n} + b^*(y) \quad \text{in } C^2_{\mathrm{loc}}(\overline{\mathbb{R}^3_+} \setminus \mathcal{S}_2),$$

where $a_1, a_2 > 0$ are some constants. In particular, for some constant $b > 0$,

$$h^*(y) = a_1 |y|^{2-n} + b + O(|y|) \quad \text{for } y \text{ close to } 0.$$

Applying Lemma 1.1 (or Corollary 1.1 of [L1]) as in the proof of Proposition 2.4, we reach a contradiction.

PROOF OF THEOREM 2.1. Let $\{u_i\}$ satisfy $\max_{\overline{\mathbb{S}^3_-}} u_i \to \infty$. After passing to a subsequence, it follows from Propositions 2.6 and 2.4 that $\{u_i\}$ has finitely many isolated simple blow up points, denoted as $\{\overline{q}^{(1)}, \dots, \overline{q}^{(k)}\}$. Let $q_i^{(j)} \to \overline{q}^{(j)}$ be the local maximum of u_i as stated in Definition 2.2. We know from Proposition 2.5, and Proposition 3.2 of [L1], that $\lim_{i\to\infty} |\nabla K_i(q_i^{(j)})| = 0$ if $\overline{q}^{(j)} \in \mathbb{S}^3_-$, and $\lim_{i\to\infty} |\nabla_{\tan} K_i(q_i^{(j)})| = 0$ if $\overline{q}^{(j)} \in \partial\mathbb{S}^3_-$. It follows from Proposition 2.3 and some standard elliptic theory that

$$u_i(q_i^{(1)}) u_i \to h \equiv \sum_{j=1}^k a_j (G_{\overline{q}^{(j)}} + G_{\hat{q}^{(j)}}) \quad \text{in } C^2_{\mathrm{loc}}(\overline{\mathbb{S}^3_-} \setminus \{\overline{q}^{(1)}, \dots, \overline{q}^{(k)}\}),$$

where $a_1, \dots, a_k > 0$ are some constants, $\widehat{q}^{(j)} \in \overline{\mathbb{S}^3_+}$ denotes the symmetric point of $\overline{q}^{(j)}$, and $G_{\overline{q}^{(j)}}$ denotes the Green function of $-\Delta_{g_0} + \frac{3}{4}$ at $\overline{q}^{(j)}$.

We first show that $\{\overline{q}^{(1)}, \dots, \overline{q}^{(k)}, \widehat{q}^{(1)}, \dots, \widehat{q}^{(k)}\}$ consists of precisely one point. Suppose the contrary; it follows from the positivity of the Green function that for some constants $a, b > 0$,

$$h(q) = aG_{\overline{q}^{(1)}} + b + O(|q - \overline{q}^{(1)}|).$$

Applying the Pokhozhaev type identity as in the proof of Proposition 2.4, we reach a contradiction. This shows that $\{u_i\}$ has precisely one boundary isolated simple blow up point. By making a suitable stereographic projection to transform $\mathbb{S}^3_-$ to $\mathbb{R}^3_+$ and q_i to 0, u_i is transformed to v_i which satisfies (2.14). Applying Lemma 1.1, we obtain

$$\int_{\mathbb{R}^3_+} y \cdot \nabla(K_i H_i^{\tau_i}) v_i^{p_i} + \frac{\tau_i}{2} \int_{\mathbb{R}^3_+} K_i H_i^{\tau_i} v_i^{p_i+1} = 0. \tag{2.16}$$

Using Lemma 2.4, Lemma 2.3 and Proposition 2.5, we have

$$\int_{\mathbb{R}^3_+} y \cdot \nabla(K_i H_i^{\tau_i}) v_i^{p_i} = \frac{\partial K_i}{\partial y_3}(0) \int_{\mathbb{R}^3_+} y_3 v_i^{p_i} + o(v_i(0)^{-2}).$$

Estimate (2.12) follows from the above, (2.16) and Lemma 2.3.

3. Proof of Theorem 0.1

We define $H = \{u \in H^1(\mathbb{S}^3) \mid u(x_1, x_2, x_3, x_4) = u(x_1, x_2, x_3, -x_4)\}$, and the inner product and norm by $\langle u, v \rangle = \int_{\mathbb{S}^3} \left(\nabla u \nabla v + \frac{3}{4} uv\right)$ and $\|u\| = \sqrt{\langle u, u \rangle}$ respectively. For $\tau > 0$ small, we set

$$I_\tau(u) = \frac{1}{2}\|u\|^2 - \frac{1}{8(6-\tau)} \int_{\mathbb{S}^3} K|u|^{6-\tau}.$$

For $P \in \mathbb{S}^3$ and $t > 0$, we define

$$\delta_{P,t}(x) = \left(\frac{t}{1 + \frac{t^2-1}{2}(1 - \cos d(P, x))} \right)^{1/2}, \quad x \in \mathbb{S}^3,$$

where $d(P, x)$ denotes the geodesic distance between $P, x \in \mathbb{S}^3$. It is well known that $\delta_{P,t}$ satisfies

$$-\Delta_{g_0} \delta_{P,t} + \tfrac{3}{4}\delta_{P,t} = \tfrac{3}{4}\delta_{P,t}^5,$$

and

$$\|\delta_{P,t}\|^2 = \tfrac{3}{4}|\mathbb{S}^3|, \quad \int_{\mathbb{S}^3} \delta_{P,t}^6 = |\mathbb{S}^3|.$$

For $\overline{P} \in \mathcal{K}^-$ and $\varepsilon_0 > 0$ suitably small, let

$$\Omega_{\varepsilon_0}(\overline{P}) = \{(\alpha, t, P) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \partial\mathbb{S}^3_- \mid |\alpha - (6/K(P))^{1/4}| < \varepsilon_0, \\ |P - \overline{P}| < \varepsilon_0,\ t > 1/\varepsilon_0\}.$$

It follows from [BC] that for any $u \in H$ satisfying $\|u - \widetilde{\alpha}\delta_{\widetilde{P},\widetilde{t}}\| < \varepsilon_0/2$, for some $(\widetilde{\alpha}, \widetilde{t}, \widetilde{P}) \in \Omega_{\varepsilon_0/2}$, we have a unique representation

$$u = \alpha\delta_{P,t} + v$$

with $(\alpha, t, P) \in \Omega_{\varepsilon_0}$ and

$$\langle v, \delta_{P,t} \rangle = \left\langle v, \frac{\partial \delta_{P,t}}{\partial P^{(l)}} \right\rangle = \left\langle v, \frac{\partial \delta_{P,t}}{\partial t} \right\rangle = 0. \tag{3.1}$$

We work in some orthonormal basis near $\overline{P}$, and $\partial/\partial P^{(l)}$ denotes the corresponding derivatives. By uniqueness, we know that $P \in \partial\mathbb{S}^3_-$, and hence $v \in H$. We denote the set of $v \in H$ satisfying (3.1) by $E_{t,P}$. It follows that in a small tubular neighborhood (independent of τ) of $\{\alpha\delta_{P,t} \mid (\alpha, t, P) \in \Omega_{\varepsilon_0/2}\}$ in H, (α, t, P, v) is a good parametrization. In the new parameters, we write

$$J_\tau(\alpha, t, P, v) = I_\tau(u) \quad \text{for } u = \alpha\delta_{P,t} + v.$$

For a suitably large constant A and suitably small constants ε_0, ν_0, set

$$\Sigma_\tau(\overline{P}) = \{(\alpha, t, P, v) \in \Omega_{\varepsilon_0/2} \times H \mid 1/A < t\tau < A,\ v \in E_{t,P},\ \|v\| < \nu_0\}.$$

Without confusion we use the same notation for

$$\Sigma_\tau(\overline{P}) = \{u = \alpha\delta_{P,t} + v \mid (\alpha, t, P, v) \in \Sigma_\tau(\overline{P})\}.$$

PROPOSITION 3.1. *For $K \in \mathcal{A} \cap C^2(\overline{\mathbb{S}^3_-})$, assume that $K|_{\partial\mathbb{S}^3_-}$ is a Morse function. Let $0 < \alpha < 1$. Then there exist some positive constants $\varepsilon_0, \nu_0 \ll 1$ and $A, R \gg 1$, depending only on K and α, such that, when $\tau > 0$ is sufficiently small,*

$$u \in \mathcal{O}_R \equiv \{w \in C^{2,\alpha}(\overline{\mathbb{S}^3_-}) \mid 1/R < w < R \text{ on } \mathbb{S}^3_-,\ \|w\|_{C^{2,\alpha}(\mathbb{S}^3_-)} < R\},$$

or $u \in \Sigma_\tau(\overline{P})$ for some $\overline{P} \in \mathcal{K}^-$, for all u satisfying $u \in H$, $u > 0$ a.e., and $I'_\tau(u) = 0$.

PROOF. This follows from Theorem 2.1, Proposition 2.3, and some standard elliptic estimates.

THEOREM 3.1. *For $K \in \mathcal{A} \cap C^2(\overline{\mathbb{S}^3_-})$, assume that $K|_{\partial\mathbb{S}^3_-}$ is a Morse function. Then for $\tau > 0$ sufficiently small, and $\overline{P} \in \mathcal{K}^-$, I_τ has a unique critical point in $\Sigma_\tau(\overline{P})$, which is nondegenerate with Morse index $3 - i(\overline{P})$. It follows that*

$$\deg_{H^1}(I'_\tau, \Sigma_\tau(\overline{P}), 0) = (-1)^{3 - i(\overline{P})}. \tag{3.2}$$

Here $i(\overline{P})$ denotes the Morse index of $K|_{\partial\mathbb{S}^3_-}$ at $\overline{P}$.

PROPOSITION 3.2. *For $\tau > 0$ small and $(\alpha, t, P, 0) \in \Sigma_\tau(\overline{P})$ with $\overline{P} \in \mathcal{K}^-$, there exists a unique minimizer $\overline{v} = \overline{v}_\tau(\alpha, t, P) \in E_{t,P}$ of $J_\tau(\alpha, t, P, v)$. Furthermore,*

$$\|\overline{v}\| \le C\tau|\log\tau|, \quad \langle \partial_v J_\tau(\alpha, t, P, v), v\rangle \ne 0, \quad \forall (\alpha, t, P, v) \in \Sigma_\tau(\overline{P}), v \ne \overline{v},$$

and $(\tau, \alpha, t, P) \mapsto \overline{v}_\tau(\alpha, t, P)$ is a C^2 map to H.

PROOF. It follows from a direct calculation, using (3.1), that

$$J_\tau(\alpha, t, P, v) = \frac{3|\mathbb{S}^3|\alpha^2}{8} - \frac{\alpha^{6-\tau}}{8(6-\tau)} \int_{\mathbb{S}^3} K\delta_{P,t}^{6-\tau} + f_\tau(v) + Q_\tau(v, v) + O(\|v\|^3),$$

where

$$\begin{aligned} f_\tau(v) &= -\frac{\alpha^{5-\tau}}{8} \int_{\mathbb{S}^3} K\delta_{P,t}^{5-\tau} v, \\ Q_\tau(\varphi, v) &= \frac{1}{2}\langle \varphi, v\rangle - \frac{(5-\tau)\alpha^{4-\tau}}{16} \int_{\mathbb{S}^3} K\delta_{P,t}^{4-\tau} \varphi v. \end{aligned}$$

It is well known that for some $\delta_0 > 0$, $Q_0(v, v) \ge \delta_0\|v\|^2$ for all $v \in E_{t,P}$. It follows, after some elementary calculations, that for $\tau > 0$ small we have

$$Q_\tau(v, v) \ge (\delta_0/2)\|v\|^2, \quad \forall (\alpha, t, P, v) \in \Sigma_\tau(\overline{P}).$$

Using (3.1), the Sobolev embedding theorem and Lemma A in the Appendix, we have

$$\begin{aligned} f_\tau(v) = &- \frac{\alpha^{5-\tau}}{8} \int_{\mathbb{S}^3} [K - K(P)]\delta_{P,t}^5 v + O(\|\delta_{P,t}^5 - \delta_{P,t}^{5-\tau}\|_{L^{6/5}}\|v\|) \\ &= O(\| \,|\cdot - P|\delta_{P,t}^5\|_{L^{6/5}} + \|\delta_{P,t}^5 - \delta_{P,t}^{5-\tau}\|_{L^{6/5}})\|v\| \\ &= O(\tau|\log\tau|)\|v\|. \end{aligned}$$

It follows that $\|f_\tau\| = O(\tau|\log\tau|)$. The existence, uniqueness and C^2 dependence of the minimizer $\overline{v} = \overline{v}_\tau(\alpha, t, P)$ as stated in Proposition 3.2 follow from standard functional analysis arguments.

PROOF OF THEOREM 3.1. We will only prove (3.2). The full strength of Theorem 3.1 can be proved by some further essentially elementary, even though somewhat tedious, argument. Set $\beta = \alpha - (6/K(P))^{1/4}$. It follows from (3.1) and Lemma A in the Appendix that

$$
\begin{aligned}
(3.3)\quad & \frac{\partial}{\partial \alpha} J_\tau(\alpha,t,P,v)\\
=\ & \frac{3|\mathbb{S}^3|\alpha}{4} - \frac{1}{8}\alpha^{5-\tau}\int_{\mathbb{S}^3} K\delta_{P,t}^{6-\tau} - \frac{(5-\tau)\alpha^{4-\tau}}{8}\int_{\mathbb{S}^3} K\delta_{P,t}^{5-\tau} v + O(\|v\|^2)\\
=\ & \frac{3|\mathbb{S}^3|\alpha}{4} - \frac{1}{8}\alpha^{5-\tau}K(P)\int_{\mathbb{S}^3}\delta_{P,t}^6 + O(\|\,|\cdot - P|\delta_{P,t}^{6-\tau}\|_{L^1}) + O(\|\delta_{P,t}^6 - \delta_{P,t}^{6-\tau}\|_{L^1})\\
& + O(\|\,|\cdot - P|\delta_{P,t}^{5-\tau}\|_{L^{6/5}}\|v\|) + O(\|\delta_{P,t}^5 - \delta_{P,t}^{5-\tau}\|_{L^{6/5}}\|v\|) + O(\|v\|^2)\\
=\ & -3|\mathbb{S}^3|\beta + O(\tau|\log\tau|) + O(\beta^2) + O(\|v\|^2).
\end{aligned}
$$

Similarly, noticing that several integrals vanish due to oddness with respect to certain directions, e.g., $\int_{\mathbb{S}^3}\delta_{P,t}^{5-\tau}\frac{\partial\delta_{P,t}}{\partial P} = 0$, we have

$$
\begin{aligned}
(3.4)\quad & \frac{\partial}{\partial P} J_\tau(\alpha,t,P,v)\\
&= -\frac{\alpha^{6-\tau}}{4}\int_{\mathbb{S}^3_-} K\delta_{P,t}^{5-\tau}\frac{\partial\delta_{P,t}}{\partial P} + O(\|v\|)\\
&= -\frac{\alpha^{6-\tau}}{4}\int_{\mathbb{S}^3_-} [K - K(P)]\delta_{P,t}^{5-\tau}\frac{\partial\delta_{P,t}}{\partial P} + O(\|v\|)\\
&= -\frac{\alpha^{6-\tau}}{4}\int_{\mathbb{S}^3_-} \nabla K(P)\cdot(\cdot - P)\delta_{P,t}^{5-\tau}\frac{\partial\delta_{P,t}}{\partial P} + O(1/t) + O(\|v\|)\\
&= -\Gamma K(P)^{-3/2}\nabla_{\tan}K(P) + O(1/t) + O(|\beta|) + O(\|v\|).
\end{aligned}
$$

Here and throughout the paper, $\Gamma > 0$ denotes various universal constants. Set

$$
\widetilde{\Sigma}_\tau = \{u = \alpha\delta_{P,t} + v \in \Sigma_\tau \mid \|v\| < \tau|\log\tau|^3,\ |\beta| < \tau|\log\tau|^2\}.
$$

Using Proposition 3.1, Proposition 3.2 and (3.3), we know that $I'_\tau(u) \neq 0$ for all $u \in \Sigma_\tau \setminus \widetilde{\Sigma}_\tau$. In the following, we only make calculations for $u = \alpha\delta_{P,t} + v \in \widetilde{\Sigma}_\tau$.

A calculation yields

$$
\begin{aligned}
\frac{\partial}{\partial t} J_\tau(\alpha,t,P,v) &= -\frac{\alpha}{8}\int_{\mathbb{S}^3} K|\alpha\delta_{P,t} + v|^{4-\tau}(\alpha\delta_{P,t} + v)\frac{\partial\delta_{P,t}}{\partial t}\\
&= -\frac{\alpha}{8}\int_{\mathbb{S}^3} K\{(\alpha\delta_{P,t})^{5-\tau} + (5-\tau)(\alpha\delta_{P,t})^{4-\tau}v\}\frac{\partial\delta_{P,t}}{\partial t}\\
&\quad + O\Big(\|v\|^2\Big\|\frac{\partial\delta_{P,t}}{\partial t}\Big\|\Big).
\end{aligned}
$$

Noticing that $\int_{\mathbb{S}^3} \delta_{P,t}^4 v \frac{\partial \delta_{P,t}}{\partial t} = \frac{1}{5}\frac{\partial}{\partial t}\int_{\mathbb{S}^3} \delta_{P,t}^5 v = 0$, we have

$$\begin{aligned}
&\int_{\mathbb{S}^3} K\delta_{P,t}^{4-\tau} v \frac{\partial \delta_{P,t}}{\partial t} \\
&\quad = \int_{\mathbb{S}^3} [K - K(P)]\delta_{P,t}^{4-\tau} v \frac{\partial \delta_{P,t}}{\partial t} + \int_{\mathbb{S}^3} K(P)[\delta_{P,t}^{4-\tau} - \delta_{P,t}^4] v \frac{\partial \delta_{P,t}}{\partial t} \\
&\quad = O\bigg(\| \,|\cdot - P|\delta_{P,t}^{4-\tau}\|_{L^{3/2}}\|v\|\bigg\|\frac{\partial \delta_{P,t}}{\partial t}\bigg\|\bigg) + O\bigg(\|\delta_{P,t}^4 - \delta_{P,t}^{4-\tau}\|_{L^{3/2}}\|v\|\bigg\|\frac{\partial \delta_{P,t}}{\partial t}\bigg\|\bigg) \\
&\quad = o(t^{-2}).
\end{aligned}$$

It follows from the above and from $6\int_{\mathbb{S}^3} \delta_{P,t}^5 \frac{\partial \delta_{P,t}}{\partial t} = \frac{\partial}{\partial t}\int_{\mathbb{S}^3} \delta_{P,t}^6 = 0$ that

$$\begin{aligned}
(3.5)\qquad &\frac{\partial}{\partial t} J_\tau(\alpha, t, P, v) \\
&= -\frac{\alpha^{6-\tau}}{4}\int_{\mathbb{S}^3_-} K\delta_{P,t}^{5-\tau}\frac{\partial \delta_{P,t}}{\partial t} + o(t^{-2}) \\
&= -\frac{\alpha^{6-\tau}}{4}\int_{\mathbb{S}^3_-} [K(P) + \nabla K(P)\cdot(\cdot - P)]\delta_{P,t}^{5-\tau}\frac{\partial \delta_{P,t}}{\partial t} \\
&\quad + O\bigg(\| \,|\cdot - P|^2\delta_{P,t}^{5-\tau}\|_{L^{6/5}}\bigg\|\frac{\partial \delta_{P,t}}{\partial t}\bigg\|\bigg) + o(t^{-2}) \\
&= -K(P)\frac{\alpha^{6-\tau}}{4}\int_{\mathbb{S}^3_-} \delta_{P,t}^{5-\tau}\frac{\partial \delta_{P,t}}{\partial t} \\
&\quad + \frac{\alpha^{6-\tau}}{4}\frac{\partial K}{\partial \nu}(P)\int_{\mathbb{S}^3_-} |(\cdot - P)_4|\delta_{P,t}^{5-\tau}\frac{\partial \delta_{P,t}}{\partial t} + o(t^{-2}) \\
&= \Gamma\sqrt{K(P)}\,\frac{\tau}{t} - \Gamma K(P)^{-3/2}\frac{\partial K}{\partial \nu}(P)t^{-2} + O(|\beta|t^{-2}) + o(t^{-2}).
\end{aligned}$$

At $u = \alpha\delta_{P,t} + v \in \widetilde{\Sigma}_\tau$,

$$T_u H = E_{t,P} \oplus \operatorname{span}\bigg\{\delta_{P,t}, \frac{\partial \delta_{P,t}}{\partial t}, \frac{\partial \delta_{P,t}}{\partial P}\bigg\}.$$

We write $I'_\tau(u) \in T_u H$ as

$$I'_\tau(u) = \xi + \eta,$$

where $\xi \in E_{t,P}$ and $\eta \in \operatorname{span}\{\delta_{P,t}, \partial\delta_{P,t}/\partial t, \partial\delta_{P,t}/\partial P\}$. For all $\varphi \in E_{t,P}$,

$$\langle \xi, \varphi\rangle = I'_\tau(u)\varphi = f_\tau(\varphi) + 2Q_\tau(\varphi, v) + \langle V_v(\tau, \alpha, t, P, v), \varphi\rangle,$$

where V_v is some function satisfying $\|V_v(\tau, \alpha, t, P, v)\| \le C\|v\|^2$. Taking $\varphi = v$, we get

$$\|\xi\| \ge \delta_0\|v\| - \|f_\tau\| - O(\|v\|^2) \ge \frac{\delta_0}{2}\|v\| - \|f_\tau\|.$$

It follows from (3.3) that

$$\langle \eta, \delta_{P,t}\rangle = \frac{\partial}{\partial \alpha} I_\tau(u) = -3|\mathbb{S}^3|\beta + V_\alpha(\tau,\alpha,t,P,v),$$

where V_α satisfies $|V_\alpha(\tau,\alpha,t,P,v)| \le C\tau|\log\tau|$. It follows from (3.5) that

$$\left\langle \eta, \frac{\partial \delta_{P,t}}{\partial t}\right\rangle = \frac{1}{\alpha}\frac{\partial}{\partial t} I_\tau(u) = \Gamma K(P)^{3/4}\frac{\tau}{t} - \Gamma K(P)^{-5/4}\frac{\partial K}{\partial \nu}(P)t^{-2} + V_t(\tau,\alpha,t,P,v),$$

where V_t satisfies $|V_t(\tau,\alpha,t,P,v)| = o(t^{-2})$. It follows from (3.4) that

$$\left\langle \eta, \frac{\partial \delta_{P,t}}{\partial P}\right\rangle = \frac{1}{\alpha}\frac{\partial}{\partial P} I_\tau(u) = -\Gamma K(P)^{-5/4}\nabla_{\tan} K(P) + V_P(\tau,\alpha,t,P,v),$$

where V_P satisfies $|V_P(\tau,\alpha,t,P,v)| = C(\tau + |\beta| + \|v\|) = o(1)$.

It is well known that $I'_\tau(u) = \xi + \eta$ is of the form $\mathrm{id} + \text{compact}$ in H. We first define $P(\theta)$ as the geodesic trajectory on $\partial\mathbb{S}^3_-$ with $P(1) = P$ and $P(0) = \overline{P}$. Define

$$X_\theta = \xi_\theta + \eta_\theta, \quad 0 \le \theta \le 1,$$

as follows. For all $\varphi \in E_{t,P}$, $0 \le \theta \le 1$,

$$\begin{aligned}
\langle \xi_\theta, \varphi\rangle &= \theta f_\tau(\varphi) + (1-\theta)\langle v, \varphi\rangle + 2\theta Q_\tau(\varphi, v) + \theta\langle V_v(\tau,\alpha,t,P,v), \varphi\rangle,\\
\langle \eta_\theta, \delta_{P,t}\rangle &= -3|\mathbb{S}^3|\beta + \theta V_\alpha,\\
\left\langle \eta_\theta, \frac{\partial \delta_{P,t}}{\partial t}\right\rangle &= \Gamma K(P(\theta))^{3/4}\frac{\tau}{t} - \Gamma K(P(\theta))^{-5/4}\frac{\partial K}{\partial \nu}(P(\theta))t^{-2} + \theta V_t,\\
\left\langle \eta, \frac{\partial \delta_{P,t}}{\partial P}\right\rangle &= -\Gamma K(P(\theta))^{-5/4}\nabla_{\tan} K(P) + tV_P.
\end{aligned}$$

It is easy to see that X_θ is well defined in $\widetilde{\Sigma}_\tau$. It follows from the Sobolev compact embedding theorem, the explicit form of V_v, V_α, V_t, V_P, $A^{-1} < t\tau < A$, and the estimates we have obtained that X_θ is of the form $\mathrm{id} + \text{compact}$. Furthermore, it is not difficult to see that X_θ $(0 \le \theta \le 1)$ is an admissible homotopy with $X_\theta|_{\partial\widetilde{\Sigma}_\tau} \ne 0$. It follows that

$$\deg_{H^1}(X_1, \widetilde{\Sigma}_\tau, 0) = \deg_{H^1}(X_0, \widetilde{\Sigma}_\tau, 0).$$

It is easy to see that

$$\deg_{H^1}(X_0, \widetilde{\Sigma}_\tau, 0) = (-1)^{3 - i(\overline{P})}.$$

We have thus established (3.2).

Proof of Theorem 0.1 (and the justification of the definition of Index: $\mathcal{A} \to \mathbb{Z}$). Part (a) follows from Theorem 2.1. For $K \in \mathcal{A} \cap C^2(\overline{\mathbb{S}^3_-})$, $K|_{\partial\mathbb{S}^3_-}$ being a Morse function, (0.3) follows from Theorem 3.1 and properties of the Leray–Schauder degree as in [L2]. Now the definition of Index can be justified by the above and the homotopy invariance of the Leray–Schauder degree. For the same reason, (0.3) holds for all $K \in \mathcal{A}$. Part (b) can be derived from (a) by an argument similar to that in [L3].

Appendix

LEMMA A. *Let $A > 1$ be some positive constant, $\tau > 0$, and $1/A < t\tau < A$. Then*

$$\| \, |\cdot - P|\delta_{P,t}^{5}\|_{L^{6/5}(\mathbb{S}^3)} \le C/t, \quad \| \, |\cdot - P|\delta_{P,t}^{5-\tau}\|_{L^{6/5}(\mathbb{S}^3)} \le C/t,$$
$$\int_{\mathbb{S}^3} |\cdot - P|^a \delta_{P,t}^{6-\tau} = \begin{cases} \Gamma t^{-\tau/2} + o(1), & a = 0, \\ \Gamma t^{-a} + o(t^{-a}), & 0 < a < 3, \end{cases}$$
$$\int_{\mathbb{S}^3} |\cdot - P|^a \delta_{P,t}^{6} = \Gamma t^{-a} + o(t^{-a}), \quad 0 < a < 3,$$
$$\int_{\mathbb{S}^3} \delta_{P,t}^{6} |1 - \delta_{P,t}^{-\tau}|^a \le C(a)(\tau|\log\tau|)^a, \quad 0 \le a < 3,$$
$$\left\| \frac{\partial \delta_{P,t}}{\partial t} \right\| = \Gamma/t, \quad \int_{\mathbb{S}^3} \delta_{P,t}^{5-\tau} \frac{\partial \delta_{P,t}}{\partial t} = -\Gamma\tau/t + o(\tau/t),$$
$$\int_{\mathbb{S}^3} |(\cdot - P)_4| \delta_{P,t}^{5-\tau} \frac{\partial \delta_{P,t}}{\partial t} = -\Gamma t^{-2} + o(t^{-2}),$$
$$\int_{\mathbb{S}^3} |\cdot - P|^2 \delta_{P,t}^{5-\tau} \left| \frac{\partial \delta_{P,t}}{\partial P} \right| \le C/t,$$

where $o(1)$ denotes some quantity which tends to 0 as t tends to infinity, and C denotes some constant depending only on A. We also recall that $\Gamma > 0$ denotes various universal constants.

PROOF. This follows from straightforward calculations.

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