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NEW METHOD FOR LARGE QUASIPERIODIC NONLINEAR OSCILLATIONS WITH FIXED FREQUENCIES FOR THE NONDISSIPATIVE SECOND TYPE DUFFING EQUATION

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Dedicated to Professor Nirenberg on his birthday

1. Introduction

In this paper we will use a new partial differential equation method to find a new family of quasiperiodic solutions of fixed frequencies for the forced second type nondissipative Duffing equation which can be written as

$$\ddot{u} + au - bu^3 = f(t), \tag{1}$$

where $a > 0, b > 0$, and $f(t)$ is assumed to be a quasiperiodic function with given prescribed rationally independent frequencies $\omega_1, \ldots, \omega_m$. The solutions found will have frequencies proportional to $\omega_1, \ldots, \omega_m$. First we solve the equation

$$\ddot{u} + \beta^2 u - \frac{b\beta^2}{a} u^3 = \frac{\beta^2}{a} f(t), \tag{2}$$

where a and b are arbitrary positive numbers and β satisfies the condition $0 < \beta < 2/I$, where I is the maximum length of a segment with direction $(\omega_1, \ldots, \omega_m)$ cut out by the boundary of the torus $T^m = [-\pi, \pi]^m$.

We derive a nonlinear partial differential equation for the generating function $U(x)$ of the tentative smooth solutions $u(t)$ for (2):

$$\sum_{i,j=1}^{m} \omega_i \omega_j \frac{\partial^2 U}{\partial x_i \partial x_j} + \beta^2 U - \frac{b\beta^2}{a} U^3 = \frac{\beta^2}{a} F(x), \tag{3}$$

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where $f(t) = F(\omega_1 t, \dots, \omega_m t), -\infty < t < \infty$, both $F(x)$ and $U(x)$ are defined on the torus T^m and are periodic in each variable x_i, $i = 1, \dots, m$, with period 2π.

Many authors put very restrictive conditions on the frequencies $\omega_1, \dots, \omega_m$, namely infinitely many Diophantine conditions: for all integers $j_1, \dots, j_m$ satisfying $\sum_{\mu=1}^m |j_\mu| > 0$,

$$\left| \sum_{\mu=1}^m j_\mu \omega_\mu + j_0 \right| \geq C_0^{-1} \left(\sum_{\mu=1}^m |j_\mu| \right)^{-\tau},$$

where $j_0 = 0, 1, 2, C_0, \tau$ are fixed positive numbers. In this paper, we remove the Diophantine conditions on these frequencies. In order to do this we need an additional condition. Suppose $f(t)$ is a quasiperiodic function and $F(x)$ is the generating function of $f(t)$, i.e. $f(t) = F(\omega_1 t, \dots, \omega_m t)$. We will assume $f(0) = F(x_0)$, where $x_0 = (x_0^1, \dots, x_0^m)$ can be anywhere on the torus T^m except on a set of measure zero; thus equation (1) will have a smooth solution $u(t) = U(x_0^1 + \omega_1 t, \dots, x_0^m + \omega_m t)$ on the trajectory $\{x_0 + \omega t = (x_0^1 + \omega_1 t, \dots, x_0^m + \omega_m t) : -\infty < t < \infty\}$ on the torus for almost every $x_0 \in T^m$.

In our paper we construct a Hilbert space $P_{1,2}(T^m)$ of functions U defined on T^m, periodic in each variable with period 2π and such that

$$\|U\|^2 = \int_{T^m} \left(U^2 + \left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i} \right)^2 \right) < \infty.$$

We use $P_{1,2}^0(T^m)$ to denote the closed subspace of $P_{1,2}(T^m)$ which is the closure of $C_0^\infty(T^m)$ under the norm $\|\cdot\|$.

By minimizing the functional $F_2(U)$ defined by

$$F_2(U) = \int_{T^m} \left[\frac{\left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i} \right)^2 - \beta^2 U^2}{2} + \frac{b\beta^2}{4a} U^4 + \frac{\beta^2}{a} F(x) U \right] dx$$

on M, where

$$M = \{U \in P_{1,2}^0(T^m) : U \in L^4(T^m)\},$$

we get a family of weak solutions $U(x)$ for equation (3).

Here a weak solution of (3) is defined as follows. If $U \in P_{1,2}^0(T^m)$ satisfies

$$\int_{T^m} \left[\left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i} \right) \left(\sum_{i=1}^m \omega_i \frac{\partial V}{\partial x_i} \right) + \beta^2 UV - \frac{b\beta^2}{a} U^3 V - \frac{b\beta^2}{a} FV \right] dx = 0$$

for all $V \in C_0^\infty(T^m)$, we call $U(x)$ a *weak solution* of equation (3) in $P_{1,2}^0(T^m)$.

In our paper we first prove the following theorem:

THEOREM 1. *There exists a weak solution for the following partial differential equation which corresponds to the second type Duffing equation:*

$$\sum_{i,j=1}^m \omega_i \omega_j \frac{\partial^2 U}{\partial x_i \partial x_j} + \beta^2 U - \frac{b\beta^2}{a} U^3 = \frac{\beta^2}{a} F(x) \tag{4}$$

in the space $P^0_{1,2}$, provided that $F \in L^2(T^m)$. Here a and b are arbitrary positive real numbers and β is any number satisfying $0 < \beta < 2/I$, where I is the maximum length of a segment with direction $(\omega_1, \ldots, \omega_m)$ bounded by the sides of the torus.

In the second part of this paper we will prove our main theorem:

THEOREM 2. *For any $a > 0$, $b > 0$, and each β as in Theorem 1, the second type Duffing equation*

$$\ddot{u} + \beta^2 u - \frac{b\beta^2}{a} u^3 = \frac{\beta^2}{a} f(t)$$

has a smooth solution $u(t)$ with prescribed rationally independent frequencies $\omega_1, \ldots, \omega_m$ on the trajectory $\{x_0 + \omega t = (x_0^1 + \omega_1 t, \ldots, x_0^m + \omega_m t) : -\infty < t < \infty\}$ on the torus for almost every $x_0 = (x_0^1, \ldots, x_0^m) \in T^m$ provided $F \in C^1(T^m)$ and $f(0) = F(x_0)$.

Thus finally we will have

THEOREM 3. *For any $a, b > 0$, and each β as in Theorem 1, the general second type Duffing equation (1) has a family of smooth solutions $u(t)$ with prescribed rationally independent frequencies $\frac{\sqrt{a}}{\beta}\omega_1, \ldots, \frac{\sqrt{a}}{\beta}\omega_m$ on the trajectory $\{x_0 + \omega t = (x_0^1 + \omega_1 t, \ldots, x_0^m + \omega_m t) : -\infty < t < \infty\}$ on the torus for almost every $x_0 = (x_0^1, \ldots, x_0^m) \in T^m$ provided $F \in C^1(T^m)$ and $f(0) = F(x_0)$.*

PROOF. (1) can be solved in two steps. We first solve

$$\ddot{u} + \beta^2 u - \frac{b\beta^2}{a} u^3 = \frac{\beta^2}{a} f$$

for all β as in the statement. Then by scaling t into $\frac{\sqrt{a}}{\beta} t$, we get a solution u of

$$\ddot{u} + au - bu^3 = f$$

with frequencies $\left(\frac{\sqrt{a}}{\beta}\omega_1, \ldots, \frac{\sqrt{a}}{\beta}\omega_m\right)$.

Previous work on quasiperiodic solutions of nondissipative Duffing equations includes the 1965 paper of Moser. He was the first to use the K.A.M. theory to find quasiperiodic solutions of the forced Duffing equations using Diophantine restrictions. Thus his solutions are not valid for all parameters a, b. Moser's solution is of small amplitude and Moser in fact requires a, b to satisfy certain conditions.

On the other hand, Moser's quasiperiodic solutions are not shown to be the minimizers of any functionals, so they differ substantially from our solutions. Moser's solutions can be described for K.A.M. approximations. We will describe the relationships between the solutions obtained here and his solutions in another paper.

2. An analogue of the Poincaré Inequality for the space $P^0_{1,2}$

In Section 1 we defined the space $P^0_{1,2}(T^m)$ as the completion of $C^\infty_0(T^m)$ under the norm

$$\|U\|^2_{P_{1,2}} = \int_{T^m}\bigg(U^2 + \bigg(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i}\bigg)^2\bigg) < \infty.$$

In this section first we prove the following lemma:

LEMMA 1 (An analogue of the Poincaré Inequality for $P^0_{1,2}(T^m)$). *For every $U \in P^0_{1,2}$,*

$$\int_{T^m}\bigg(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i}\bigg)^2 \geq \alpha^2 \int_{T^m} U^2,$$

where $\alpha = 2/I$, and I is the maximum length of a segment with direction $(\omega_1,\dots,\omega_m)$ bounded by the sides of the torus.

PROOF. If we make an orthogonal transformation of the coordinate system from $\{x_1,\dots,x_m\}$ to $\{t,y_2,\dots,y_m\}$ such that the direction of the t axis is $(\omega_1,\dots,\omega_m)$, we can denote each point on the torus as (t,y'), where $y' = (y_2,\dots,y_m)$. Let A be the projection of T^m to the hyperplane $t = 0$. Then for each $U \in P^0_{1,2}$,

$$\int_{T^m}\bigg(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i}\bigg)^2 dx = \int_A\int_{l_{y'}}\bigg(\frac{dU(t,y')}{dt}\bigg)^2 dt\,dy',$$

where $l_{y'}$ denotes the line segment with direction $(\omega_1,\dots,\omega_m)$ passing through $y' \in A$. (For simplicity we will use l to denote $l_{y'}$ later on.) We claim that for almost every $y' \in A$, $U(t,y')$ belongs to $W^0_{1,2}(l)$. In fact, since $U \in P^0_{1,2}$, there is a sequence Φ_n such that $\{\Phi_n\} \subset C^\infty_0(T^m)$ and

$$\lim_{n\to\infty}\int_{T^m}\bigg[\sum_{i=1}^m \omega_i \frac{\partial}{\partial x_i}(U-\Phi_n)\bigg]^2 dx = 0.$$

That is,

$$\lim_{n\to\infty}\int_A\int_l\bigg(\frac{dU}{dt}-\frac{d\Phi_n}{dt}\bigg)^2 dtdy' = 0.$$

Hence $\int_l\big(\frac{dU}{dt}-\frac{d\Phi_n}{dt}\big)^2 dt$ converges to zero in measure on A. Therefore we can find a subsequence of $\big\{\int_l\big(\frac{dU}{dt}-\frac{d\Phi_n}{dt}\big)^2 dt\big\}$, still denoted by $\big\{\int_l\big(\frac{dU}{dt}-\frac{d\Phi_n}{dt}\big)^2 dt\big\}$, which converges to zero almost everywhere on A. That means that for almost every $y' \in A$,

$$\lim_{n\to\infty}\int_l\bigg(\frac{dU}{dt}-\frac{d\Phi_n}{dt}\bigg)^2 dt = 0.$$

Therefore the claim is true.

Since for each $\phi \in C_0^\infty(l)$ and $t \in l$,

$$\phi^2 = \int_{t_0}^t \frac{\phi\dot{\phi}}{2},$$

where we suppose $\phi(t_0) = 0$, we finally get

$$\|\dot{\phi}\|_{L^2(l)} \geq \frac{2}{|l|}\|\phi\|_{L^2(l)}.$$

This inequality holds for every function in $W^0_{1,2}(l)$, therefore for almost every $y' \in A$,

$$\left\|\frac{dU(t,y')}{dt}\right\|_{L^2(l)} \geq \frac{2}{|l|}\|U(t,y')\|_{L^2(l)}.$$

If we denote by I the maximum length of the line segments l on the torus which have the direction $(\omega_1, \dots, \omega_m)$, then

$$\left\|\frac{dU(t,y')}{dt}\right\|_{L^2(l)} \geq \frac{2}{I}\|U(t,y')\|_{L^2(l)}.$$

Squaring and integrating on A, we finally get

$$\int_{T^m}\left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i}\right)^2 \geq \frac{4}{I^2}\int_{T^m} U^2.$$

By setting $\alpha^2 = 4/I^2$ we have finished the proof of the lemma.

LEMMA 2. *For $0 < \beta < \alpha$, where α is as in Lemma 1,*

$$\|U\|^2 = \int_{T^m}\left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i}\right)^2 dx - \beta^2 \int_{T^m} U^2\, dx$$

is an equivalent norm in $P^0_{1,2}(T^m)$.

PROOF. First we show that $\|\cdot\|$ is a norm in $P^0_{1,2}$. In fact, for any $U, V \in P^0_{1,2}$,

$$(U,V) = \int_{T^m}\left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i}\right)\left(\sum_{i=1}^m \omega_i \frac{\partial V}{\partial x_i}\right) dx - \beta^2 \int_{T^m} UV\, dx$$

is an inner product in $P^0_{1,2}$. It is obvious that the product $(\cdot,\cdot)$ has the following properties: (i) symmetry, (ii) linearity in the first variable, (iii) $(U,U) > 0$ when $U \neq 0$. We only need to prove that if $(U,U) = 0$, then $U = 0$ a.e. on the torus. By Lemma 1,

$$(U,U) = \int_{T^m}\left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i}\right)^2 dx - \beta^2 \int_{T^m} U^2\, dx \geq (\alpha^2 - \beta^2)\int_{T^m} U^2\, dx,$$

so that $(U,U) = 0$ will force that $U = 0$ a.e. on the torus. Therefore $\|U\|^2 = (U,U)$ is a norm on $P^0_{1,2}$.

It is obvious that

$$\|U\|^2_{P_{1,2}} = \int_{T^m} \left[U^2 + \left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i}\right)^2\right] dx$$
$$\geq \int_{T^m} \left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i}\right)^2 dx - \beta^2 \int_{T^m} U^2\, dx = \|U\|^2.$$

On other hand, let r satisfy $\beta^2/\alpha^2 < r < 1$. Then

$$\int_{T^m} \left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i}\right)^2 dx - \beta^2 \int_{T^m} U^2\, dx$$
$$\geq (1-r)\int_{T^m} \left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i}\right)^2 dx + r\int_{T^m} \left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i}\right)^2 dx - \beta^2 \int_{T^m} U^2\, dx$$
$$\geq (1-r)\int_{T^m} \left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i}\right)^2 dx + (\alpha^2 r - \beta^2)\int_{T^m} U^2\, dx.$$

Therefore

$$\|U\|^2 \geq \min\{1-r, \alpha^2 r - \beta^2\}\|U\|^2_{P_{1,2}}.$$

So we conclude that the norms $\|\cdot\|$ and $\|\cdot\|_{P_{1,2}}$ are equivalent.

3. A weak solution

In this section we will get a weak solution of the partial differential equation

$$\sum_{i,j=1}^m \omega_i\omega_j \frac{\partial^2 U}{\partial x_i \partial x_j} + \beta^2 U - \frac{b\beta^2}{a}U^3 = \frac{\beta^2}{a}F(x). \tag{5}$$

This equation corresponds to the second Duffing equation

$$\ddot{u} + \beta^2 u - \frac{b\beta^2}{a}u^3 = \frac{\beta^2}{a}f(t), \tag{6}$$

where $0 < \beta < 2/I$, $I = \max\{|l| : l$ is the segment with direction $(\omega_1, \dots, \omega_m)$ and $|l|$ denotes the length of the segment $l\}$. We have $u(t) = U(x_0 + \omega t) = U(x_0^1 + \omega_1 t, \dots, x_0^m + \omega_m t)$, $f(t) = F(x_0 + \omega t) = F(x_0^1 + \omega_1 t, \dots, x_0^m + \omega_m t)$ for almost every $x_0 = (x_0^1, \dots, x_0^m) \in T^m$, also $\{x_0 + \omega t = (x_0^1 + \omega_1 t, \dots, x_0^m + \omega_m t) : -\infty < t < \infty\}$ is the trajectory on the torus.

We use the minimization method to get a minimum point of $F_2(U)$ in $M \subset P^0_{1,2}$, where

$$M = \{U \in P^0_{1,2}(T^m) : U \in L^4(T^m)\}$$

and

$$F_2(U) = \int_{T^m} \left[\frac{\left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i}\right)^2 \beta^2 U^2}{2} + \frac{b\beta^2}{4a}U^4 + \frac{\beta^2}{a}FU\right] dx.$$

Our theorem is the following:

THEOREM 4. *There is a point $U_0 \in M \subset P^0_{1,2}$ such that*

$$F_2(U_0) = \inf_{U\in M} F_2(U)$$

provided $F \in L^2(T^m)$.

PROOF. We divide the proof into 3 steps:

(i) $F_2(U)$ is coercive and bounded below.
(ii) The minimizing sequence has a weakly convergent subsequence.
(iii) The weak limit of this subsequence is a minimum point of $F_2(U)$ in M.

In the proof we will take the norm of the space as

$$\|U\|^2 = \int_{T^m} \left[\left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i} \right)^2 - \beta^2 U^2 \right] dx.$$

(i) To prove that $F_2(U)$ is coercive and bounded below, it is sufficient to prove that there is a constant c such that

$$F_2(U) \geq \frac{1}{2}\|U\|^2 + \frac{b\beta^2}{4a} \int_{T^m} \left(U^2 - \frac{1}{b} \right)^2 - c.$$

In fact, since

$$\int_{T^m} FU \leq \|U\|_{L^2}\|F\|_{L^2} \leq \frac{\int F^2 + \int U^2}{2}$$

it follows that

$$\begin{aligned} F_2(U) &= \int_{T^m} \left[\frac{\left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i} \right)^2 - \beta^2 U^2}{2} + \frac{b\beta^2}{a} \left(\frac{U^4}{4} + \frac{FU}{b} \right) \right] dx \\ &\geq \frac{1}{2}\|U\|^2 + \frac{b\beta^2}{4a} \int_{T^m} \left[U^4 - \frac{2}{b}(U^2 + F^2) \right] dx \\ &= \frac{1}{2}\|U\|^2 + \frac{b\beta^2}{4a} \int_{T^m} \left(U^2 - \frac{1}{b} \right)^2 + c, \end{aligned}$$

where $c = -\frac{\beta^2}{4ab}(2\pi)^m - \frac{\beta^2}{2a}\int_{T^m} F^2\,dx$. Thus $F_2(U)$ is coercive and bounded from below.

(ii) Let $\{U_n\}$ be a minimizing sequence in M. Also, we assume $F_2(U_n) \leq C$, where C is some positive constant. Since $F_2(U)$ is coercive and bounded below, $\{U_n\}$ is uniformly bounded in $P_{1,2}$ norm, i.e. there is a positive constant $K > 0$ such that

$$\|U_n\|_{P_{1,2}} \leq K.$$

Therefore if $\|\cdot\|$ denotes the equivalent norm as before, we have

$$\begin{aligned} \frac{1}{4}\int_{T^m} U_n^4 &\leq \frac{1}{2}\|U_n\|^2 + \frac{1}{4}\int_{T^m} U_n^4\,dx + \int_{T^m} FU_n\,dx + \|U\|_{P_{1,2}}\|F\|_{L^2} \\ &\leq C + K\|F\|_{L^2}. \end{aligned}$$

If we write $B = 4(C + K\|F\|_{L^2})$, then

$$\int_{T^m} U_n^4\,dx \le B.$$

Since $\{U_n\}$ is a bounded sequence in $P^0_{1,2}(T^m)$, $\{U_n\}$ has a weakly convergent subsequence $\{U'_n\}$ with weak limit U in $P^0_{1,2}(T^m)$; we still denote this subsequence as $\{U_n\}$. (Since the whole space $P^0_{1,2}$ is weakly closed, $U \in P^0_{1,2}$.) Now we prove that U belongs to $L^4(T^m)$.

By the Banach–Saks Theorem,

$$\left\| \frac{\sum_{i=1}^n U_i}{n} - U \right\|_{P_{1,2}} \to 0,$$

hence

$$\left\| \frac{\sum_{i=1}^n U_i}{n} - U \right\|_{L^2} \to 0.$$

There is a subsequence of $\sum_{i=1}^n U_i/n$ that we still denote by $\sum_{i=1}^n U_i/n$ such that $\sum_{i=1}^n U_i/n \to 0$ a.e. on T^m. By Fatou's Lemma we find that

$$\int_{T^m} U^4\,dx \le \varliminf_{n\to\infty} \int_{T^m} \left(\frac{\sum_{i=1}^n U_i}{n}\right)^4 dx \le \varliminf_{n\to\infty} \frac{\sum_{i=1}^n \int_{T^m} U_i^4\,dx}{n} \le B.$$

Therefore $U \in L^4$ and $U \in M$.

(iii) We can rewrite the functional $F_2(U)$ as

$$F_2(U) = \tfrac{1}{2}\|U\|^2 + F^*(U),$$

where $\|U\|$ is the equivalent norm in $P^0_{1,2}$, and

$$F^*(U) = \int_{T^m} \left[\frac{b\beta^2}{4a} U^4 + \frac{\beta^2}{a} FU\right] dx.$$

Since $F^*(U)$ is a convex functional on M which is weakly lower semicontinuous, the norm $\|\cdot\|$ is also weakly lower semicontinuous. Therefore

$$\begin{aligned} C = \inf_M F_2(U) &= \lim_{n\to\infty} F_2(U_n) \ge \varliminf_{n\to\infty} \left[\tfrac{1}{2}\|U_n\|^2 + F^*(U_n)\right] \\ &\ge \tfrac{1}{2} \varliminf_{n\to\infty} \|U_n\|^2 + \varliminf_{n\to\infty} F^*(U_n) \ge \tfrac{1}{2}\|U\|^2 + F^*(U) = F_2(U). \end{aligned}$$

That means U is a minimum point of $F_2(U)$ on M.

Thus we have the following theorem:

Theorem 5. *If $F \in L^2(T^m)$, then the equation*

$$\sum_{i,j=1}^m \omega_i\omega_j \frac{\partial^2 U}{\partial x_i \partial x_j} + \beta^2 U - \frac{b\beta^2}{a} U^3 = \frac{\beta^2}{a} F(x) \tag{8}$$

has a weak solution U in $P^0_{1,2}$ for $0 < \beta < 2/I$, where I is as in Lemma 1.

PROOF. For each $\Phi \in C_0^\infty(T^m) \subset M$, we have

$$\frac{dF_2}{dt}(U+t\Phi)\bigg|_{t=0} = 0.$$

That means that

$$\int_{T^m} \left[\left(\sum_{i=1}^m \omega_i \frac{\partial U}{\partial x_i}\right)\left(\sum_{i=1}^m \omega_i \frac{\partial \Phi}{\partial x_i}\right) + U\Phi - U^3\Phi - F\Phi\right] = 0$$

for all $\Phi \in C_0^\infty(T^m)$. We have finished the proof.

4. The regularity of $u(x_0 + \omega t)$

In this section we first prove the smoothness of $u(t)$ on the closed segment $l_{y'}$ for almost every $y' \in A$, and then on the whole trajectory $\{x_0 + \omega t : -\infty < t < \infty\}$ for almost every $x_0 \in T^m$. Here A is as in Section 2, i.e. we make an orthogonal change of variables from $\{x_1, \ldots, x_m\}$ to $\{t, y_2, \ldots, y_m\}$, and we let A be the projection of the torus T^m to the $(m-1)$-dimensional space $t = 0$. Now we prove the following lemma:

LEMMA 3. *For almost every $y' \in A$, $u(t) = U(t, y')$ satisfies the equation*

$$\ddot{u} + \beta^2 u - \frac{b\beta^2}{a}u^3 = \frac{\beta^2}{a}f(t) \tag{9}$$

on the closed interval $l_{y'}$, where $l_{y'}$ denotes the segment with direction $(\omega_1, \ldots$ $\ldots, \omega_m)$ bounded by the sides of the torus which passes through the point y' in A, provided $F \in C^1(T^m)$.

PROOF. Suppose $U(x)$ is the weak solution of Theorem 5 and $U(x)$ satisfies the equation

$$\sum_{i,j=1}^m \omega_i\omega_j \frac{\partial^2 U}{\partial x_i \partial x_j} + \beta^2 U - \frac{b\beta^2}{a}U^3 = \frac{\beta^2}{a}F(x) \tag{10}$$

almost everywhere on the torus. Then for almost every y' in A equation (10) holds almost everywhere on $l_{y'}$. In fact, if not, (10) cannot hold almost everywhere on T^m. Also from the proof of Lemma 1 we know that for almost every $y' \in A$, $U(t, y') \in W_{1,2}^0(l_{y'})$. Set

$$\mathcal{P} = \{l_{y'} : y' \in A,\ U(t,y') \in W_{1,2}^0(l_{y'}),\ U(t,y') \text{ satisfies (10) a.e. on } l_{y'}\}.$$

Suppose $l_{y'} \in \mathcal{P}$ for a fixed $y' \in A$. Then $u(t) = U(t, y') \in W_{1,2}^0(l_{y'})$. Therefore $u(t) \in C(l_{y'})$, and we can assume that there is a positive constant e such that $|u(t)| \le e$ on $l_{y'}$ for this fixed $y' \in A$.

Since $F \in C^1(T^m)$ we can assume that $|F(x)| \le d$ on the torus for some constant $d > 0$. By (10) we see that

$$\left(\int_{l_{y'}} \ddot{u}^2 \, dt \right)^{1/2} \le \beta^2 \left[\int_{l_{y'}} \left(-u + \frac{b}{a} u^3 + \frac{1}{a} f(t) \right)^2 \right]^{1/2}$$
$$\le \beta^2 \left[\int_{l_{y'}} \left(e + \frac{b}{a} e^3 + \frac{1}{a} d \right)^2 \right]^{1/2} < \infty.$$

Therefore $u(t) \in W_{2,2}(l_{y'})$. By the Sobolev theory the space $W_{2,2}(l_{y'})$ is compactly embedded in $C^1(l_{y'})$, and so $\dot{u}$ is bounded on $l_{y'}$. Thus by the condition that $F \in C^1(T^m)$ we find that $\int_{l_{y'}} \left[-\beta^2 \dot{u} + \frac{3b\beta^2}{a} \dot{u} u^2 + \frac{\beta^2}{a} \dot{f} \right]^2$ is bounded since the integrand is. Therefore the weak derivative d^3u/dt^3 exists and

$$\left[\int_{l_{y'}} \left(\frac{d^3 u}{dt^3} \right)^2 \right]^{1/2} \le \left(\int_{l_{y'}} \left[-\beta^2 \dot{u} + \frac{3b\beta^2}{a} \dot{u} u^2 + \frac{\beta^2}{a} \dot{f} \right]^2 \right)^{1/2} < \infty.$$

Then $u \in C^2(l_{y'})$. We have thus finished the proof of Lemma 3.

PROOF OF THEOREM 2. Let E be the set of all $x \in T^m$ such that there is at least one segment bounded by the sides of the torus that does not belong to $\mathcal{P}$ on $\{x + \omega t : -\infty < t < \infty\}$, where $\mathcal{P}$ is as defined in Lemma 2. If E is a set of non-zero m-dimensional measure, the projection of E to A will have non-zero $(m-1)$-dimensional measure. Let G denote this projection. There is one and only one trajectory $\{y' + \omega t : -\infty < t < \infty\}$ passing through each point $y' \in G \subset A$. Partition $\mathbb{R}^m$ into countably many cubes obtained by periodically translating the cube of length 2π, centered at the origin. Denote these cubes by $\{T_i\}_{i=1}^{\infty}$ and let A_i be the image in T_i of A under this periodic translating for each i. Also let E_i be the set of all points y' of A_i such that $l_{y'}$ does not belong to $\mathcal{P}_i$, where

$$\mathcal{P}_i = \{ l_{y'} : y' \in A_i, \ U(t, y') \in W^0_{1,2}(l_{y'}), \ U(t, y') \text{ satisfies (10) a.e. on } l_{y'} \}.$$

The projection of $\bigcup_{i=1}^{\infty} E_i$ to A is the set G. By assumption, $\mu(G) \neq 0$, therefore there is at least one i such that E_i has non-zero $(m-1)$-measure. This means that there is a subset G' of G with non-zero measure such that if $y' \in G'$, then $l_{y'}$ does not belong to $\mathcal{P}$, which is a contradiction to Lemma 3. We have finished the proof of Theorem 2.

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