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RELAXED YANG–MILLS FUNCTIONAL OVER 4-MANIFOLDS

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Dedicated to Louis Nirenberg on the occasion of his 70th birthday

We give suitable completions of the space of principal G-bundles over M and the space of smooth connections on them, where G is a compact, simple, simply connected Lie group and M is a 4-dimensional compact orientable manifold. We also introduce a natural energy defined in such spaces and consider variational problems on them.

0. Introduction

We begin by considering the following example.

Let M be an n-dimensional compact, orientable Riemannian manifold, G a simple, simply connected compact Lie group and $P \to M$ a principal G-bundle. We denote by $\mathcal{A}(P)$ the space of smooth connections on P. For $A \in \mathcal{A}(P)$, we define the *Yang–Mills energy* $\mathrm{YM}(A)$ by the formula

$$\mathrm{YM}(A) = \int_M |F_A|^2 * 1,$$

where $F_A = dA + A \wedge A$ is the curvature of A. The pointwise norm on F_A is the $\mathrm{Aut}(P)$-invariant norm on the vector bundle $\Lambda^2 T^* M \otimes \mathrm{Ad}(P)$ which is induced from the adjoint action invariant norm on $g =$ the Lie algebra of G, and the Riemannian metric of M. (See [5] for foundations of Yang–Mills theory.)

Let us consider the following problem.

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Problem A. *Does there exist* $\widehat{A} \in \mathcal{A}(P)$ *such that* $\mathrm{YM}(\widehat{A}) = \inf\{\mathrm{YM}(A) : A \in \mathcal{A}(P)\} \equiv m(P)$*? In other words, does* YM *attain its inf in* $\mathcal{A}(P)$*?*

The most natural approach to the above problem is the direct methods in the calculus of variations, that is, we consider minimizing sequences $\{A_i\} \subset \mathcal{A}(P)$, i.e., $\mathrm{YM}(A_i) \to m(P)$. If there exists a subsequence $\{A_{n_i}\} \subset \{A_i\}$ such that $\{A_{n_i}\}$ converges to some connection $A \in \mathcal{A}(P)$ in some sense, then A will be a minimizing connection. But, in general, this method does not work well in the space $\mathcal{A}(P)$. The problem is that, in general, we cannot extract a subsequence which converges (in C^∞-topology) to some connection in $\mathcal{A}(P)$. C^∞-topology is too strong to obtain the solution of Problem A.

For this reason, some people consider Problem A in the Sobolev category instead of C^∞-category.

We denote by $L^2_1(\mathcal{A}(P))$ the space of L^2_1-connections on P, where L^p_k is the Sobolev space of functions with k derivatives which are p-integrable. We denote by $L^2_2(M; \mathrm{Aut}(P))$ the space of L^2_2-gauge transformations.

By the Sobolev embedding theorem, L^2_2 is embedded in C^0 if and only if $\dim M \le 3$. In this case, $\varphi \in L^2_2(M; \mathrm{Aut}(P))$ does not change the topology of P. Moreover, when $\dim M \le 3$, the Sobolev category is a suitable class for Problem A, that is, the following theorem of Uhlenbeck [13] holds:

Theorem 0 (Global weak compactness theorem of Uhlenbeck). *Suppose* $\dim M \le 3$*. Let* $\{A_i\} \subset \mathcal{A}(P)$ *be such that* $\sup_{i \ge 1} \int_M |F_{A_i}|^2 {*1} < \infty$*. Then there is a subsequence* $\{A_{n_i}\} \subset \{A_i\}$ *and* $\{\varphi_i\} \subset L^2_2(M; \mathrm{Aut}(P))$ $(\hookrightarrow C^0(M; \mathrm{Aut}(P)))$ *such that* $\{\varphi_i^* A_{n_i}\}$ *converges weakly in* $L^2_1(M)$*.*

In the setting of the above theorem, we assume $\varphi_i^* A_{n_i} \rightharpoonup A$ weakly in $L^2_1(M)$. By the Sobolev embedding theorem, $L^2_1 \hookrightarrow L^6$ in dimensions 2 and 3, and we have (passing to a subsequence if necessary)

$$F_{\varphi_i^* A_{n_i}} = \varphi_i^* F_{A_{n_i}} \rightharpoonup F_A \quad \text{weakly in } L^2(M).$$

Thus by the weak lower semicontinuity of the L^2-norm we obtain

$$\int_M |F_A|^2 * 1 \le \liminf_{i \to \infty} \int_M |F_{A_{n_i}}|^2 * 1.$$

This global weak compactness theorem does not hold when the dimension of M is greater than 3. See [9] how weak compactness breaks down in the critical dimension $\dim M = 4$. (We remark that when $\dim M \ge 4$, L^2_2 is not embedded in C^0.) We may, therefore, reasonably conclude that the Sobolev class is not suitable for Problem A when the dimension of M is greater than or equal to 4. The Sobolev class is too weak to obtain the solution of Problem A.

The purpose of this paper is to give suitable classes of principal bundles and connections for considering Problem A in the critical dimension $\dim M = 4$, and to give a natural energy on such connections.

For $\dim M = 5$, related problems are considered in [8].

Our basic approach to Problem A is the following. For simplicity, throughout this paper we only consider the case $G = SU(2)$, but our results and arguments also hold with slight modifications when G is a simple, simply connected compact Lie group.

Let $P \to M$ be a principal $SU(2)$-bundle, where M is a 4-dimensional compact orientable Riemannian manifold. Principal $SU(2)$-bundles $\{P\}$ over M are classified by the 2nd Chern class $C_2(P) \in H^4(M;\mathbb{Z})$ of the associated vector bundles $\{\eta(P)\}$, where $\eta(P) = P \times_{SU(2)} su(2)$ and $SU(2)$ acts on $su(2) =$ the Lie algebra of $SU(2)$ via the adjoint action. That is, the isomorphism class of $P \leftrightarrow C_2(P) \in H^4(M;\mathbb{Z})$ is a 1-1 correspondence. See, for example, [5, Appendix E]. By the Chern–Weil formula [7], $C_2(P)$ is given by

$$C_2(P) = \Big[-\frac{1}{8\pi^2}\operatorname{tr}(F_A \wedge F_A)\Big],$$

where $A \in \mathcal{A}(P)$.

Our idea is to "complete" the space

$$\mathcal{P} = \{C_2(P) \in H^4(M;\mathbb{Z}) : P \to M \text{ a principal } SU(2)\text{-bundle}\},$$

using the above fact, by a suitable (weak) topology instead of completing $\{P\}$, and we think of this completion $\mathcal{P}^2$ as the space of "generalized" principal bundles.

It should be mentioned that similar difficulties occur when we consider the problem of minimization of the Dirichlet integral of mappings with prescribed homotopy class between two manifolds. In fact, our motivation comes from works of Bethuel, Brezis and Coron [2] and Giaquinta, Modica and Souček [6], where they treat the Dirichlet integral of mappings between two manifolds and obtain suitable extensions of "maps" (that is, the theory of "cartesian currents" introduced in [6]) and "energies" (that is, the theory of "relaxed energies" introduced in [2], see also [6]). See [2], [3], [6] and references therein for more details.

This paper is organized as follows: In §1 we introduce a natural topology in $\mathcal{P}$ and a completion $\mathcal{P}^2$ of it. We give a characterization of the space $\mathcal{P}^2$. We also introduce the space $\mathcal{A}(\mathcal{P}^2)$ of connections on $\mathcal{P}^2$. In §2 we extend the Yang–Mills functional to the space $\mathcal{A}(\mathcal{P}^2)$ by the method of relaxation. Finally, in §3 we show that Problem A is always solvable in our setting by the direct methods of the calculus of variations.

1. The classes $\mathcal{P}^2$ and $\mathcal{A}(\mathcal{P}^2)$

Let M be a compact orientable Riemannian manifold of dimension 4. We denote by $\mathcal{P}_k$ the space of all principal $SU(2)$-bundles over M with 2nd Chern number $-k$. Any two principal $SU(2)$-bundles in $\mathcal{P}_k$ are isomorphic.

With $P \in \mathcal{P}_k$ and $A \in \mathcal{A}(P)$ we associate $C_2(A)$ by the formula

$$C_2(A) = \frac{1}{8\pi^2}\operatorname{tr}(F_A \wedge F_A).$$

(We will not distinguish a cohomology class and its representative.) The cohomology class of $C_2(A)$ is independent of A and in fact $\int_M C_2(A) = k$. We identify P with $[C_2(A)] \in H^4(M;\mathbb{Z})$. We set $\mathcal{P} = \bigcup_{-\infty<k<\infty} \mathcal{P}_k$, that is, $\mathcal{P}$ is the space of all principal $SU(2)$-bundles over M.

For $P \in \mathcal{P}$ and $A \in \mathcal{A}(P)$, we can consider $C_2(A) \in \Lambda^4 T^*M$ as a Radon measure on M as follows:

For $\varphi \in C^0(M)$ we define $C_2(A) \in \mathcal{R}(M)$ by

$$C_2(A)(\varphi) = \int_M C_2(A) \wedge \varphi,$$

where $\mathcal{R}(M)$ is the space of Radon measures on M.

We denote by $\mathcal{A}_k$ the space of C^∞-connections on some principal bundle P in $\mathcal{P}_k$. That is, $A \in \mathcal{A}_k$ if and only if there exists $P \in \mathcal{P}_k$ such that $A \in \mathcal{A}(P)$. We also set $\mathcal{A} = \bigcup_{-\infty<k<\infty} \mathcal{A}_k$.

Definition 1.1. We define the space $\mathcal{P}_k^2 \subset \mathcal{R}(M)$ as follows. $\omega \in \mathcal{P}_k^2 \subset \mathcal{R}(M)$ if and only if there exists $\{A_i\} \subset \mathcal{A}_k$ such that:

(1) $\sup_{i\ge 1} \int_M |F_{A_i}|^2 * 1 < \infty$,
(2) $\frac{1}{8\pi^2}\operatorname{tr}(F_{A_i} \wedge F_{A_i}) \rightharpoonup \omega$ in the weak* topology of $\mathcal{R}(M)$.

We also set $\mathcal{P}^2 = \bigcup_{-\infty<k<\infty} \mathcal{P}_k^2$.

Remark 1.2. If $\{A_i\} \subset \mathcal{A}_k$ satisfies $\sup_{i\ge 1} \int_M |F_{A_i}|^2 * 1 < \infty$, then $\left\{\frac{1}{8\pi^2}\operatorname{tr}(F_{A_i} \wedge F_{A_i})\right\}$ is a bounded sequence in $\mathcal{R}(M)$. Thus there exists a subsequence $\{A_{n_i}\} \subset \{A_i\}$ such that $\left\{\frac{1}{8\pi^2}\operatorname{tr}(F_{A_{n_i}} \wedge F_{A_{n_i}})\right\}$ converges to some $\omega \in \mathcal{R}(M)$ in the weak* topology of $\mathcal{R}(M)$.

First, we examine the structure of $\mathcal{P}_k^2$. Our main result is

Theorem 1.3. *$\omega \in \mathcal{P}_k^2$ if and only if*

$$\omega = \frac{1}{8\pi^2}\operatorname{tr}(F_A \wedge F_A) + \sum_{j=1}^{q} d_j \delta_{a_j},$$

where

(1) *A is an L^2_1-connection on some principal $SU(2)$-bundle $Q \to M$,*
(2) *$q, d_1, \dots, d_q \in \mathbb{Z}$, $q \geq 0$,*
(3) *$a_1, \dots, a_q \in M$ and δ_{a_j} is the Dirac measure at a_j,*
(4) *$q(Q) + \sum_{j=1}^q d_j = k$, where $q(Q)$ is the topological quantum number of Q, i.e., $(-1)\times$ 2nd Chern number of Q.*

To prove this theorem, we quote the following two important theorems which are due to Uhlenbeck [13], [14] and Sedlacek [9].

THEOREM 1.4 (Uhlenbeck [13], Sedlacek [9]). *Let $\{A_i\} \subset \mathcal{A}_k(P)$ be such that*

$$\sup_{i\geq 1} \int_M |F_{A_i}|^2 * 1 < \infty.$$

Then there exist $a_1, \dots, a_p \in M$, an open covering $\{U_\alpha\}$ of $M \setminus \{a_1, \dots, a_p\}$ and sections $\sigma_\alpha(i) : U_\alpha \to P$ such that $\{\sigma^*_\alpha(i) A_i\}$ is weakly compact in $L^2_1(T^*U_\alpha \otimes su(2))$.

THEOREM 1.5 (Uhlenbeck [14]). *Let $b_1, \dots, b_q \in M$. Assume N is conformally equivalent to $M \setminus \{b_1, \dots, b_q\}$, that is, there exists a conformal diffeomorphism $f : N \to M \setminus \{b_1, \dots, b_q\}$. Let $Q \to N$ be a principal $SU(2)$-bundle and A an $L^2_{1,\mathrm{loc}}$-connection on Q such that $F_A \in L^2(\Lambda^2 T^*N \otimes su(2))$. Then there exist a principal $SU(2)$-bundle $\widetilde{Q} \to M$ and an L^2_1-connection $\widetilde{A}$ on $\widetilde{Q}$ such that A is gauge equivalent to $f^*\widetilde{A}$ over N. In particular, $\frac{1}{8\pi^2}\int_M \mathrm{tr}(F_A \wedge F_A) \in \mathbb{Z}$.*

Since the proof of Theorem 1.3 is long, we decompose it into four steps. Our proof is inspired by the proof of Theorem E.1 in [4].

PROOF OF THEOREM 1.3. STEP 1. Suppose that $\{A_i\} \subset \mathcal{A}_k$ satisfies $\sup_{i\geq 1} \int_M |F_{A_i}|^2 * 1 < \infty$. By Theorem 1.4 there exist $a_1, \dots, a_p \in M$, an open covering $\{U_\alpha\}$ of $M \setminus \{a_1, \dots, a_p\}$ and sections $\sigma_\alpha(i) : U_\alpha \to P$ such that $A_\alpha(i) := \sigma^*_\alpha(i) A_i \rightharpoonup A_\alpha$ in $L^2_1(T^*U_\alpha \otimes su(2))$ for all α.

Let $C \subset M$ be a cube such that $C \cong \mathbb{B}^4 := \{x \in \mathbb{R}^4 : |x| < 1\}$, $\{\iota^* A_\alpha(i)\} \subset L^2_1(U_\alpha \cap \partial C)$ and $\{F_{\iota^* A_i}\} \subset L^2(\partial C)$ are bounded, C contains at most one a_j, and $a_j \notin \partial C$ for all j. Here $\iota : \partial C \to M$ is the inclusion.

We first consider the case $a_j \in C$.

We define the connection $A(i)$ on $P|_C \times [0,1]\big|_{\partial(C\times[0,1])} \to \partial(C \times [0,1])$ as follows:

We set $A^*_i := \iota^* A_i$. Since $\{F_{A^*_i}\}$ is bounded in $L^2(\Lambda^2 T^*(\partial C) \otimes su(2))$, by the global weak compactness theorem of Uhlenbeck, there exists $\varphi_i \in L^2_2(\Lambda^2 T^*(\partial C) \otimes su(2))$ ($\hookrightarrow C^0(\partial C; \mathrm{Aut}(P|_{\partial C}))$) such that

$$\varphi^*_i A^*_i \rightharpoonup \widetilde{A} \quad \text{weakly in } L^2_1(\partial C).$$

On the other hand, there exists $\varphi \in L^2_2(\partial C; \mathrm{Aut}(P|_{\partial C}))$ such that $\varphi^*(\iota^* A) = \widetilde{A}$. Here $A = \{A_\alpha\}$. Since $C \setminus \{a_j\}$ is diffeomorphic to $S^3 \times (0,1)$, φ_i, φ can be extended to $C \setminus \{a_j\}$. We also denote by φ_i, φ such extensions. Then $\varphi_i, \varphi \in L^2_{2,\mathrm{loc}}(C \setminus \{a_j\}; \mathrm{Aut}(P|_{C\setminus\{a_j\}}))$.

We write

$$\partial(C \times [0,1]) = (C \times \{0\}) \cup (C \times \{1\}) \cup (\partial C \times [0,1]) = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2.$$

We define $A(i)$ by

$$A(i) = \begin{cases} \varphi^* A & \text{on } \Gamma_0, \\ \varphi_i^* A_i & \text{on } \Gamma_1, \\ t\varphi_i^* A_i^* + (1-t)\varphi^* A^* & \text{on } \Gamma_2. \end{cases}$$

For simplicity we write $\varphi_i^* A_i^* = \widetilde{A}_i$, $\varphi^* A^* = \widetilde{A}$.

LEMMA 1.6. *There exist $d_i \in \mathbb{Z}$ and $K > 0$ (independent of i) such that*

$$\left| \frac{1}{8\pi^2} \int_C \mathrm{tr}(F_{A_i} \wedge F_{A_i}) - \frac{1}{8\pi^2} \int_C \mathrm{tr}(F_A \wedge F_A) - d_i \right| \\ \leq K \left(\int_{\partial C} |F_{A_i}| \cdot |\widetilde{A}_i - \widetilde{A}| + \int_{\partial C} |F_{\widetilde{A}}| \cdot |\widetilde{A}_i - \widetilde{A}| + \int_{\partial C} |\widetilde{A}_i - \widetilde{A}|^3 \right).$$

PROOF. Since $A = \{A_\alpha\} \in L^2_{1,\mathrm{loc}}(C \setminus \{a_j\})$ we have $\varphi^* A \in L^2_{1,\mathrm{loc}}(C \setminus \{a_j\})$. On the other hand, since $F_A \in L^2(C)$, we have $F_{\varphi^* A} = \varphi^* F_A \in L^2(C)$. In the same way we obtain $\varphi_i^* A_i \in L^2_{1,\mathrm{loc}}(C \setminus \{a_j\})$ and $F_{\varphi_i^* A_i} = \varphi_i^* F_{A_i} \in L^2(C)$.

Thus $A(i)$ is a connection over $((C\setminus\{a_j\})\times\{1\}) \cup ((C\setminus\{a_j\})\times\{0\}) \cup \Gamma_2$ and $A(i) \in L^2_{1,\mathrm{loc}}(((C\setminus\{a_j\})\times\{1\}) \cup ((C\setminus\{a_j\})\times\{0\}) \cup \Gamma_2)$, $F_{A(i)} \in L^2(\partial(C\times[0,1]))$. Therefore by Theorem 1.5 there exists $d_i \in \mathbb{Z}$ such that

$$\frac{1}{8\pi^2} \int_{\partial(C\times[0,1])} \mathrm{tr}(F_{A(i)} \wedge F_{A(i)}) = d_i.$$

On the other hand,

$$\int_{\partial(C\times[0,1])} \mathrm{tr}(F_{A(i)} \wedge F_{A(i)}) = \int_C \mathrm{tr}(F_{\varphi_i^* A_i} \wedge F_{\varphi_i^* A_i}) - \int_C \mathrm{tr}(F_{\varphi^* A} \wedge F_{\varphi^* A}) \\ + \int_{\partial C\times[0,1]} \mathrm{tr}(F_{A(i)} \wedge F_{A(i)}).$$

Thus we get

$$(1) \quad \left| \frac{1}{8\pi^2} \int_C \mathrm{tr}(F_{A_i} \wedge F_{A_i}) - \frac{1}{8\pi^2} \int_C \mathrm{tr}(F_A \wedge F_A) - d_i \right| \\ \leq \left| \frac{1}{8\pi^2} \int_{\partial C\times[0,1]} \mathrm{tr}(F_{A(i)} \wedge F_{A(i)}) \right|.$$

On $\partial C \times [0,1]$, $A(i) = t\widetilde{A}_i + (1-t)\widetilde{A}$. Therefore we have

$$F_{A(i)} = td\widetilde{A}_i + (1-t)d\widetilde{A} + (\widetilde{A}_i - \widetilde{A})dt + (t\widetilde{A}_i + (1-t)\widetilde{A}) \wedge (t\widetilde{A}_i + (1-t)\widetilde{A}) \\ = tF_{\widetilde{A}_i} + (1-t)F_{\widetilde{A}} + t(t-1)(\widetilde{A}_i - \widetilde{A}) \wedge (\widetilde{A}_i - \widetilde{A}) + (\widetilde{A}_i - \widetilde{A})dt.$$

Thus we get

$$\begin{aligned}F_{A(i)} \wedge F_{A(i)} = {} & tF_{\widetilde{A}_i} \wedge (\widetilde{A}_i - \widetilde{A})dt + (1-t)F_{\widetilde{A}} \wedge (\widetilde{A}_i - \widetilde{A})dt \\ & + t(t-1)(\widetilde{A}_i - \widetilde{A}) \wedge (\widetilde{A}_i - \widetilde{A}) \wedge (\widetilde{A}_i - \widetilde{A})dt \\ & + t(\widetilde{A}_i - \widetilde{A}) \wedge dt \wedge F_{\widetilde{A}_i} + (1-t)(\widetilde{A}_i - \widetilde{A}) \wedge dt \wedge F_{\widetilde{A}} \\ & + t(t-1)(\widetilde{A}_i - \widetilde{A}) \wedge dt \wedge (\widetilde{A}_i - \widetilde{A}) \wedge (\widetilde{A}_i - \widetilde{A}).\end{aligned}$$

From (1) we obtain

$$\begin{aligned}&\left|\frac{1}{8\pi^2}\int_C \mathrm{tr}(F_{A_i} \wedge F_{A_i}) - \frac{1}{8\pi^2}\int_C \mathrm{tr}(F_A \wedge F_A) - d_i\right| \\ &\qquad \leq K\left(\int_{\partial C} |F_{\widetilde{A}_i}| \cdot |\widetilde{A}_i - \widetilde{A}| + \int_{\partial C} |F_{\widetilde{A}}| \cdot |\widetilde{A}_i - \widetilde{A}| + \int_{\partial C} |\widetilde{A}_i - \widetilde{A}|^3\right).\end{aligned}$$

This completes the proof. □

In the above argument we assume $a_j \in C$. But the result of Lemma 1.6 also holds when $a_j \notin C$. The proof is essentially the same and we omit it.

Step 2. We recall that $C_2(A_i) = \frac{1}{8\pi^2}\mathrm{tr}(F_{A_i} \wedge F_{A_i}) \in \mathcal{R}(M)$ is defined by

$$C_2(A_i)(\varphi) = \frac{1}{8\pi^2}\int_M \mathrm{tr}(F_{A_i} \wedge F_{A_i}) \wedge \varphi$$

for $\varphi \in C^0(M)$. Since $\{F_{A_i}\}$ is bounded in $L^2(\Lambda^2 T^*M \otimes su(2))$ we have

$$\sup_{i\geq 1} |C_2(A_i)|(M) < \infty,$$

where $|C_2(A_i)|$ is the total variation measure of $C_2(A_i)$. Therefore we may assume (passing to a subsequence if necessary) that

$$C_2(A_i) \rightharpoonup \mu, \quad |C_2(A_i)| \rightharpoonup \nu$$

weak* in $\mathcal{R}(M)$ for some $\mu, \nu \in \mathcal{R}(M)$.

Definition 1.7. We say that a "cube" $C \subset M$ is *good* for $\{A_i\}$ if:

(i) C is diffeomorphic to $\mathbb{B}^4$,
(ii) $\{\iota^* A_\alpha(n_i)\}$ is bounded in $L^2(\partial C \cap U_\alpha)$ and $\{F_{A_{n_i}}\}$ is bounded in $L^2(\partial C)$ for some subsequence $\{n_i\} \subset \{i\}$, where $\iota : \partial C \hookrightarrow M$ is the inclusion,
(iii) $\nu(\partial C) = 0$,
(iv) $\partial C \cap \{a_1, \ldots, a_p\} = \emptyset$.

Let $C \subset M$ be a good cube.

Lemma 1.8. *We have*

$$\mu(C) - \frac{1}{8\pi^2}\int_C \mathrm{tr}(F_A \wedge F_A) \in \mathbb{Z}.$$

PROOF. By Lemma 1.6 there exists $d_i \in \mathbb{Z}$ such that

$$(2)\quad \left| C_2(A_i)(C) - \frac{1}{8\pi^2}\int_C \operatorname{tr}(F_A \wedge F_A) - d_i \right| \\ \le K\left(\int_{\partial C} |F_{\widetilde{A}_i}| \cdot |\widetilde{A}_i - \widetilde{A}| + \int_{\partial C} |F_{\widetilde{A}}| \cdot |\widetilde{A}_i - \widetilde{A}| + \int_{\partial C} |\widetilde{A}_i - \widetilde{A}|^3\right).$$

Since $\widetilde{A}_i \rightharpoonup \widetilde{A}$ weakly in $L_1^2(\partial C)$ and $L_1^2(\partial C) \hookrightarrow_{\text{compact}} L^p(\partial C)$ for $1 \le p < 6$ (by the Sobolev embedding theorem), using Hölder's inequality we see that the right hand side of (2) converges to 0 as $i \to \infty$. On the other hand, since C is good, we have $C_2(A_i)(C) \to \mu(C)$ as $i \to \infty$. Therefore

$$\mu(C) - \frac{1}{8\pi^2}\int_C \operatorname{tr}(F_A \wedge F_A) \in \mathbb{Z}. \qquad \square$$

STEP 3. We next examine the atomic part of μ. Our main result is

LEMMA 1.9. *The atomic part of the measure μ is*

$$\sum_{l=1}^{q} d_l \delta_{b_l}$$

for some $d_l \in \mathbb{Z}$, $0 \le q < \infty$ and $b_l \in M$.

PROOF. We fix $a \in M$. Let $C_j \subset M$ $(1 \le j < \infty)$ be such that $a \in C_j$ for all j, C_j is good for all j and $\operatorname{meas}(C_j) \to 0$ as $j \to \infty$. We note that there always exists such a sequence $\{C_j\}$ for any $a \in M$. By Lemma 1.8 we have

$$\mu(C_j) - \frac{1}{8\pi^2}\int_{C_j} \operatorname{tr}(F_A \wedge F_A) = d_j \in \mathbb{Z}.$$

Letting $j \to \infty$ in the above equality we obtain $\mu(\{a\}) \in \mathbb{Z}$. Since $|\mu|(M) < \infty$, there exist at most finitely many $b_j \in M$ $(1 \le j \le q)$, $0 \le q < \infty$, which are atom points of μ. This completes the proof. $\square$

STEP 4.

LEMMA 1.10. *We have*

$$\mu = \frac{1}{8\pi^2}\operatorname{tr}(F_A \wedge F_A) + \sum_{j=1}^{q} d_j \delta_{b_j}.$$

PROOF. We set

$$m = \mu - \frac{1}{8\pi^2}\operatorname{tr}(F_A \wedge F_A) - \sum_{j=1}^{q} d_j \delta_{b_j}.$$

We prove $m \equiv 0$. For a good cube $C \subset M$, by Lemma 1.8 we have

$$(3)\qquad m(C) \in \mathbb{Z}.$$

Since m has no atomic part, by (3) there exists $\varepsilon > 0$ such that $m(C) = 0$ for all good cubes $C \subset M$ with $\operatorname{meas}(C) < \varepsilon$.

We take $r > 0$ such that $r \ll \varepsilon^{1/4}$ and $a \in M$. We claim

Claim. *For a.e.* $x \in M$, $x - C_r$ *is a good cube, where* $C_r = \{x \in M : |x_i| < r,$ $i = 1, \dots, 4\}$ ($x = (x_i)$ *is the normal coordinate of* M *centered at* a).

Proof of Claim. We check the conditions (i)–(iv) of Definition 1.7. (i) is clear. For (ii), we set

$$g(i) = |\nabla A(i)|^2 + |A(i)|^2 + |F_{A_i}|^2.$$

Since $\sup_{i \geq 1} \int_0^r (\int_{\partial(x - C_r)} g(i))\, dx_j < \infty$ for any $1 \leq j \leq 4$, by Fatou's lemma and Fubini's theorem for a.e. $x \in M$ we get

$$\liminf_{i \to \infty} \int_{\partial(x - C_r)} g(i) < \infty.$$

Thus (ii) holds. The verification of (iii) and (iv) is standard. The proof of the claim is complete.

By the above claim we obtain $\chi_{C_r} * m(x) = 0$ for a.e. $x \in M$, where χ_{C_r} is the characteristic function of C_r. On the other hand, since $r^{-4} \chi_{C_r} * m \rightharpoonup m$ weak* in $\mathcal{R}(M)$ we obtain $m \equiv 0$. Therefore we finally obtain

$$C_2(A_i) \rightharpoonup \frac{1}{8\pi^2} \mathrm{tr}(F_A \wedge F_A) + \sum_{j=1}^{q} d_j \delta_{b_j}$$

weak* in $\mathcal{R}(M)$. Here $A \in L^2_{1,\mathrm{loc}}(M \setminus \{a_1, \dots, a_p\})$ and $F_A \in L^2(\Lambda^2 T^* M \otimes su(2))$. Thus by Theorem 1.5 there exist a principal $SU(2)$-bundle $Q \to M$ and an L^2_1-connection $\widetilde{A}$ on Q such that A and $\widetilde{A}$ are gauge equivalent over $M \setminus \{a_1, \dots, a_p\}$. Since $\mathrm{tr}(F_A \wedge F_A) = \mathrm{tr}(F_{\widetilde{A}} \wedge F_{\widetilde{A}})$ in $\mathcal{R}(M)$, we may identify A with $\widetilde{A}$.

Obviously we have

$$\begin{aligned} k = C_2(A_i)(M) \to \frac{1}{8\pi^2} \int_M \mathrm{tr}(F_A \wedge F_A) + \sum_{j=1}^{q} d_j \\ = q(Q) + \sum_{j=1}^{q} d_j. \end{aligned}$$

Conversely, for a given principal $SU(2)$-bundle $Q \to M$, an L^2_1-connection A on Q, $q \geq 0$, $d_1, \dots, d_q \in \mathbb{Z}$ and $b_1, \dots, b_q \in M$ satisfying (1)–(4) of Theorem 1.3, we must show that there exist $\{A_i\} \subset \mathcal{A}_k$ such that

$$\sup_{i \geq 1} \int_M |F_{A_i}|^2 * 1 < \infty,$$

$$\frac{1}{8\pi^2} \mathrm{tr}(F_{A_i} \wedge F_{A_i}) \rightharpoonup \frac{1}{8\pi^2} \mathrm{tr}(F_A \wedge F_A) + \sum_{j=1}^{q} d_j \delta_{b_j}$$

weak* in $\mathcal{R}(M)$. This follows from the same construction as in the proof of Theorem 2.2 in §2, Step 2. This completes the proof of Theorem 1.3. $\square$

For the later purpose, we give the following corollary of Theorem 1.3.

Corollary 1.11. *Under the same notations as in Theorem* 1.3 *and its proof, assume* $|F_{A_i}|^2 \rightharpoonup \omega$ *weak* in* $\mathcal{R}(M)$. *Then*

$$\omega \geq |F_A|^2 + 8\pi^2 \sum_{j=1}^{q} |d_j| \delta_{b_j}.$$

Proof. Since $F_{A_i} \rightharpoonup F_A$ weakly in $L^2(\Lambda^2 T^* M \otimes su(2))$, by the weak lower semicontinuity of the L^2-norm we have $\omega \geq |F_A|^2$. On the other hand, for any $\varphi \in C^0(M)$, we have

$$\left| \int_M \operatorname{tr}(F_{A_i} \wedge F_{A_i}) \wedge \varphi \right| \leq \int_M |F_{A_i}|^2 |\varphi| * 1.$$

Letting $i \to \infty$ in the above inequality we obtain

$$\left| \int_M \operatorname{tr}(F_A \wedge F_A) \wedge \varphi + 8\pi^2 \sum_{j=1}^{q} d_j \varphi(b_j) \right| \leq \int_M |\varphi| \, d\omega.$$

We take $\varphi = \varphi_n$ such that $0 \leq \varphi_n \leq 1$ and $\varphi_n \searrow \chi_{\{b_j\}}$ to obtain

$$\omega(\{b_j\}) \geq 8\pi^2 |d_j| \quad (1 \leq j \leq q).$$

Since $|F_A| \perp \delta_{b_j}$ for any $1 \leq j \leq q$, we get

$$\omega \geq |F_A|^2 + 8\pi^2 \sum_{j=1}^{q} |d_j| \delta_{b_j}.$$

This completes the proof. □

Definition 1.12.

(1) For $\omega \in \mathcal{P}_k^2$, $\omega = \frac{1}{8\pi^2} \operatorname{tr}(F_A \wedge F_A) + \sum_{j=1}^{q} d_j \delta_{b_j}$, where A is an L_1^2-connection on a principal $SU(2)$-bundle $Q \to M$, we write

$$[\omega] = ([Q], (d_j, b_j)_{1 \leq j \leq q}),$$

where $[Q]$ is the isomorphism class of principal $SU(2)$-bundles with $Q \in [Q]$.

(2) We define $[\mathcal{P}_k^2] = \{[\omega] : \omega \in \mathcal{P}_k^2\}$.

(3) For $[\omega] = ([Q], (d_j, b_j)_{1 \leq j \leq q}) \in [\mathcal{P}_k^2]$ we define the space $\mathcal{A}_k^2([\omega])$ of L_1^2-connections on $[\omega]$ by $A \in \mathcal{A}_k^2([\omega])$ if and only if A is an L_1^2-connection on some principal $SU(2)$-bundle $Q' \to M$, where $Q' \in [Q]$.

(4) We define $\mathcal{A}_k^2 = \bigcup_{[\omega] \in [\mathcal{P}_k^2]} \mathcal{A}_k^2([\omega])$.

Remark 1.13. We can view $[\omega] = ([Q], (d_j, b_j)_{1 \leq j \leq q}) \in [\mathcal{P}_k^2]$ as follows:

$[\omega]$ is obtained by attaching principal $SU(2)$-bundles $P_j \to S^4$ to Q at b_j, where the 2nd Chern number of P_j is $-d_j$ and we identify b_j with the south pole of S^4.

This can be explained by the following consideration. We identify b_j with the south pole of S^4, $d_j\delta_{b_j}$ with $d_j\{b_j\} \in H_0(S^4;\mathbb{Z})$ and $d_j\{b_j\}$ with its Poincaré dual $d_j\eta_{\{b_j\}} \in H^4(S^4;\mathbb{Z})$, where $\eta_{\{b_j\}}$ is the Poincaré dual of b_j in S^4. Since $\eta_{\{b_j\}}$ generates $H^4(S^4;\mathbb{Z})$, $d_j\eta_{\{b_j\}}$ corresponds to the isomorphism class of principal $SU(2)$-bundles over S^4 with 2nd Chern number $-d_j$. Therefore $[\omega]$ can be viewed as mentioned above.

For $\omega \in \mathcal{P}_k^2$, $\omega = \frac{1}{8\pi^2}\mathrm{tr}(F_A \wedge F_A) + \sum_{j=1}^q d_j\delta_{b_j}$, we define $\|\omega\|$ by

$$\|\omega\| = \|F_A\|_{L^2(M)} + |\omega|(M),$$

where $|\omega|(M) =$ total variation of ω in M.

In the next theorem we show that $\mathcal{P}_k^2$ is bounded weakly compact with respect to the "norm" $\|\cdot\|$.

THEOREM 1.14. *Let* $\{\omega_i\} \subset \mathcal{P}_k^2$, $\omega_i = \frac{1}{8\pi^2}\mathrm{tr}(F_{A_i} \wedge F_{A_i}) + \sum_{j=1}^{q_i} d_j^i\delta_{b_j^i}$, *be such that* $\sup_{i\geq 1}||\omega_i|| < \infty$. *Then there exists a subsequence* $\{\omega_{n_i}\} \subset \{\omega_i\}$ *such that* $\omega_{n_i} \rightharpoonup \omega$ *weak* in* $\mathcal{R}(M)$ *for some* $\omega \in \mathcal{P}_k^2$.

PROOF. Since $\sup_{i\geq 1}\|\omega_i\| < \infty$, we have

(a) $\sup_{i\geq 1}\int_M |F_{A_i}|^2 * 1 < \infty$,
(b) $\sup_{i\geq 1}\sum_{j=1}^{q_i}|d_j^i| < \infty$.

By (a), as in the proof of Theorem 1.3, there exist a principal $SU(2)$-bundle $Q \to M$, an L_1^2-connection A on Q, $\widehat{d}_1, \dots, \widehat{d}_l \in \mathbb{Z}$ and $\widehat{b}_1, \dots, \widehat{b}_l \in M$ such that

$$\frac{1}{8\pi^2}\mathrm{tr}(F_{A_i} \wedge F_{A_i}) \rightharpoonup \frac{1}{8\pi^2}\mathrm{tr}(F_A \wedge F_A) + \sum_{m=1}^{l}\widehat{d}_m\delta_{\widehat{b}_m}$$

weak* in $\mathcal{R}(M)$.

On the other hand, by (b), $\sup_{i\geq 1} q_i < \infty$. (We remark that we may assume all $d_j^i \neq 0$.) Thus, passing to a subsequence if necessary, we may assume $q_i = q$ for all i. Also by (b), $\sup_{i\geq 1} d_j^i < \infty$, and we may also assume $d_j^i = d_j$ for all i. Since M is compact, passing to a subsequence if necessary, we may assume $b_j^i \to b_j \in M$ for all $1 \leq j \leq q$. Under these conditions we easily see that

$$\omega_i \rightharpoonup \frac{1}{8\pi^2}\mathrm{tr}(F_A \wedge F_A) + \sum_{m=1}^{l}\widehat{d}_m\delta_{\widehat{b}_m} + \sum_{j=1}^{q} d_j\delta_{b_j}$$

weak* in $\mathcal{R}(M)$. It is also easy to see that

$$q(Q) + \sum_{m=1}^{l}\widehat{d}_m + \sum_{j=1}^{q} d_j = k.$$

This completes the proof. □

REMARK 1.15. For $\omega_i = \frac{1}{8\pi^2}\mathrm{tr}(F_{A_i} \wedge F_{A_i}) \in \mathcal{P}_k^2$, $\sup_{i\geq 1}\|\omega_i\| < \infty$ if and only if $\sup_{i\geq 1}\|F_{A_i}\|_{L^2(M)} < \infty$.

2. Relaxed Yang–Mills functional

Let $A \in \mathcal{A}_k^2$, that is, let $A \in \mathcal{A}_k^2([\omega])$ for some $\omega \in \mathcal{P}_k^2$. Here $[\omega] = ([Q], (d_j, b_j)_{1 \le j \le q})$. We extend the Yang–Mills functional which is defined on $\mathcal{A}(P)$ (= smooth connections on the principal $SU(2)$-bundle $P \in \mathcal{P}_k$) to $\mathcal{A}_k^2$ by the method of relaxation. This is also the Lebesgue extension of the variational integral $\int_M |F_A|^2 * 1$.

DEFINITION 2.1. Let $A \in \mathcal{A}_k^2$ be as above. We define the *relaxed Yang–Mills functional* $\mathrm{YM}_{\mathrm{rel}}$ by

$$\mathrm{YM}_{\mathrm{rel}}(A) = \inf\{\liminf_{i\to\infty} \mathrm{YM}(A_i) : \{A_i\} \in S\},$$

where $\{A_i\} \in S$ if and only if $A_i \in \mathcal{A}(P_i)$ for some $P_i \in \mathcal{P}_k$ and

$$\frac{1}{8\pi^2}\mathrm{tr}(F_{A_i} \wedge F_{A_i}) \rightharpoonup \mu + \sum_{j=1}^{q} d_j \delta_{b_j}$$

and $A_i \rightharpoonup^{\star} A$ with $[\mu] = [Q]$. Here we use the notation $A_i \rightharpoonup^{\star} A$ if and only if there exist $a_1, \dots, a_p \in M$, an open covering $\{U_\alpha\}$ of $M \setminus \{a_1, \dots, a_p\}$, and sections $\sigma_\alpha(i) : U_\alpha \to P_i$ such that $\sigma_\alpha^*(i) A_i \rightharpoonup A_\alpha$ weakly in $L_1^2(U_\alpha)$, where $A = \{A_\alpha\}$.

For any $A \in \mathcal{A}_k^2$, there exists $\{A_i\} \in S$ such that $\{A_i\}$ converges to A in the above sense. (See the proof of Theorem 2.2 below.)

THEOREM 2.2. *Let* $A \in \mathcal{A}_k^2$, *where* $A \in \mathcal{A}_k^2([\omega])$ *with* $\omega \in \mathcal{P}_k^2$ *and* $[\omega] = ([Q], (d_j, b_j)_{1 \le j \le q})$. *Then*

$$\mathrm{YM}_{\mathrm{rel}}(A) = \mathrm{YM}(A) + 8\pi^2 \sum_{j=1}^{q} |d_j|.$$

PROOF. STEP 1. We first prove

$$\mathrm{YM}_{\mathrm{rel}}(A) \ge \int_M |F_A|^2 * 1 + 8\pi^2 \sum_{j=1}^{q} |d_j|.$$

Let $A_i \in \mathcal{A}(P_i)$ ($P_i \in \mathcal{P}_k$) be such that

$$\frac{1}{8\pi^2}\mathrm{tr}(F_{A_i} \wedge F_{A_i}) \rightharpoonup \mu + \sum_{j=1}^{q} d_j \delta_{b_j}, \quad A_i \rightharpoonup^{\star} A.$$

Then we must have $\mu = \frac{1}{8\pi^2}\mathrm{tr}(F_A \wedge F_A)$ as in the proof of Theorem 1.3.

We fix $\varepsilon > 0$ small enough. Then we have

$$\mathrm{YM}(A_i) = \int_M |F_{A_i}|^2 * 1 = \int_{M \setminus \bigcup_{j=1}^q B_\varepsilon(b_j)} |F_{A_i}|^2 * 1 + \int_{\bigcup_{j=1}^q B_\varepsilon(b_j)} |F_{A_i}|^2 * 1,$$

where $B_\varepsilon(b_j)$ is the geodesic ball of radius ε with center at $b_j \in M$.

Since $F_{A_i} \rightharpoonup F_A$ weakly in $L^2(\Lambda^2 T^*M \otimes \mathrm{Ad}(P))$, we have

$$\liminf_{i\to\infty} \int_{M\setminus\bigcup_{j=1}^q B_\varepsilon(b_j)} |F_{A_i}|^2 * 1 \geq \int_{M\setminus\bigcup_{j=1}^q B_\varepsilon(b_j)} |F_A|^2 * 1. \tag{4}$$

On the other hand, by Corollary 1.11 we have (passing to a subsequence if necessary) $|F_{A_i}|^2 \rightharpoonup \omega$, where ω satisfies

$$\omega \geq |F_A|^2 + 8\pi^2 \sum_{j=1}^q |d_j| \delta_{b_j}.$$

So we have

$$\liminf_{i\to\infty} \int_{\bigcup_{j=1}^q B_\varepsilon(b_j)} |F_{A_i}|^2 * 1 \geq \int_{\bigcup_{j=1}^q B_\varepsilon(b_j)} |F_A|^2 * 1 + 8\pi^2 \sum_{j=1}^q |d_j|. \tag{5}$$

Combining (4) and (5) we obtain

$$\liminf_{i\to\infty} \int_M |F_{A_i}|^2 * 1 \geq \int_M |F_A|^2 * 1 + 8\pi^2 \sum_{j=1}^q |d_j|. \tag{6}$$

Since $\{A_i\} \in S$ is arbitrary we get

$$\mathrm{YM}_{\mathrm{rel}}(A) \geq \int_M |F_A|^2 * 1 + 8\pi^2 \sum_{j=1}^q |d_j|.$$

STEP 2. Next we show

$$\mathrm{YM}_{\mathrm{rel}}(A) \leq \int_M |F_A|^2 * 1 + 8\pi^2 \sum_{j=1}^q |d_j|.$$

For this it is sufficient to show that there exist $A_i \in \mathcal{A}(P_i)$, where $P_i \in \mathcal{P}_k$, such that

$$\frac{1}{8\pi^2}\mathrm{tr}(F_{A_i} \wedge F_{A_i}) \rightharpoonup \frac{1}{8\pi^2}\mathrm{tr}(F_A \wedge F_A) + \sum_{j=1}^q d_j \delta_{b_j}, \quad A_i \rightharpoonup^* A$$

and

$$\int_M |F_{A_i}|^2 * 1 \to \int_M |F_A|^2 * 1 + 8\pi^2 \sum_{j=1}^q |d_j|.$$

We now construct such a sequence. The construction can be summarized as follows: A_i is constructed by gluing self-dual (if $d_j < 0$) or anti-self-dual (if $d_j > 0$) connections $A_j^\star$ on principal $SU(2)$-bundles $P_j \to S^4$ $(j = 1, \ldots, q)$ to $A \in \mathcal{A}_k^2(Q)$ at b_j, where P_j is a principal $SU(2)$-bundle with 2nd Chern number $-d_j$ and we identify b_j with the south pole of S^4. The details are as follows:

Let $P_j \to S^4$ be a principal $SU(2)$-bundle whose 2nd Chern number is $-d_j$ and $A_j^\star$ a self-dual (if $d_j < 0$) or anti-self-dual (if $d_j > 0$) connection on P_j.

Let $\alpha : S^4 \to S^4$ be the inversion of S^4 through its equatorial S^3, which is fixed.

We denote by $z_j : B_j \to \mathbb{R}^4$ the normal coordinate of M, where $B_j \subset M$ is a geodesic ball with $B_j \cap B_l = \emptyset$ (if $j \neq l$). We identify $S^4 \setminus \{n\}$ (n is the north pole of S^4) with $\mathbb{R}^4$ by the stereographic projection from the north pole. We also identify B_j with $z_j(B_j) \subset \mathbb{R}^4 = S^4 \setminus \{n\}$ and $b_j \in M$ with the south pole s of S^4 which is also identified with $0 \in \mathbb{R}^4$ by the stereographic projection from the north pole. For simplicity we assume $B_j = \{x : |x| < 1\} \subset \mathbb{R}^4$. Since $Q|_{B_j} \to B_j$ and $\alpha^* P_j|_{B_j} \to B_j$ are trivial, we may glue these principal $SU(2)$-bundles over B_j and obtain the principal $SU(2)$-bundle $Q - \sum_{j=1}^q P_j \to M$. By the construction

$$q\Big(Q - \sum_{j=1}^q P_j\Big) = q(Q) - \sum_{j=1}^q q(P_j) = q(Q) + \sum_{j=1}^q d_j = q(P) = k.$$

Next we define a family $\{A_\varepsilon\}$ of connections on $Q - \sum_{j=1}^q P_j$ as follows. First let $\varrho \in C_0^\infty(\mathbb{R})$ be such that

$$\varrho(x) = \varrho(|x|) = \begin{cases} 0 & \text{if } |x| \leq 1/2, \\ 1 & \text{if } |x| \geq 1. \end{cases}$$

We set $\varrho_\varepsilon(x) = \varrho(\varepsilon^{-1/2}|x|)$. We denote by $T_\varepsilon : \mathbb{R}^4 \to \mathbb{R}^4$ the dilation $x \mapsto \varepsilon x$. Let θ_j be the flat product connection on $Q - \sum_{j=1}^q P_j|_{B_j} = Q - P_j|_{B_j}$.

Over B_j define

$$A_\varepsilon = \theta_j + T_{\varepsilon^{-1}}^*[\varrho_\varepsilon T_\varepsilon^* A + \alpha^*(\varrho_\varepsilon T_\varepsilon^* A_i)]. \tag{7}$$

Set $\widehat{M} = M \setminus \bigcup_{j=1}^q \{p \in M : \operatorname{dist}(p, b_j) > 1/2\}$. Over $\widehat{M}$, $Q - \sum_{j=1}^q P_j$ is identified with P. Over $\widehat{M}$ we set

$$A_\varepsilon = A. \tag{8}$$

Set $\widehat{B}_j = \{x \in \mathbb{R}^4 : |x| < \varepsilon^{3/2}/2\}$. Over $\widehat{B}_j$ we set

$$A_\varepsilon = \alpha^* T_{\varepsilon^2}^* A_j. \tag{9}$$

Then (7)–(9) give a family of connections $\{A_\varepsilon\}$ on $Q - \sum_{j=1}^q P_j$.

By the same calculations as in Taubes [12] we get

$$\frac{1}{8\pi^2}\operatorname{tr}(F_{A_\varepsilon} \wedge F_{A_\varepsilon}) \rightharpoonup \frac{1}{8\pi^2}\operatorname{tr}(F_A \wedge F_A) + \sum_{j=1}^q d_j \delta_{b_j}, \tag{10}$$

$$\int_M |F_{A_\varepsilon}|^2 * 1 \to \int_M |F_A|^2 * 1 + \sum_{j=1}^q \int_{S^4} |F_{A_j^\star}|^2 * 1. \tag{11}$$

Since $A_j^\star$ is a self-dual (or anti-self-dual) connection, we have

$$\int_{S^4} |F_{A_j^\star}|^2 * 1 = 8\pi^2 |d_j|. \tag{12}$$

Combining (11) and (12) we get

$$\int_M |F_{A_\varepsilon}|^2 * 1 \to \int_M |F_A|^2 * 1 + 8\pi^2 \sum_{j=1}^{q} |d_j|.$$

Since the space of C^∞-connections is dense in the L_1^2-connections there exist $A_i \in \mathcal{A}(P_i)$, $P_i \in \mathcal{P}_k$, such that

$$\frac{1}{8\pi^2}\mathrm{tr}(F_{A_i} \wedge F_{A_i}) \rightharpoonup \frac{1}{8\pi^2}\mathrm{tr}(F_A \wedge F_A) + \sum_{j=1}^{q} d_j \delta_{b_j},$$
$$\int_M |F_{A_i}|^2 * 1 \to \int_M |F_A|^2 * 1 + 8\pi^2 \sum_{j=1}^{q} |d_j|.$$

Thus we get

$$\mathrm{YM}_{\mathrm{rel}}(A) \le \int_M |F_A|^2 * 1 + 8\pi^2 \sum_{j=1}^{q} |d_j|.$$

Combining Step 1 and Step 2, we finally obtain

$$\mathrm{YM}_{\mathrm{rel}}(A) = \int_M |F_A|^2 * 1 + 8\pi^2 \sum_{j=1}^{q} |d_j|.$$

This completes the proof. □

In the next theorem we prove that $\mathrm{YM}_{\mathrm{rel}}$ is weakly lower semicontinuous with respect to a suitable weak topology.

THEOREM 2.3. *Let* $\{A_i\} \subset \mathcal{A}_k^2$ *and* $A_\infty \in \mathcal{A}_k^2$ *be such that* $A_i \in \mathcal{A}_k^2([\omega_i])$ $(\omega_i \in \mathcal{P}_k^2)$ *and* $A_\infty \in \mathcal{A}_k^2([\omega_\infty])$ $(\omega_\infty \in \mathcal{P}_k^2)$. *Assume* $\omega_i = \frac{1}{8\pi^2}\mathrm{tr}(F_{A_i} \wedge F_{A_i}) + \sum_{j=1}^{q_i} d_j^i \delta_{b_j^i}$, $\sup_{i\ge1} \|\omega_i\| < \infty$, $\omega_i \rightharpoonup \omega_\infty$ *and* $A_i \rightharpoonup^\star A_\infty$. *Then*

$$\mathrm{YM}_{\mathrm{rel}}(A_\infty) \le \liminf_{i\to\infty} \mathrm{YM}_{\mathrm{rel}}(A_i).$$

PROOF. This is proved as in the proof of Theorem 1.14. Since $\sup_{i\ge1}\|\omega_i\| < \infty$, we have

(a) $\sup_{i\ge1} \int_M |F_{A_i}|^2 * 1 < \infty$,
(b) $\sup_{i\ge1} \sum_{j=1}^{q_i} |d_j^i| < \infty$.

As in the proof of Theorem 1.14 there exist a principal $SU(2)$-bundle $\widehat{Q} \to M$, an L_1^2-connection $\widehat{A}$ on $\widehat{Q}$, $\widehat{d}_1, \dots, \widehat{d}_l \in \mathbb{Z}$ and $\widehat{b}_1, \dots, \widehat{b}_l \in M$ such that

$$\frac{1}{8\pi^2}\mathrm{tr}(F_{A_i} \wedge F_{A_i}) \rightharpoonup \frac{1}{8\pi^2}\mathrm{tr}(F_{\widehat{A}} \wedge F_{\widehat{A}}) + \sum_{m=1}^{l} \widehat{d}_m \delta_{\widehat{b}_m}.$$

As in the proof of Theorem 1.14 there exist $q \in \mathbb{Z}$, $c_1, \dots, c_q \in \mathbb{Z}$ and $p_1, \dots, p_q \in M$ such that

$$\sum_{j=1}^{q_i} d_j^i \delta_{b_j^i} \rightharpoonup \sum_{j=1}^{q} c_j \delta_{p_j}. \tag{13}$$

Then we have

$$\omega_\infty = \frac{1}{8\pi^2}\mathrm{tr}(F_{A_\infty} \wedge F_{A_\infty}) + \sum_{m=1}^{l} \widehat{d}_m \delta_{\widehat{b}_m} + \sum_{j=1}^{q} c_j \delta_{p_j}.$$

Therefore $\widehat{A} = A_\infty$ and

$$[\omega_\infty] = ([Q_\infty], (\widehat{d}_1, \dots, \widehat{d}_l, c_1, \dots, c_q; \widehat{b}_1, \dots, \widehat{b}_l, p_1, \dots, p_q)).$$

We may assume, as in the proof of Theorem 1.14, that $q_i \equiv q$ and $d_j^i \equiv c_j$ $(1 \le j \le q)$ for all i. Fix $\varepsilon > 0$ small enough. We have

$$\int_M |F_{A_i}|^2 * 1 = \int_{M \setminus \bigcup_{m=1}^{l} B_\varepsilon(\widehat{b}_m)} |F_{A_i}|^2 * 1 + \int_{\bigcup_{m=1}^{l} B_\varepsilon(\widehat{b}_m)} |F_{A_i}|^2 * 1. \tag{14}$$

Since $F_{A_i} \rightharpoonup F_{A_\infty}$ weakly in L^2, we get

$$\liminf_{i\to\infty} \int_{M \setminus \bigcup_{m=1}^{l} B_\varepsilon(\widehat{b}_m)} |F_{A_i}|^2 * 1 \ge \int_{M \setminus \bigcup_{m=1}^{l} B_\varepsilon(\widehat{b}_m)} |F_{A_\infty}|^2 * 1. \tag{15}$$

We may assume, passing to a subsequence if necessary, that $|F_{A_i}|^2 \rightharpoonup \nu$ weak* in $\mathcal{R}(M)$. By Corollary 1.11, we have

$$\nu \ge |F_{A_\infty}|^2 + 8\pi^2 \sum_{m=1}^{l} |\widehat{d}_m| \delta_{\widehat{b}_m}.$$

Therefore we obtain

$$\liminf_{i\to\infty} \int_{\bigcup_{m=1}^{l} B_\varepsilon(\widehat{b}_m)} |F_{A_i}|^2 * 1 \ge \int_{\bigcup_{m=1}^{l} B_\varepsilon(\widehat{b}_m)} |F_{A_\infty}|^2 * 1 + 8\pi^2 \sum_{m=1}^{l} |\widehat{d}_m|. \tag{16}$$

From (15) and (16) we get

$$\liminf_{i\to\infty} \int_M |F_{A_i}|^2 * 1 \ge \int_M |F_A|^2 * 1 + 8\pi^2 \sum_{m=1}^{l} |\widehat{d}_m|. \tag{17}$$

From (13) and (17) we finally obtain

$$\begin{aligned}\liminf_{i\to\infty} \mathrm{YM}_{\mathrm{rel}}(A_i) &\ge \int_M |F_A|^2 * 1 + 8\pi^2 \sum_{m=1}^{l} |\widehat{d}_m| + 8\pi^2 \sum_{j=1}^{q} |c_j| \\ &\ge \mathrm{YM}_{\mathrm{rel}}(A_\infty).\end{aligned}$$

This completes the proof. □

3. Minimization problems in $\mathcal{A}_k^2$

From Theorems 2.3 and 1.14 we obtain the following existence result.

THEOREM 3.1. *There exists* $A \in \mathcal{A}_k^2$ *such that* $\mathrm{YM}_{\mathrm{rel}}(A) = \inf\{\mathrm{YM}_{\mathrm{rel}}(A) : A \in \mathcal{A}_k^2\} \equiv m_k$.

PROOF. Let $\{A_i\} \subset \mathcal{A}_k^2$ be a minimizing sequence, i.e.,

$$\mathrm{YM}_{\mathrm{rel}}(A_i) \to m_k \quad \text{as } i \to \infty.$$

We assume $A_i \in \mathcal{A}_k^2([\omega_i])$, $\omega_i = \frac{1}{8\pi^2}\mathrm{tr}(F_{A_i} \wedge F_{A_i}) + \sum_{j=1}^{q_i} d_j^i \delta_{b_j^i}$.

Since $\sup_{i\geq 1} \mathrm{YM}_{\mathrm{rel}}(A_i) < \infty$, we obtain $\sup_{i \geq 1} \|\omega_i\| < \infty$. Thus we may assume, passing to a subsequence if necessary, that $\omega_i \rightharpoonup \omega_\infty$ weak* in $\mathcal{R}(M)$ for some $\omega_\infty \in \mathcal{P}_k^2$ and $A_i \rightharpoonup^\star A_\infty$ for some $A_\infty \in \mathcal{A}_k^2([\omega_\infty]) \subset \mathcal{A}_k^2$ (by Theorem 1.14). By Theorem 2.2 we get

$$\mathrm{YM}_{\mathrm{rel}}(A_\infty) \leq \liminf_{i\to\infty} \mathrm{YM}_{\mathrm{rel}}(A_i) \equiv m_k.$$

Therefore $A_\infty \in \mathcal{A}_k^2$ is a $\mathrm{YM}_{\mathrm{rel}}$-minimizer in $\mathcal{A}_k^2$. □

But in general, A_∞ obtained above is not a YM-minimizer in $\mathcal{A}(P)$ for some $P \in \mathcal{P}_k$. We next study when A_∞ is a *classical* solution, that is, when A_∞ is a minimizer of YM in $\mathcal{A}(P)$ for some $P \in \mathcal{P}_k$. We assume A_∞ is an L_1^2-connection on some principal $SU(2)$-bundle $Q_\infty \to M$.

We first prepare some lemmas.

LEMMA 3.2. *We have* $m_k \equiv \inf\{\mathrm{YM}_{\mathrm{rel}}(A) : A \in \mathcal{A}_k^2\} = \inf\{\mathrm{YM}(A) : A \in \mathcal{A}(P)$ *for some* $P \in \mathcal{P}_k\}$.

PROOF. This follows from the proof of Theorem 2.2. □

LEMMA 3.3. *Let* $A_\infty \in \mathcal{A}_k^2$ *be a* $\mathrm{YM}_{\mathrm{rel}}$*-minimizer obtained by Theorem* 3.1. *Assume* $A_\infty \in \mathcal{A}_k^2([\omega_\infty])$ *for some* $\omega_\infty \in \mathcal{P}_k^2$. *Then* A_∞ *is a* YM*-minimizer in the class* $\mathcal{A}_k^2([\omega_\infty])$.

PROOF. Assume to the contrary that A_∞ is not a YM-minimizer in $\mathcal{A}_k^2([\omega_\infty])$. Then there exists $A \in \mathcal{A}_k^2([\omega_\infty])$ and $\varepsilon > 0$ such that

$$\mathrm{YM}(A) < \mathrm{YM}(A_\infty) - \varepsilon. \tag{18}$$

Since C^∞-connections are dense in the L_1^2-connections and $L_1^2 \hookrightarrow L^4$ in dimension 4, we may assume that A is a C^∞-connection. In this case, as in the proof of Theorem 2.2, we find $A_i \in \mathcal{A}(P_i)$, $P_i \in \mathcal{P}_k$ such that

$$\mathrm{YM}(A_i) \to \mathrm{YM}(A) + 8\pi^2 \sum_{j=1}^{q} |d_j| \quad \text{as } i \to \infty,$$

where $[\omega_\infty] = ([Q_\infty], (d_j, b_j)_{1\le j\le q})$. So by (18) we obtain

$$\begin{aligned}\mathrm{YM}(A) + 8\pi^2 \sum_{j=1}^{q} |d_j| &< \mathrm{YM}(A_\infty) + 8\pi^2 \sum_{j=1}^{q} |d_j| - \varepsilon \\ &= \mathrm{YM}_{\mathrm{rel}}(A_\infty) - \varepsilon = m_k - \varepsilon.\end{aligned}$$

By Lemma 3.2 we get $m_k \le m_k - \varepsilon$. This contradiction completes the proof. □

We are now ready to state (and prove) our main theorem in this section.

THEOREM 3.4. *Let $A_\infty \in \mathcal{A}_k^2$ be a $\mathrm{YM}_{\mathrm{rel}}$-minimizer in $\mathcal{A}_k^2$. Assume that*

$$m_k < m_\ell + 8\pi^2|\ell - k| \quad \text{for } |\ell - k| \le |k|,\ \ell \ne k.$$

Then $A_\infty \in L_1^2(\mathcal{A}(P))$ for some $P \in \mathcal{P}_k$ and A_∞ is a YM-minimizing connection in $L_1^2(\mathcal{A}(P))$.

PROOF. Let $Q_0 = M \times SU(2) \to M$ be the product bundle over M and θ be a flat connection on it. We fix a point $a \in M$. We set $[\omega_0] = ([Q_0], (k, a)) \in \mathcal{P}_k^2$. We denote by ℓ the topological quantum number of Q_∞. Since $\theta \in \mathcal{A}([\omega_0]) \subset \mathcal{A}_k^2$, we obtain

$$8\pi^2|\ell - k| \le \mathrm{YM}(A_\infty) + 8\pi^2|\ell - k| \le \mathrm{YM}_{\mathrm{rel}}(A_\infty) \le \mathrm{YM}_{\mathrm{rel}}(\theta) = 8\pi^2|k|.$$

Thus we get $|\ell - k| \le |k|$. We shall prove $\ell = k$.

Assume $\ell \ne k$. Then by Lemmas 3.2 and 3.3 we have

$$m_\ell + 8\pi^2|\ell - k| \le \mathrm{YM}_{\mathrm{rel}}(A_\infty) = m_k < m_\ell + 8\pi^2|\ell - k|.$$

But this is a contradiction. Therefore $\ell = k$, which completes the proof. □

REMARK 3.5. (a) We always have $m_k \le m_\ell + 8\pi^2|\ell - k|$ for all $k, \ell \in \mathbb{Z}$.

(b) For $A \in \mathcal{A}(P)$, we always have $\mathrm{YM}(A) \ge 8\pi^2|k|$, where k is the 2nd Chern number of a principal $SU(2)$-bundle $P \to M$ and equality holds if and only if A is a self-dual (or anti-self-dual) connection. Of course in this case A is a YM-minimizer in the class $\mathcal{A}(P)$. The existence of self-dual (or anti-self-dual) connections is studied in [1], [10] and [11]. See also [5]. Taubes [10] proved the existence theorem under the assumption that the intersection matrix of M is positive definite. If M does not satisfy this assumption, in general, there is no self-dual (or anti-self-dual) connection on P. See [5].

(c) Theorem 3.4 is related to the result of Sedlacek [9, Theorem 7.1]. In our setting, the existence problem is reduced to the regularity problem.

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