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CONNECTED SUM CONSTRUCTIONS FOR CONSTANT SCALAR CURVATURE METRICS

Rafe Mazzeo[(†)] — Daniel Pollack[(‡)] — Karen Uhlenbeck[(⋆)]

Dedicated to Louis Nirenberg on the occasion of his 70th birthday

We give a general procedure for gluing together possibly noncompact manifolds of constant scalar curvature which satisfy an extra nondegeneracy hypothesis. Our aim is to provide a simple paradigm for making "analytic" connected sums. In particular, we can easily construct complete metrics of constant positive scalar curvature on the complement of certain configurations of an even number of points on the sphere, which is a special case of Schoen's [S1] well-known, difficult construction. Applications of this construction produces metrics with prescribed asymptotics. In particular, we produce metrics with cylindrical ends, the simplest type of asymptotic behaviour. Solutions on the complement of an infinite number of points are also constructed by an iteration of our construction.

I. Introduction

It is now a well-entrenched procedure in geometric analysis to construct new solutions to nonlinear PDE by gluing together known solutions: an approximate solution is constructed, then perturbed to an exact solution using analytic methods. One of the early spectacular instances of this is Taubes" patching of instantons [T]. More recent instances are too numerous to list here.

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The geometric problem we wish to examine here is the possibility of gluing together manifolds of constant scalar curvature satisfying a certain nondegeneracy condition to obtain a new constant scalar curvature metric on the connected sum. The precise notion of nondegeneracy will be given in §2 below, but when the manifolds are compact, possibly with boundary, it coincides with the invertibility of the Jacobi operator (which is weaker than stability). The main result is:

Theorem. *Let (X_1, g_1) and (X_2, g_2) be any two manifolds, possibly with boundary, with complete metrics g_1, g_2 of constant scalar curvature $n(n-1)$. Suppose also that the metrics g_i satisfy the nondegeneracy condition* (2.12) *and either* (2.15) *or* (2.16)–(2.17) *below. Then for any points $p_i \in X_i$, the connected sum $X_1 \#_\varepsilon X_2$ obtained by excising small ε-balls around the p_i and identifying boundaries, carries a complete nondegenerate metric g_ε of constant scalar curvature $n(n-1)$.*

The problem of gluing nondegenerate compact constant scalar curvature manifolds has already been studied by Joyce [J], so our primary interest is with noncompact manifolds. Furthermore, we will focus exclusively on manifolds with constant positive scalar curvature (CPSC). The simplest of these that we wish to treat as "summands" to be glued are the Delaunay metrics on the complement of two points in the sphere S^n. These are conformally equivalent to elements of an explicit one-parameter family of rotationally symmetric metrics interpolating between the cylinder and an infinite bead of spheres strung out along a common axis. These metrics satisfy the nondegeneracy condition, and the metrics we construct on the connected sum of any finite (or even infinite) number of these are conformally flat, hence may be uniformized and regarded as complete metrics on $S^n \setminus \Lambda$, where Λ is a discrete collection of points of even cardinality. Complete CPSC metrics on $S^n \setminus \Lambda$, for Λ finite, were originally constructed in Schoen's well-known and difficult paper [S1]. One motivation for our construction is to provide a simple proof of a special case of his result. The solutions of this type which we construct here are called dipole metrics, because their singular sets Λ are widely separated pairs of closely spaced points.

One contribution of this paper is a general formulation of nondegeneracy of the scalar curvature operator on manifolds with complete CPSC metrics. The importance of the nondegeneracy hypothesis is clear, for example, in the analysis of the moduli space $\mathcal{M}_\Lambda$ of complete CPSC metrics on $S^n \setminus \Lambda$, where Λ is a submanifold. When any component of Λ has positive dimension, $\mathcal{M}_\Lambda$ is infinite-dimensional [MPa1], but when Λ is a finite set of points, $\mathcal{M}_\Lambda$ is a finite-dimensional real analytic set [MPU]. If $g \in \mathcal{M}_\Lambda$ is a nondegenerate solution, then in a neighborhood of it, $\mathcal{M}_\Lambda$ is a smooth (in fact, real analytic) manifold. In particular, when Λ is finite, this neighborhood is of dimension equal to the cardinality of Λ. Unfortunately, we had previously been unable to establish the nondegeneracy of the solutions constructed in [S1], so a number of simple statements about the moduli space theory remained hypothetical. The nondegeneracy of our dipole solutions clarify many of these moduli space issues. This is discussed further in §4. Even when Λ is positive-dimensional, nondegeneracy has important ramifications for the Dirichlet problem parametrizing the infinite-dimensional moduli space $\mathcal{M}_\Lambda$.

It is possible, in certain circumstances, to glue metrics not satisfying the nondegeneracy conditions. The main instance is Schoen's construction [S1] mentioned earlier, cf. also [P1] for an analogous construction in the compact case. The summands in these constructions are standard spheres, for which the Jacobi operator is definitely not invertible. In a forthcoming paper [MPa2], a new and simpler proof of Schoen's theorem will be given; the simplification relies on the observation that here too there is an underlying nondegenerate gluing procedure.

It is well known that there are close relationships between the problems concerning CPSC metrics and constant mean curvature (CMC) surfaces in $\mathbb{R}^3$. Indeed, closely related in form to [S1], but substantially different in many technical details, is Kapouleas' famous construction [K] of CMC surfaces, both compact and noncompact, in $\mathbb{R}^3$. It is possible to adapt the ideas here to construct noncompact CMC surfaces; these surfaces are topologically identical, but geometrically quite different from many of the surfaces obtained by Kapouleas. Because of the simplicity of the CPSC construction, relative to that for CMC surfaces, we defer the CMC construction to a subsequent paper.

In §2 we first discuss the main examples of nondegenerate CPSC metrics and then, motivated by these examples, give an abstract definition of nondegeneracy. Next, in §3, we use this to prove the main gluing theorem. In §4 we apply this to the special case where the summands in the gluing construction are Delaunay metrics on the cylinder, and the ramifications of this theorem for the moduli space theory of [MPU]. We also introduce here the "unmarked moduli space" of CPSC metrics on the complement of any collection of k distinct points in S^n and prove, analogously to [MPU], that it is a real analytic set; finally, we relate the nondegeneracy of elements in this unmarked moduli space to the problem of showing that solutions other than the ones obtained in §3 by the gluing theorem are nondegenerate.

II. Nondegeneracy: examples and definitions

In this section we set up the notation used for the remainder of the paper and then give a precise definition of the nondegeneracy of solutions. We first motivate this definition by describing in some detail the key examples which led to it.

Let (M, g_0) be a fixed complete Riemannian manifold, which we do not assume to have constant positive scalar curvature. Suppose g is a complete CPSC manifold conformal to g_0. We express the conformal factor by writing $g = u^{4/(n-2)} g_0$. Let $R(g_0)$ and $R(g)$ denote the scalar curvatures of g_0 and g. Then it is well known that

$$\Delta_{g_0} u - \frac{n-2}{4(n-1)} R(g_0) u + \frac{n-2}{4(n-1)} R(g) u^{(n+2)/(n-2)} = 0. \tag{2.1}$$

Denote the left side of this equation, when $R(g)$ is replaced by the constant R, by $N_{g_0}(u)$. Much of the analysis of CPSC metrics near g revolves around the linearization L of N_{g_0} at u:

$$\begin{aligned} L_{g_0} v &= \frac{\partial}{\partial t}\bigg|_{t=0} N_{g_0}(u + tv) \\ &= \Delta_{g_0} v - \frac{n-2}{4(n-1)} R(g_0) v + \frac{n+2}{4(n-1)} R u^{4/(n-2)} v. \end{aligned} \tag{2.2}$$

In a special case, where $g = g_0$ and $R = n(n-1)$, this operator takes the form

$$L_g v = \Delta_g v + nv. \tag{2.3}$$

For convenience we let L denote the linearization; whether it is relative to g_0 or g will be clear from the context. Our interest in this section is in the mapping properties of L.

When M is compact, L is self-adjoint, and in this case it is said to be *nondegenerate* provided $0 \notin \operatorname{spec}(L)$. This is equivalent to either the injectivity or surjectivity of $L : H^{s+2}(M) \to H^s(M)$ for any s. Although it is the surjectivity that is used in the nonlinear analysis, it is usually easier to check injectivity. For example, it is clear that the sphere S^n with its standard metric is degenerate because L annihilates the restrictions of linear functions on $\mathbb{R}^{n+1}$ to S^n.

When M is noncompact, the precise formulation of nondegeneracy is more subtle since in all the known examples 0 *is* in the spectrum of L. Rather than exclude these, we must examine the mapping properties of L more closely. Before stating the correct abstract formulation of nondegeneracy, we present the two key examples motivating this definition.

Delaunay metrics. The punctured sphere $M = S^n \backslash \{p_1, p_2\}$ with its standard metric has CPSC but is incomplete. However, it is conformal to the complete CPSC product metric $g = dt^2 + d\theta^2$ on $\mathbb{R}_t \times S^{n-1}_\theta$. There is a one-parameter family of complete CPSC metrics g_ε, $0 < \varepsilon \le \overline{u}$ with $\overline{u} = ((n-2)/n)^{(n-2)/4}$, conformal to g and with g a constant multiple of $g_{\bar{u}}$. For each $\varepsilon \in (0, \overline{u}]$ we have $R(g_\varepsilon) = n(n-1)$ and g_ε is rotationally invariant with respect to the S^{n-1} factor and periodic in t. Because of their similarity to the CMC surfaces of revolution discovered by Delaunay [D] these are called *Delaunay solutions*, although it was

Fowler [F1], [F2] who first studied the differential equation of which these are solutions. Here $g_\varepsilon = u_\varepsilon^{4/(n-2)} g$, and we have normalized so that at $t = 0$, $u'_\varepsilon = 0$ and $u''_\varepsilon \geq 0$. In general, these metrics also have a translation parameter which is relevant to the analysis, as will be apparent below. These solutions are discussed at length in [MPU], to which we refer for details on the discussion below (cf. also [S2]).

As $\varepsilon \to 0$, the supremum of u_ε is uniformly bounded, but the infimum tends to zero. Geometrically, the metrics g_ε develop a sequence of evenly spaced "necks" which separate almost spherical regions. As $\varepsilon \to 0$, these metrics converge to a "string of pearls"—a sequence of round spheres of radius one adjoined at their poles and arranged along a fixed axis. We will denote $(\mathbb{R} \times S^{n-1}, g_\varepsilon)$ by D_ε.

The linearization $L_\varepsilon = \Delta_\varepsilon + n$ at any g_ε is self-adjoint. It has periodic coefficients, hence its spectrum is pure absolutely continuous; there is no point spectrum, i.e. no eigenfunctions in L^2. This last assertion may be seen rather concretely. Separating variables according to the eigenfunction decomposition of Δ_θ, we reduce to analyzing each of the ordinary differential operators L_j induced on the eigenspaces. When $j > 0$, i.e. when the eigenfunction $\psi_j(\theta)$ on S^{n-1} is nonconstant, any solution of $L_j\phi = 0$ grows exponentially in one direction or the other, as may be determined by simple estimates [MPU]. On the other hand, the two linearly independent solutions of $L_0\phi = 0$ are temperate, and so we must examine them further to ensure that they are not in L^2. Fortunately they can be determined explicitly by differentiating u_ε with respect to either the translation parameter t, or the Delaunay parameter ε. Call these ϕ_0^+ and ϕ_0^-, respectively; then ϕ_0^+ is periodic, hence bounded, while ϕ_0^- grows linearly in t.

It is crucial in what follows that these temperate Jacobi fields are integrable, i.e. they arise as derivatives of one-parameter families of conformally related CPSC metrics.

Although L_ε does not have closed range on L^2, it does have this property when considered as an operator on certain weighted Sobolev or Hölder spaces. This was proved for the weighted Sobolev spaces H^s_δ in [MPU], and for the weighted Hölder spaces in [MPa2]. There are advantages and disadvantages in using either of these spaces: the Hölder spaces are better suited to the nonlinearity, but for the various duality arguments we use, the Sobolev spaces are more convenient. Thus, for $\delta, s \in \mathbb{R}$, define

$$H^s_\delta(D_\varepsilon) = \{\phi = e^{\delta\sqrt{1+t^2}}\widetilde{\phi} : \widetilde{\phi} \in H^s(D_\varepsilon)\}, \tag{2.4}$$

where H^s is the standard (global) Sobolev space on D_ε with respect to g_ε, and t is the "cylindrical length" coordinate on $\mathbb{R} \times S^{n-1}$. Note that the geometric length along D_ε depends on ε; for each $\varepsilon > 0$ it is commensurate with t, but not uniformly so as $\varepsilon \to 0$. The spaces H^s are algebras provided $s > n/2$, and so we shall always make this restriction when needed for the nonlinear aspects of our problem.

Now

$$L_\varepsilon : H^{s+2}_\delta(D_\varepsilon) \to H^s_\delta(D_\varepsilon) \tag{2.5}$$

is bounded for any s and δ, but when $\delta = 0$ it does not have closed range. In fact, $L_\varepsilon\phi = 0$ has a two-dimensional family of temperate solutions (namely the span of ϕ_0^+ and ϕ_0^-) and these may be used to construct an orthonormal sequence $\phi^{(j)}$ in $H^0_0 = L^2$ (or any H^s_0) with $\|L_\varepsilon\phi^{(j)}\| \to 0$; this is one of the standard criteria for showing the range is not closed.

We prove in [MPU] that there is a monotone sequence $\delta_j \to \infty$, depending on ε and with $\delta_0 = 0$, for which the map (2.5) is Fredholm provided $\delta \notin \{\pm\delta_j\}$. The values of the δ_j are exactly those for which the ordinary differential operators L_j have solutions of $L_j\phi = 0$ growing exactly like $e^{\pm(\delta/P_\varepsilon)t}$, where P_ε is the period of the metric g_ε. (This is demonstrated in Proposition 4.8 of [MPU] under the assumption that $P_\varepsilon = 1$. A simple rescaling leads to this version.) Thus the same argument as above shows that L_j, and hence L_ε, cannot have closed range on $H^s_{\pm\delta_j}$. The main content of this result is that L_ε has closed range when $\delta \neq \pm\delta_j$.

There is no solution of $L_j\phi = 0$ which decays faster than $e^{-|t|\delta}$, for any fixed $\delta > 0$, as $t \to \pm\infty$; for $j \geq 1$ this follows from the maximum principle (cf. [MPU] for the case $j = 1$ which is somewhat more subtle), while for $j = 0$ it follows because we know the solutions explicitly. This implies that (2.5) is injective provided $\delta < 0$. By duality and elliptic regularity, (2.5) has dense range if $\delta > 0$; when $\delta > 0$ and $\delta \neq \delta_j$, the range is also closed, hence (2.5) is then surjective.

We have established that L_ε is surjective on H^{s+2}_δ for $\delta > 0$, $\delta \neq \delta_1, \delta_2, \ldots$ Unfortunately, none of these spaces are suitable for the nonlinear problem: if ϕ grows like $e^{t\delta}$ then $(1+\phi)^{(n+2)/(n-2)}$ grows even faster. It is possible to obtain surjectivity on a smaller space. Let χ be a cutoff function which equals one for $t \geq 1$ and zero for $t \leq -1$, and define the "deficiency subspace" W by

$$W = \operatorname{span}\{\chi\phi_0^+, \chi\phi_0^-\}. \tag{2.6}$$

(2.7) Proposition. *The map*

$$L_\varepsilon : H^{s+2}_{-\delta}(D_\varepsilon) \oplus W \to H^s_{-\delta}(D_\varepsilon)$$

is surjective for any $\delta < \delta_1$.

An analogous result is proved in [MPU] for more general manifolds of CPSC with k asymptotically Delaunay ends, and with respect to the weighted Sobolev spaces, but is quite simple to prove for the D_ε. Suppose $f \in H^{s+2}_{-\delta}(D_\varepsilon)$, and let f_j be its eigencomponents with respect to the Laplacian on S^{n-1}. A solution u_j

of $L_j u_j = f_j$ may be constructed for each j by "integrating in from $-\infty$" in the standard ODE variation of parameters formula. These solutions u_j will decay like $e^{-|t|\delta}$ as $t \to -\infty$. Since $\delta < \delta_1$, u_j must also decay like $e^{-|t|\delta}$ as $t \to +\infty$ for $j \geq 1$, for if it did not, then the difference between this solution and any other solution v_j of $L_j v_j = f_j$ would be in the nullspace of L_j, hence not in H^{s+2}_{δ}. This argument fails for $j = 0$, so u_j can be written, for $t \gg 0$, as a sum of two terms, one in $H^{s+2}_{-\delta}$ and one in W.

We still need to check that the nonlinear operator N_{g_ε} maps $H^{s+2}_{-\delta} \oplus W$ to $H^s_{-\delta}$. Clearly N_{g_ε} maps $H^{s+2}_{-\delta}$ to $H^s_{-\delta}$. To ensure that it also carries W to $H^s_{-\delta}$ we need to modify the definition of this map slightly. In fact, since elements $(a\chi\phi_0^+, b\chi\phi_0^-) \in W$ correspond, for $t \geq 1$, to the infinitesimal variations of one-parameter families of Delaunay metrics, we can define a two-parameter family of metrics $\widetilde{g}_{\varepsilon,a,b}$ on the cylinder such that for $t \leq -1$, $\widetilde{g}_{\varepsilon,a,b} = g_\varepsilon$ and for $t \geq 1$, $\widetilde{g}_{\varepsilon,a,b} = g_{\varepsilon+d_\varepsilon(b)}(t - \tau_\varepsilon(a))$. Here $d_\varepsilon : \mathbb{R} \to (-\varepsilon, \overline{u} - \varepsilon)$ and $\tau_\varepsilon : \mathbb{R} \to (-P_\varepsilon/2, P_\varepsilon/2)$ are monotone, smooth, surjective functions such that $d_\varepsilon(0) = \tau_\varepsilon(0) = 0$. The map $(a,b) \mapsto \widetilde{g}_{\varepsilon,a,b}$ induced by τ_ε and d_ε can be regarded as an exponential map to the space of Delaunay metrics on the half cylinder from the tangent plane at the point g_ε. By judicious choices of the functions d_ε and τ_ε and the definition of $\widetilde{g}_{\varepsilon,a,b}$ in $-1 < t < 1$, we can insure that if $\widetilde{g}_{\varepsilon,a,b} = \widetilde{u}^{\,4/(n-2)}_{\varepsilon,a,b} g_\varepsilon$, then

$$\chi\phi_0^+ = \chi\phi_0^+(\varepsilon) = \frac{d}{da}\widetilde{u}_{\varepsilon,a,b}\Big|_{(a,b)=(0,0)},$$

$$\chi\phi_0^- = \chi\phi_0^-(\varepsilon) = \frac{d}{db}\widetilde{u}_{\varepsilon,a,b}\Big|_{(a,b)=(0,0)}.$$

We define a new operator by

$$N^{(a,b)}_{g_\varepsilon}(\phi) = \Delta_{\widetilde{g}_{\varepsilon,a,b}}\phi - \frac{n-2}{4(n-1)}R(\widetilde{g}_{\varepsilon,a,b})(1+\phi) + \frac{n(n-2)}{4}(1+\phi)^{(n+2)/(n-2)} \tag{2.8}$$

for $\phi \in H^{s+2}_{-\delta}$ and $(a,b) \in \mathbb{R}^2$. Finally, setting

$$N_{g_\varepsilon}(\phi, a\chi\phi_0^+, b\chi\phi_0^-) \equiv N^{(a,b)}_{g_\varepsilon}(\phi), \tag{2.9}$$

we see that $N_{g_\varepsilon} : H^{s+2}_{-\delta} \oplus W \to H^s_{-\delta}$ is a well defined real analytic map.

Complete CPSC metrics on $X \setminus \Lambda$. In the last subsection we considered singular Yamabe metrics on the sphere with two points removed. Rather different solutions were constructed in [MPa1] and [MS]. These are complete CPSC metrics on $M = X \setminus \Lambda$, where Λ is a finite disjoint union of submanifolds Λ_i without boundary, (X, g_0) is compact of nonnegative scalar curvature, and $\dim \Lambda_i = k_i$ with $1 \leq k_i \leq (n-2)/2$. The upper bound on the dimensions k_i

is, by a theorem of Schoen and Yau [SY], a necessary condition. Note that we temporarily abandon the convention that g_0 is complete on $M = X \setminus \Lambda$ here.

The completeness of $g = u^{4/(n-2)} g_0$ on $M \setminus \Lambda$ necessitates that u tends to infinity rather strongly on approach to Λ. A detailed study of this singular behavior is given in [M1]. Let r denote a smooth function on M which is everywhere positive, and which agrees with the polar distance function (with respect to g_0) on a tubular neighborhood of Λ. The solutions u constructed in [MPa1] and [MS] are asymptotic, to leading order, to $Ar^{(2-n)/2}$ as $r \to 0$, with A a constant depending only on dimension. It is shown in [M1] that in a neighborhood of each component Λ_i these solutions have more refined asymptotics $u \sim Ar^{(2-n)/2}(1 + O(r^{k_i/2}))$. For convenience, in the rest of this section assume that Λ has only one component of dimension k.

The linearized scalar curvature operator relative to one of these CPSC metrics g has the form (2.3). If $k^2 \leq 4(n-2)(n-2k-2)$ (cf. [MS]), the continuous spectrum of L contains 0, hence again L does not have closed range on L^2. As before, it is appropriate to let L act on weighted Sobolev or Hölder spaces. Although both [MPa1] and [MS] use Hölder spaces, we shall use Sobolev spaces as above. Once again, $H^s_\delta(M, g)$ is defined to be the space of functions $\phi = r^{-\delta + k/2}\widetilde{\phi}$, where $\widetilde{\phi}$ is in the uniform global Sobolev space H^s on M with respect to the complete metric $r^{-2} g_0$ (or, equivalently, with respect to g). Note the change of sign and shift of weight parameter relative to the previous definition. Then it follows from the theory of [M2] that

$$L : H^{s+2}_\delta(M) \to H^s_\delta(M) \tag{2.10}$$

has closed range for all $\delta \notin \{\pm\delta_j\}$, where as before $0 = \delta_0 < \delta_1 < \ldots \to \infty$. Unlike the situation for the Delaunay metrics, (2.10) will not be Fredholm, even when it has closed range. Indeed, for $\delta < 0$ it has infinite-dimensional kernel, but at most finite-dimensional cokernel, while for $\delta > 0$ it has infinite-dimensional cokernel and at most finite-dimensional kernel.

The CPSC metrics constructed in [MS] and [MPa1] are nondegenerate in the sense that (2.10) is surjective if $0 = \delta_0 < \delta < \delta_1$ (and hence for all $\delta > 0$, $\delta \neq \delta_j$). As shown in [MS], this implies that every sufficiently small element of the nullspace of L in H^{s+2}_δ for $0 < \delta < \delta_1$ is integrable, i.e. is the tangent vector of a one-parameter family of solutions u_t. Notice that because of the shift in the weight parameter here, the space on which L is surjective contains only decaying functions, so unlike before we do not need to separate off the nullspace (or deficiency subspace) as in (2.7) to obtain a space on which the nonlinear operator N_g acts.

CPSC metrics on manifolds with boundary. Our construction also applies to CPSC manifolds with boundary, either compact or noncompact. The issue here is the mapping properties of L on Sobolev or Hölder spaces with Dirichlet boundary values. A geometrically natural boundary condition for the nonlinear problem is to require the boundary in the induced metric to have constant mean curvature. This has been studied extensively by Escobar [E] and others. Two key examples are the spherical cap S^n_r of radius r and the Delaunay metrics on the half cylinder, D^α_ε, which are simply the restrictions of the D_ε to $t \geq \alpha$. The mean curvature of the boundary is a constant depending on r and α; when $r = \pi/2$, $\alpha = 0$ the boundaries are not only minimal but totally geodesic.

The half-Delaunay metrics are all nondegenerate, as defined below. This follows from simple modifications of the previous discussion of the full Delaunay metrics. On the other hand, the spherical cap S^n_r is nondegenerate only when $r \neq \pi/2$. When $r = \pi/2$, L has a one-dimensional nullspace consisting of the linear functions vanishing on the boundary, hence by duality is not surjective.

Nondegeneracy. Having described in some detail the main examples of CPSC manifolds for which the Jacobi operators L are in some sense surjective, we now abstract these properties and formulate a general notion of nondegeneracy sufficiently flexible for the gluing construction.

Suppose (M, g) is a noncompact, complete Riemannian manifold of CPSC. The standard Sobolev spaces H^s are defined relative to the Riemannian measure and connection. We shall assume that there exists a weight function $0 < \alpha \in \mathcal{C}^\infty(M)$ the powers of which define a scale of weighted L^2 and Sobolev spaces. Thus we define

$$H^s_\delta(M) = \{v = \alpha^\delta \widetilde{v} : \widetilde{v} \in H^s(M)\}. \tag{2.11}$$

The dual of H^s_δ is naturally identified with $H^{-s}_{-\delta}$.

The main nondegeneracy hypothesis is that there exists a weight parameter $\delta > 0$ such that for all $s \in \mathbb{R}$ there exists a constant $C = C_s > 0$ for which

$$\|\phi\|_{s+2,-\delta} \leq C\|L\phi\|_{s,-\delta} \tag{2.12}$$

for every $\phi \in \mathcal{C}^\infty_0(M)$. This implies that

$$L : H^{s+2}_{-\delta} \to H^s_{-\delta} \tag{2.13}$$

is injective and has closed range. It also gives some analytic control on the behavior of L on the ends of M. By duality we see that

$$L : H^{s+2}_\delta \to H^s_\delta \tag{2.14}$$

is surjective. (It is precisely this last assertion which, though still true, would be a bit more difficult to obtain if we were using Hölder spaces.)

In some cases, such as for the problem on $M \setminus \Lambda$ where all components of Λ are of positive dimension, this is the only hypothesis needed because, for some neighborhood of zero $\mathcal{U} \subset H^{s+2}_\delta$, and for $\delta > 0$ sufficiently small,

$$N_g : \mathcal{U} \to H^s_\delta \tag{2.15}$$

is well defined and has surjective linearization. In other cases, such as for the Delaunay metrics, N_g does not map elements of H^{s+2}_δ to H^s_δ and so we need to find another space on which the linearization is surjective and on which N_g is well-behaved. Thus we assume the existence of a "deficiency space" $W \subset H^{s+2}_\delta$, composed of elements of the form $\chi\phi$, for some fixed cutoff function χ, where $\phi \in H^{s+2}_\delta$ and $L\phi = 0$ outside some compact set, such that

$$L : H^{s+2}_{-\delta} \oplus W \to H^s_{-\delta} \tag{2.16}$$

is surjective. There is no loss of generality in assuming that (2.16) is an isomorphism, because we can always restrict to the orthogonal complement of the intersection of the nullspace of L on H^{s+2}_δ with $H^{s+2}_{-\delta} \oplus W$, no element of which is contained in $H^{s+2}_{-\delta}$ by hypothesis. We also require that the elements of W are "asymptotically integrable," which we take to mean that elements of W are derivatives of one-parameter families of exact solutions of N_g outside a compact set. The validity of this condition was discussed in detail for the Delaunay metrics. Rather than formulate the asymptotic integrability more specifically, we refer to these examples and single out as the second nondegeneracy hypothesis the only consequence that we require, namely that

$$N_g : \mathcal{U} \to H^s_{-\delta} \tag{2.17}$$

is surjective onto a neighborhood of 0 with surjective linearization (2.16), where $\mathcal{U}$ is a neighborhood of the origin in $H^{s+2}_{-\delta} \oplus W$.

It turns out that this second condition is rather less general than it might appear. In fact, it is not hard to show that (2.16) implies that W must be finite-dimensional. For if it were not, then one could construct an orthonormal sequence $\{\chi\phi_j\} \subset W$, of fixed H^{s+2}_δ norm one, which decays to zero uniformly on any compact set. This would contradict the closedness of the range of (2.16).

We say that a metric g is *nondegenerate* if the linearization at 1 of the nonlinear operator N_g for g is nondegenerate, in the sense that (2.12) and either (2.15) or (2.16)–(2.17) hold. Note that in the two main examples indicated above nondegeneracy holds provided that L has no kernel in L^2. The analytic control at infinity which improves this to (2.12) and (2.16) is provided by the strong asymptotics which these solutions exhibit (cf. [MPU], [M1] and [M2]).

III. The gluing construction

Our aim in this section is to state and prove the gluing theorems for manifolds with nondegenerate CPSC metrics. Thus let (M_1, g_1) and (M_2, g_2) be two complete CPSC manifolds (possibly with boundary). In the next subsection we will construct a one-parameter family of approximate solution metrics g_ε on the connected sum $M_1 \#_\varepsilon M_2$, where the parameter ε corresponds to the size of the connecting neck and is assumed to be small. The approximate solution metric g_ε has CPSC except on a neighborhood of this neck.

(3.1) Theorem. *Suppose that (M_1, g_1) and (M_2, g_2) are two nondegenerate CPSC manifolds. Then for some $\varepsilon_0 > 0$ and all $0 < \varepsilon < \varepsilon_0$, there exists a function $u \in H^{s+2}_{-\delta}(M_1 \#_\varepsilon M_2, g_\varepsilon)$ such that $\overline{g}_\varepsilon = (1+u)^{4/(n-2)} g_\varepsilon$ is nondegenerate with CPSC.*

In the next section we will make more refined statements about the global geometry of these new metrics and the implications of this construction for the moduli spaces.

In the rest of this section we shall prove Theorem (3.1). The proof has several steps. We first construct the approximate solution metrics g_ε. In the next two steps, which are the heart of the proof, we show that the g_ε are nondegenerate and that (right) inverses for the linearized scalar curvature operators are uniformly bounded as $\varepsilon \to 0$. The rather simple indirect method used here is the main novel ingredient in this paper. Finally, using this nondegeneracy, we perturb g_ε to an exact solution using a standard iteration argument.

Approximate solutions. Let (M_i, g_i), $i = 1, 2$, be two nondegenerate complete CPSC manifolds. Fix points $p_i \in M_i$ and small metric balls $B_{2\alpha_i}(p_i)$. Let (r_i, θ_i) be Riemannian polar coordinates about p_i. Then for each $\varepsilon \in (0, 1)$ identify the annulus $B_{\alpha_1}(p_1) \setminus B_{\varepsilon\alpha_1}(p_1)$ with $B_{\alpha_2}(p_2) \setminus B_{\varepsilon\alpha_2}(p_2)$ by the relation $(r_1, \theta_1) \sim (r_2, \theta_2)$ if $\theta_1 = \theta_2$ and $r_1 r_2 = \varepsilon \alpha_1 \alpha_2$. This is the connected sum $M_\varepsilon \equiv M_1 \#_\varepsilon M_2$; the points p_i and radii α_i are suppressed in this notation, although both metrically and conformally this data is important.

We first consider the case where the metrics g_i are conformally flat in the balls $B_{2\alpha_i}(p_i)$. In this case the analysis is the most transparent. It is useful to rephrase the problem relative to new, conformally equivalent, background metrics $g_{i,c}$ on $M_i \setminus \{p_i\}$. Geometrically we deform the conformally flat metrics g_i in $B_{\alpha_i}(p_i)$ to half-infinite cylinders. The connected sum M_ε is then given by identifying these cylinders at a certain distance. The metric degeneration of M_ε as $\varepsilon \to 0$ now corresponds to the lengthening of this cylindrical tube. More specifically, by (temporary) hypothesis, $g_i = v_i^{4/(n-2)} \delta$ in $B_{2\alpha_i}(p_i)$, where δ is the standard Euclidean metric. We set $g_{i,c} = u_i^{-4/(n-2)} g_i$, where

$$u_i = \varrho_i + (1-\varrho_i)((n-2)/n)^{(2-n)/4} r^{(n-2)/2} v_i,$$

for some smooth cutoff function $\varrho_i \geq 0$ with $\varrho_i = 1$ on $M_i \setminus B_{2\alpha_i}(p_i)$ and $\varrho_i = 0$ in $B_{\alpha_i}(p_i)$. Then in $B_{\alpha_i}(p_i) \setminus \{p_i\}$, $g_{i,c}$ is isometric to the cylindrical metric $\frac{n-2}{n}(dt_i^2 + d\theta_i^2)$. The normalizing constant is chosen so that this cylinder has scalar curvature $R = n(n-1)$.

In this smaller ball replace the variable r_i by $t_i = -\log r_i$, and let $T = T(\varepsilon) = -\log \varepsilon$. Then the identification between the two annular regions is given by $(t_1, \theta_1) \sim (t_2, \theta_2)$ if $\theta_1 = \theta_2$ and $t_1 + t_2 = A_1 + A_2 + T$, where $A_i = -\log \alpha_i$ and $A_i \leq t_i \leq A_i + T$. We now alternately denote the connected sum M_ε by M_T. Because we have assumed that the g_i are conformally flat on $B_{2a_i}(p_i)$, this identification map is an isometry with respect to the metrics $g_{i,c}$, hence there is a naturally induced metric $g_{c,T}$ on M_T. We let C_T denote the cylindrical region where $A_i \leq t_i \leq A_i + T$.

The approximate solution metric g_T on M_T is defined in terms of the conformal factor

$$u_T = \chi_1 u_1 + \chi_2 u_2,$$

using nonnegative cutoff functions $\{\chi_1, \chi_2\}$ on C_T, where $\chi_i \equiv 1$ for $t_i \leq A_i + T/2 - 1$ and $\chi_i = 0$ for $t_i \geq A_i + T/2 + 1$ (here we regard χ_i as a function on M_i). Then u_T extends naturally to all of M_T, and we define the approximate solution metric by

$$g_T = u_T^{4/(n-2)} g_{c,T}. \tag{3.2}$$

Note that $g_T = g_i$ on $M_i \setminus B_{c(\varepsilon)\alpha_i}(p_i)$, where $c(\varepsilon) = c\sqrt{\varepsilon}$.

If the metrics g_i are not conformally flat in the balls $B_{2\alpha_i}(p_i)$, the construction of g_T is almost identical, but is no longer conformally natural, i.e. the conformal class of g_T depends on choices of cutoff functions as well as T. In $B_{2\alpha_i}(p_i)$ we can choose a normal coordinate system in terms of which $g_i = \delta + h_i$, where h_i is small in some fixed norm. In $B_{2\alpha_i}(p_i) \setminus B_{\alpha_i}(p_i)$ we deform h_i to zero and simultaneously deform δ to $((n-2)/n)r^{-2}\delta$. These metrics may now be joined as before.

We define the unweighted Sobolev spaces H^s with respect to the metrics $g_{c,T}$; of course, the norms on these spaces depend on T, although this effect may be localized to C_T. We may assume that the weight functions α_i on M_i are identically one in a large neighborhood of the points p_i; these extend naturally over C_T and we may define a new weight function α on M_T. Using it, we define the weighted Sobolev spaces H^s_δ on the connected sum.

The metric g_T does not have CPSC, and we can easily estimate the error term

$$f_T = \Delta_{g_{c,T}} u_T - \frac{n-2}{4(n-2)} R(g_{c,T}) u_T + \frac{n(n-2)}{4} u_T^{(n+2)/(n-2)}. \tag{3.3}$$

In fact, (in the conformally flat case) g_T does have scalar curvature $n(n-1)$ except in the middle of C_T, where $t_i \in [A_i + T/2 - 1, A_i + T/2 + 1]$. Since $u_i = O(r_i^{(n-2)/2})$ and $r_i = e^{-t_i}$, it is clear that

$$\|f_T\|_s \leq Ce^{-T/2}$$

for any s. We do not need to use a weighted norm here because f_T is supported on C_T where $\alpha = 1$. In the non-conformally flat case, there is an additional error term incurred by cutting off h_i. By choosing the normal coordinates correctly, we can also bound this extra error term, which is now supported near the boundary of C_T, by $Ce^{-T/2}$.

Nondegeneracy of the approximate solution. In order to be able to perturb g_T to a CPSC metric, we need to establish nondegeneracy of the linearization of the scalar curvature operator for $g_{c,T}$ at u_T. Although the definitions (2.12) and (2.15)–(2.17) of nondegeneracy were given only for CPSC metrics, these hypotheses make perfect sense here.

(3.4) Proposition. *There exists a $T_0 > 0$ such that for all $T \geq T_0$, the metric g_T of* (3.2) *is nondegenerate.*

Before embarking on the proof, we make some preliminary observations about the linearizations of the scalar curvature operator on M_1, M_2 and M_T. For any metric g, the conformal Laplacian

$$\mathcal{L}_g = \Delta_g - \frac{n-2}{4(n-1)} R(g_0)$$

is the linear part of N_g in (2.1). It is conformally equivariant in the sense that if $g' = u^{4/(n-2)}g$, then for any $\phi \in \mathcal{C}^\infty$,

$$\mathcal{L}_g(u\phi) = u^{(n+2)/(n-2)}\mathcal{L}_{g'}\phi. \tag{3.5}$$

A special case of this equality is when $\phi = 1$, in which case (3.5) reduces to (2.1).

Next, suppose $g' = u^{4/(n-2)}g$, and let L_g be the linearization of N_g at u. The relationship between L_g and $L_{g'}$ is not as simple as (3.5) in general, but it is when both g and g' have the same (constant) scalar curvature. Indeed, if $R(g) = R(g') = n(n-1)$, then $\mathcal{L} = \Delta - n(n-2)/4$ for g or g'. Since $L_g = \mathcal{L}_g + (n(n+2)/4)u^{4/(n-2)}$, we have

$$\begin{aligned} L_g(u\phi) &= \mathcal{L}_g(u\phi) + \frac{n(n+2)}{4}u^{4/(n-2)}(u\phi) \\ &= u^{(n+2)/(n-2)}\left(\mathcal{L}_{g'}\phi + \frac{n(n+2)}{4}\phi\right) = u^{(n+2)/(n-2)}L_{g'}\phi. \end{aligned} \tag{3.6}$$

In particular, away from the transition regions $B_{2\alpha_i}(p_i) \setminus B_{\alpha_i}(p_i)$, (3.6) applies to either of the two pairs of metrics g_i, $g_{i,c}$, with $u = u_i$. The linearizations corresponding to these two metrics will be denoted by L_i and $L_{i,c}$.

(3.7) Lemma. *Suppose $L_{i,c}\phi = 0$ for some function ϕ on $M_i \setminus \{p_i\}$, and suppose that ϕ is bounded on the deleted neighborhood $B_{\alpha_i}(p_i) \setminus \{p_i\}$. Then $u_i^{-1}\phi$ extends smoothly across p_i on M_i, and in particular, $|\phi| \le Ce^{-(n-2)t_i/2} = Cr_i^{(n-2)/2}$ on this neighborhood.*

Proof. By (3.6),

$$u_i^{(n+2)/(n-2)} L_i(u_i^{-1}\phi) = L_{i,c}\phi = 0,$$

and so, letting $\psi = u_i^{-1}\phi$, we see that $L_i\psi = 0$ on $B_{\alpha_i}(p_i) \setminus \{p_i\}$. Since ϕ is bounded, $|\psi| \le Cr^{(2-n)/2}$. Thus ψ extends to a weak solution of $L_i\psi = 0$ on all of M_i, and by a standard removable singularities theorem, extends smoothly across p_i. □

One other result is needed. Let L_T denote the linearization of $N_{g_{c,T}}$ at u_T.

(3.8) Lemma. *Suppose that ϕ solves $L_T\phi = 0$ on the cylindrical region C_T, and furthermore suppose that $\|\phi\|_{L^2(A)} \le 1$, where A is the union of two annular neighborhoods, one about each of the boundary components of C_T. Then $|\phi| \le C$ on all of C_T, where C is independent of T.*

Proof. Since $g_{c,T}$ is a product metric in C_T,

$$L_T = \frac{n}{n-2}\left(\frac{\partial^2}{\partial t^2} + \Delta_\theta\right) - \frac{n(n-2)}{4} + \frac{n(n+2)}{4} u_T^{(n+2)/(n-2)}$$

there. Provided we adjust the annular region appropriately, we can ensure that the term of order zero in this operator is strictly negative. The result then follows from the maximum principle, and in fact we can take $C = 1$. □

It is not strictly necessary that the term of order zero is negative on the whole cylinder; it is only necessary that it is nonnegative on a compact set not growing in size as T gets large. We leave details to the reader.

Proof of Proposition (3.4). We first show that the linearization L_T of $N_{g_{c,T}}$ at u_T is injective on $H^{s+2}_{-\delta}$ for any s, and we argue by contradiction. Suppose that there exists a sequence $T_j \to \infty$ and a function $\phi_j \in H^{s+2}_{-\delta}(M_{T_j})$ such that $L_{T_j}\phi_j = 0$. The weight $-\delta$ of course refers to growth behavior on any other ends of M_1 and M_2. Choose compact neighborhoods $K_i \subset M_i \setminus \{p_i\}$ containing $\partial B_{\alpha_i}(p_i)$ and normalize ϕ_j so that

$$\max_{i=1,2}\{\|\phi_j\|_{H^{s+2}(K_1)}, \|\phi_j\|_{H^{s+2}(K_2)}\} = 1.$$

In particular, on one or the other of these sets, ϕ_j has Sobolev norm uniformly bounded below; by passing to a subsequence we can assume that this takes place on K_1. Since we can assume that $s > n/2$, by elliptic regularity, we also get uniform supremum bounds for ϕ_j on K_1 and K_2.

We can now take the limit as $j \to \infty$ to obtain a limit ϕ, which is a function on the disjoint union $M_1 \setminus \{p_1\} \sqcup M_2 \setminus \{p_2\}$. By the uniform lower bound, ϕ is nontrivial on K_1 and solves $L_{1,c}\phi = 0$ there. Using Lemma (3.8) we see that ϕ is bounded along the cylindrical end of $M_1 \setminus \{p_1\}$, hence, by Lemma (3.7), the function $\psi = u_1^{-1}\phi$ extends smoothly to all of M_1 and satisfies $L_1\psi = 0$. It is also the case that $\psi \in H^{s+2}_{-\delta}(M_1)$. To see this let χ be a cutoff function equaling 1 outside $B_{2\alpha_1}(p_1)$, vanishing near p_1, and with the support of $\nabla\chi$ in K_1. Then

$$L_T(\chi\phi_j) = L_{1,c}(\chi\phi_j) = (\Delta\chi)\phi_j + 2\nabla\chi \cdot \nabla\phi_j.$$

The right hand side is compactly supported and uniformly bounded in $H^s_{-\delta}(M_1)$. By (2.12) we obtain a uniform bound for $\chi\phi_j$ in $H^{s+2}_{-\delta}$, and since $u_1 = 1$ outside $B_{2\alpha_1}(p_1)$ it is easy to see that $\psi \in H^{s+2}_{-\delta}(M_1)$, as claimed. This is a contradiction, since ψ is nontrivial and L_1 satisfies (2.12). This proves the injectivity of L_T for T sufficiently large. Now we may patch together the estimates (2.12) from M_1 and M_2 to obtain, for each T,

$$\|\phi\|_{s+2,-\delta} \leq C(\|L_T\phi\|_{s,-\delta} + \|\phi\|_{0,K}),$$

where the final term is the L^2 norm of ϕ on a fixed compact set K. Finally, it is standard that the injectivity of L_T shows that this final term must be bounded by a fixed constant multiple (possibly depending on T) of $\|L_T\phi\|_{s+2,-\delta}$, so we have shown that (M_T, g_T) satisfies (2.12) for T sufficiently large.

If we are in a case where (2.15) holds, then we have proved that g_T is nondegenerate already. Thus suppose that either M_1 or M_2, or both, have deficiency spaces W_1 and W_2 satisfying (2.16). Let $W = W_1 \oplus W_2$. We must show that L_T satisfies (2.16) for T sufficiently large. For clarity, we denote the restricted (or extended, depending on your viewpoint) map in (2.16) by $\widetilde{L}_T$. We shall verify that $\widetilde{L}_T$ is surjective by computing its adjoint $\widetilde{L}_T^*$ and showing that it has no nullspace. Note that $\widetilde{L}_T$ obviously has closed range because it is a finite-dimensional extension of an operator with closed range.

We formulate this somewhat more generally in the

(3.9) LEMMA. *Assuming the general setup of* §2 *above, the map* L *in* (2.16) *is surjective if and only if for every* $\phi \in \mathcal{B} \equiv \ker(L) \cap H^s_\delta$*, the linear functional on* W *defined by*

$$\int (Lw)\phi \tag{3.10}$$

for $w \in W$ *is not identically zero.*

PROOF. The surjectivity of the operator in (2.16), which we denote by $\widetilde{L}$ temporarily, is equivalent to the injectivity of its adjoint $\widetilde{L}^*$. We first compute this adjoint. If $(v,w) \in H^{s+2}_{-\delta} \oplus W$ and $f \in H^s_{-\delta}$, then $\widetilde{L}^*$ is defined by

$$\int \widetilde{L}(v+w) f \alpha^{2\delta} = \int v(\alpha^{-2\delta} L \alpha^{2\delta} f)\alpha^{2\delta} + \int L(w)(\alpha^{2\delta} f).$$

Note that we cannot integrate the second term by parts because w does not decay, even though Lw is compactly supported. Now multiplication by $\alpha^{2\delta}$ defines the natural isomorphism between $H^s_{-\delta}$ and H^s_δ; since L is self-adjoint when $\delta = 0$ we see that $\alpha^{-2\delta} L \alpha^{2\delta}$ is canonically identified with L on H^s_δ.

Now if f is in the nullspace of $\widetilde{L}^*$, then setting $w = 0$ and letting v range over all of $H^{s+2}_{-\delta}$ we see that $\alpha^{-2\delta} L \alpha^{2\delta} f = 0$. Equivalently, $\phi = \alpha^{2\delta} f$ is in the nullspace of L in H^s_δ. Now letting $v = 0$ we see that ϕ is orthogonal (in unweighted L^2) to Lw for every $w \in W$.

On the other hand, if $\phi \in H^s_\delta$ with $L\phi = 0$ and $\int (Lw)\phi = 0$ for every $w \in W$ then $f = \alpha^{-2\delta}\phi$ is in the nullspace of $\widetilde{L}^*$. $\square$

We return to the proof of Proposition (3.4). We need to show that $\widetilde{L}^*_T$ is injective for T sufficiently large. As usual, assume not, so that there exists a sequence T_j tending to infinity and corresponding elements ϕ_j in the nullspace $\mathcal{B}$ such that $\int \phi_j L_T w = 0$ for all $w \in W$. (Note that the space W is independent of T because its elements are supported away from C_T.) Normalize the sequence as before, so that its norm on one of the two compact sets $K_i \subset M_i$ is one. By the finite-dimensionality of W the restriction (of a subsequence) of the ϕ_j converges on $M_1 \setminus \{p_1\}$, say, to a nontrivial element ϕ. As before, $\psi = u_1^{-1}\phi$ extends smoothly to all of M_1 and solves $L_1\psi = 0$. This is a contradiction since $\int \psi L_1 w_1 = 0$ for all $w_1 \in W_1$ implies by Lemma (3.9) and the surjectivity of L_1 in (2.16) on M_1 that $\psi = 0$. $\square$

Uniform surjectivity of L_T. By Proposition (3.4) and duality, L_T is surjective on $H^s_\delta(M_T)$ for all $T \geq T_0$. This means that there exists some right inverse

$$G_T : H^s_\delta(M_T) \to H^{s+2}_\delta(M_T). \tag{3.11}$$

Because L_T is not injective on H^{s+2}_δ, there are many choices for the map (3.11). The canonical choice is the one which has range agreeing with the range of the adjoint map L^*_T. Henceforth we assume that the map (3.11) satisfies this condition.

To understand this choice better, note that by what we have proved, $L_T L^*_T$ is an isomorphism, hence has a unique inverse $\mathcal{G}_T$. Since $L_T L^*_T \mathcal{G}_T = I$, we see that G_T must be $L^*_T \mathcal{G}_T$ because both are right inverses of L_T with range contained in the range of L^*_T. We also note that by the same formalism as discussed in the last subsection using the weight function α, we may identify this adjoint L^*_T with $\alpha^{2\delta} L_T \alpha^{-2\delta}$.

Now we may restrict G_T to $H^s_{-\delta}$. Unfortunately, its range may not coincide with $H^{s+2}_{-\delta} \oplus W$. Thus for a given $f \in H^s_{-\delta}$ we have found two possibly distinct solutions of $Lu = f$, namely the solution $v + w \in H^{s+2}_{-\delta} \oplus W$ and $G_T f$. The difference between these solutions is an element $\phi \in \mathcal{B}$, the nullspace of L_T in H^{s+2}_{δ}. Our ultimate goal is to show that $\|v + w\|$ is bounded by a multiple of $\|f\|$, uniformly in T. We do this in two steps, first showing that the norm of G_T, and then the correction term ϕ, are uniformly bounded.

(3.12) Proposition. *The norm of the map G_T in* (3.11) *is uniformly bounded as $T \to \infty$.*

Proof. The proof, once again, is indirect. Thus we assume that the result is false, so that for some sequence $T_j \to \infty$ there are functions $f_j \in H^s_\delta$ with $\|f_j\|_{s,\delta} \to 0$ such that $\|G_{T_j} f_j\|_{s+2,\delta} = 1$. Since each $G_{T_j} f_j = \psi_j$ is in the range of $L^*_{T_j}$, there exist functions $v_j \in H^{s+4}_\delta$ with $L^*_{T_j} v_j = \psi_j$. Because $\|\psi_j\| = 1$ and because of the boundedness of the $L^*_{T_j}$ on Sobolev spaces, we know that

$$\|v_j\|_{s+4,\delta} \geq C.$$

Our goal is to show that some subsequence of the v_j converges to a function $v \in H^{s+4}_\delta$ which is nontrivial on at least one of M_1 or M_2. Suppose for definiteness that $v \neq 0$ on M_1. Because of the boundedness of $L^*_{T_j}$ on these Sobolev spaces, we can also assume that ψ_j converges to $\psi \in H^{s+2}_\delta$, where $L^*_{1,c} v = \psi$; also $\|f_j\|_{s,\delta} \to 0$, so that $L_{1,c}\psi = 0$ and hence $L_{1,c} L^*_{1,c} v = 0$. Using Lemma (3.7) in the same way as before we see that $u_1^{-1}\psi \equiv \phi$ is smooth on M_1, so that $|\psi| \leq Ce^{(2-n)t/2}$ on the cylindrical end of M_1. This allows us to integrate by parts to conclude that

$$0 = \langle v, L_{1,c} L^*_{1,c} v\rangle = \langle \psi, \psi\rangle$$

and so $\psi = 0$. But now $L^*_{1,c} v = 0$, or equivalently, $L_1^*(u_1^{-1} v) = 0$. But $u_1^{-1} v \in H^{s+4}_\delta(M_1)$ and we have already established that L_1^* is injective on this space (or rather, we established that L_1 is injective on $H^{s+4}_{-\delta}$, which is the same). This is a contradiction, hence the v_j cannot converge as claimed, and the maps G_T must be uniformly bounded.

To finish the proof we must show that the v_j converge in H^{s+4}_δ. First we transform the problem so that we may work in $H^s_{-\delta}$ and so avail ourselves of the estimate (2.12). Let $\widetilde{v}_j = \alpha^{-2\delta} v_j \in H^{s+4}_{-\delta}$, and define $\widetilde{\psi}_j$ similarly. Then $L_{T_j}\widetilde{v}_j = \widetilde{\psi}_j$. Let χ be a smooth, nonnegative cutoff function on either M_1 or M_2 which equals one outside $B_{2\alpha_i}(p_i)$ and zero inside $B_{\alpha_i}(p_i)$. By computing $L_{T_j}(\chi\widetilde{v}_j)$ we see that

$$\|\chi\widetilde{v}_j\|_{s+4,-\delta} \leq C(\|\chi\widetilde{\psi}_j\|_{s+2,-\delta} + \|\widetilde{v}_j\|_{s+3,K}),$$

where K is some compact set containing the support of $\nabla\chi$ and $\|\cdot\|_{s,K}$ is the Sobolev H^s norm on K.

We now show that $\|\widetilde{v}_j\|_{s+3,K}$ is bounded away from zero and infinity as $j\to\infty$. The upper bound will imply, by (2.12), that $\|\widetilde{v}_j\|_{s+4,-\delta}\leq C$. The lower bound will imply that $\widetilde{v}_j$ does not converge to zero on K. This will show that $\widetilde{v}_j$, hence also v_j, must converge to a nonzero function, which we know from above cannot happen.

There are two cases. Suppose first that $\|\widetilde{v}_j\|_{s+3,K}\to\infty$. Rescale $\widetilde{v}_j$ and $\widetilde{\psi}_j$ by the factor $\|\widetilde{v}_j\|^{-1}_{s+3,K}$ to obtain functions $\overline{v}_j$, $\overline{\psi}_j$ with $L_{T_j}\overline{v}_j=\overline{\psi}_j$, and $\|\overline{v}_j\|_{s+3,K}=1$, $\|\overline{\psi}_j\|_{s+2,-\delta}\to 0$. Since $\overline{v}_j$ has fixed norm on a fixed compact set, $\|\chi\overline{v}_j\|_{s+4,-\delta}\leq C$ by (2.12), so we may pass to a limit and obtain a function $\overline{v}\in H^{s+4}_{-\delta}$ with $L_{j,c}\overline{v}=0$ for $j=1,2$. Furthermore, the restriction of $\overline{v}$ to at least one of the M_i, say M_1, is nonzero. The boundedness of $\overline{v}_j$ on K, which contains a neighborhood of the boundary of C_{T_j}, implies by Lemma (3.8) that $\overline{v}_j$, hence $\overline{v}$ too, are uniformly bounded on C_{T_j}. As before, this shows that $\overline{w}=u_1^{-1}\overline{v}\in H^{s+4}_{-\delta}(M_1)$. But $L_1\overline{w}=0$, which is a contradiction since $\overline{w}\neq 0$.

The other case is that $\|\widetilde{v}_j\|_{s+3,K}\to 0$. Use the same cutoff function χ as above (say on M_1) to compute that

$$L_{T_j}L^*_{T_j}(\chi v_j)=\chi f_j+[L_{T_j}L^*_{T_j},\chi]v_j\equiv h_j. \tag{3.13}$$

By hypothesis, both terms on the right tend to zero in H^s_δ. But $L_{T_j}=L_1$ on the support of χ, and if the v_j are not convergent in H^{s+4}_δ, then from (3.12) we see that $L_1L_1^*$ has closed range. But this is a contradiction, since L_1^* has closed range and, for any operator A, AA^* has closed range if and only if A does.

This completes the proof of Proposition (3.12). $\square$

The second part of the uniform surjectivity is the

(3.14) Proposition. *Suppose that* $f\in H^s_{-\delta}$ *and let* $v+w$ *be the* (*unique*) *solution of* $L_T(v+w)=f$ *in* $H^{s+2}_{-\delta}\oplus W$. *Then the function* ϕ *in the nullspace of* L_T *in* H^{s+2}_δ *defined by* $\phi=(v+w)-G_Tf$ *satisfies* $\|\phi\|\leq C\|f\|_{s,-\delta}$.

We have not specified in which space the norm of ϕ is to be taken. However, since $\mathcal{B}$ is finite-dimensional, all choices are equivalent. To be definite, we take it as the L^2 norm over a compact set $K=K_1\cup K_2$ where $K_i\subset M_i\setminus B_{\alpha_i}(p_i)$.

Proof. Again suppose not, so that for some sequence T_j tending to infinity there is an element $f_j\in H^s_{-\delta}$ with $\|f_j\|_{s,-\delta}\to 0$ such that the corresponding ϕ_j satisfies $\|\phi_j\|=1$. Then we may take a limit and get a nontrivial function ϕ on $M_1\setminus\{p_1\}$ which is bounded on the cylindrical end and satisfies $L_{1,c}\phi=0$. Then, as before, $\psi=u_1^{-1}\phi$ is smooth across p_1 and solves $L_1\psi=0$. But since $\phi_j=v_j+w_j-G_{T_j}f_j$ and $\|f_j\|$ tends to zero, we see that $\psi\in H^{s+2}_{-\delta}\oplus W_1$, which is a contradiction since we assumed that L_1 has no nullspace here. $\square$

Proof of Theorem (3.1). It is now a relatively simple matter to complete the proof of the gluing theorem. In fact, using either (2.15) or (2.17) as appropriate, the nonlinear step is trivial. Recall that we wish to find a small function $v \in H^{s+2}_{\delta}$ or pair $v = (\widetilde{v}, w)$ near zero in $H^{s+2}_{-\delta} \oplus W$ such that

$$N_{g_{c,T}}(u_T + v) = 0. \tag{3.15}$$

We shall suppose that we are in the case where (2.17) applies, to be definite. If the w component were trivial, this would be equivalent to requiring that $(1+\widetilde{v})^{4/(n-2)} g_T$ have CPSC, but in general w cannot be treated as a simple conformal factor (cf. (2.9)). In general, (3.15) is equivalent to the condition that $(1+\widetilde{v})^{4/(n-2)} g^w_T$ has CPSC $n(n-1)$, where g^w_T is defined in analogous fashion to $\widetilde{g}_{\varepsilon,a,b}$ in §2.

Denote the left side of (3.15) by $N(v)$. Expanding in a Taylor series shows that

$$N(v) = f_T + L_T v + Q_T(v),$$

where f_T is the error term (3.3), L_T is the linearization of N at u_T, hence acts by $L_T(v) = L_T(\widetilde{v} + w)$, and Q_T is a quadratically small remainder term in v, which depends uniformly on u_T, hence on T. The inverse $\widetilde{G}_T$ of $\widetilde{L}_T$ is an isomorphism from $H^s_{-\delta}$ to $H^{s+2}_{-\delta} \oplus W$, and so we can rewrite (3.15) now as

$$v = -G_T(f_T + Q_T(v)), \quad v \in V. \tag{3.16}$$

It is now standard to solve (3.16), for example by showing that the map

$$\eta \to \mathcal{T}(\eta) = -G_T(f_T + Q_T(\eta))$$

is a contraction on a ball of radius σ about 0 in $H^{s+2}_{-\delta} \oplus W$, hence has a unique fixed point. In fact, by taking T large enough we have $\|f_T\| \le Ce^{-T/2} \le \sigma/(2A)$ and $\|Q_T(\eta)\| \le C\sigma^2 \le \sigma/(2A)$ for $\|\eta\| \le \sigma$, where A is a uniform bound for the norm of G_T. Thus $\mathcal{T}$ maps the ball of radius σ into itself. Furthermore,

$$\|\mathcal{T}(\eta_1) - \mathcal{T}(\eta_2)\| \le A\|Q_T(\eta_1) - Q_T(\eta_2)\| \le 2A\sigma\|\eta_1 - \eta_2\|,$$

hence if $\sigma < 1/(4A)$, $\mathcal{T}$ is a contraction mapping and there is a unique $v \in V$ satisfying (3.16) and (3.15).

The only remaining fact to check is that the solution metric g obtained here is nondegenerate. By the locality of W, (2.16) and (2.17) are immediate. (2.12) can be proved by contradiction. It suffices to prove that the linearization of N at g is injective on $H^s_{-\delta}$ provided T is sufficiently large. The proof of this is identical to that of Proposition (3.5). This completes the proof of Theorem (3.1). $\square$

We remark that the solution v of (3.15) and (3.16) is unique in $H^{s+2}_{-\delta} \oplus W$, but it is not the unique solution near zero in H^{s+2}_{δ}. In fact, since we are working orthogonal to $\mathcal{B}\cap(H^{s+2}_{-\delta}\oplus W)$, where $\mathcal{B}\equiv\ker(L)\cap H^s_\delta$, we see that the solutions in H^{s+2}_δ are parametrized, as in [MPU], by the elements of $\mathcal{B}$.

IV. Dipole metrics and applications to the moduli spaces

In this section we wish to discuss the ramifications of Theorem (3.1) in the special case where the component manifolds (M_i, g_i) are conformally cylinders with Delaunay metrics. As established in §2, these solutions are nondegenerate, and the deficiency subspace W may be localized to one end of each cylinder. Furthermore, it is clear that this gluing procedure can be iterated, because the glued solutions are again nondegenerate. This leads us to

(4.1) Theorem. *Let (M_i, g_{ε_i}), $i=1,2,\dots,$ be any sequence of Delaunay manifolds. Choose points p_i^+, p_i^- and sufficiently small balls $B_{\alpha_i^\pm}(p_i^\pm)$ on each M_i. Then for each $N\geq 2$, and for gluing parameters η_j sufficiently small, there is a nondegenerate metric $g^{(N)}(\eta)$ with CPSC on the iterated connected sum*

$$M^{(N)}(\eta)\equiv M_1\,\#_{\eta_1}\,M_2\,\#_{\eta_2}\dots\#_{\eta_{N-1}}\,M_N,$$

obtained by gluing the neighborhood $B_{\alpha_i^+}(p_i^+)$ to $B_{\alpha_{i+1}^-}(p_{i+1}^-)$, which is near to the connected sum metric. This metric may be obtained inductively, by gluing (M_N, g_{ε_N}) to $(M^{(N-1)}(\eta'), g^{(N-1)}(\eta'))$, where $\eta'=(\eta_1,\dots,\eta_{N-1})$. Furthermore, if the sequence $\{\eta_j\}$ tends to zero rapidly enough, then the Riemannian manifolds $(M^{(N)}(\eta), g^{(N)}(\eta))$ converge on compact sets to a manifold M with infinitely generated homology group H^{n-1}, and a Riemannian metric g on M of CPSC.

Proof. The only statement that needs proving is the convergence as $N\to\infty$. By the Harnack inequality, it suffices to show that in this iterative process, the norm of the conformal factor does not blow up or degenerate on some fixed compact set $K\subset M_1$. But in the gluing procedure we have shown that we can make the conformal factor as small as desired on the set K by choosing T sufficiently large. Thus, when gluing M_N onto the previously constructed connected sum, we choose T_N so that the conformal factor on K is no larger than 2^{-N-2}. The net change of all the conformal factors is then no larger than $1/2$, hence the procedure converges. $\square$

In [MPU] we studied the moduli space $\mathcal{M}_\Lambda$ of all CPSC metrics g on the complement of a fixed singular set Λ in S^n. The basic result is that $\mathcal{M}_\Lambda$ is a real analytic set. As such, it is endowed with a real analytic stratification into smooth analytic varieties, but without further information it is impossible to control the dimensions of these strata or of the whole moduli space. In particular, it is conceivable that $\mathcal{M}_\Lambda$ could be a single point. However, if one can prove that some $g\in\mathcal{M}_\Lambda$ is nondegenerate, then a neighborhood of any nondegenerate solution

g is a real analytic manifold of dimension $\#\Lambda$. This shows that the connected component of $\mathcal{M}_\Lambda$ containing g is of this dimension, and this nondegenerate solution is in the principal stratum. Unfortunately, it was not at all clear that any of those solutions previously known to exist, i.e. those constructed by Schoen, are nondegenerate. The original motivation for this paper was to construct nondegenerate solutions, at least for certain singular sets Λ, and this has now been accomplished. Thus we have proved

(4.2) Corollary. *Let $(M^{(N)}, g^{(N)})$ be one of the solutions obtained in Theorem* (4.1) *by gluing together N cylinders with their Delaunay metrics. This manifold is conformally flat and may be uniformized as a domain in the sphere $\Omega = S^n \setminus \Lambda$, for some set Λ with $2N$ elements. The resulting conformally flat metric $g = g^{(N)}$ is a nondegenerate element of the moduli space $\mathcal{M}_\Lambda$. The component of $\mathcal{M}_\Lambda$ containing g is a $2N$-dimensional real analytic set and g is in the principal, top-dimensional, stratum.*

The singular sets Λ we obtain by this gluing procedure are very special: they contain N pairs of closely spaced points, with the distance between each pair bounded from below. We call these *dipole configurations*. In the next subsection we indicate how we may use the nondegeneracy of dipole metrics to deduce nondegeneracy of elements in other moduli spaces $\mathcal{M}_\Lambda$, where Λ is not necessarily a dipole configuration.

We end this subsection by observing that by the construction we can prescribe the geometry of half of the ends of a dipole solution. This follows immediately by observing that the elements in the parameter space W are supported on only half (i.e. $\{(t_j, \theta_j) : t_j > -1\}$) of each cylinder.

(4.3) Corollary. *Let g be a dipole metric constructed by gluing together the N Delaunay solutions $D_{\varepsilon_1}, \dots, D_{\varepsilon_N}$. Let the singular set $\Lambda \subset S^n$ be written as a set of pairs $\{(x_1, y_1), \dots, (x_N, y_N)\}$. Then the asymptotic Delaunay parameter for g at x_j is exactly ε_j and the Delaunay parameter at y_j is very close to ε_j. In particular, since we can prescribe the Delaunay parameters at x_j, by choosing $\varepsilon_j = \overline{u}$ for $j = 1, \dots, N$, there exist dipole solutions with N asymptotically cylindrical ends.*

This shows that the dipole solutions are geometrically very different than the solutions constructed by Schoen, where the Delaunay parameters at each singular point are very close to zero.

The unmarked moduli space $\mathbb{M}_k$. As a final topic, and to further elucidate the impact of the dipole solutions on the moduli space theory, we shall consider the unmarked moduli space

$$\mathbb{M}_k = \{(g, \Lambda) : \Lambda \in \mathcal{C}_k \text{ and } g \in \mathcal{M}_\Lambda\}$$

of conformally flat CPSC metrics on the complement of any k points in the sphere. Here $\mathcal{C}_k$ is the configuration space of k distinct points in S^n and $\mathcal{M}_\Lambda$ is the moduli space of solutions on $S^n \setminus \Lambda$ studied in [MPU]. There is a natural map

$$\pi : \mathbb{M}_k \to \mathcal{C}_k.$$

In this subsection we prove that $\mathbb{M}_k$ is a real analytic set. We will only sketch the proof since it is very close to the proof of the analogous result in [KMP]; note that we use this rather than the proof in [MPU] because it is much more direct and simple to modify. A consequence of this result is that $\mathbb{M}_k$ admits a stratification into smooth real analytic manifolds. The dimensions of these strata, particularly the maximal dimension, are difficult to obtain without more information. However, there is a nondegeneracy condition which is less stringent than the one we have used before, such that if $(g, \Lambda) \in \mathbb{M}_k$ is nondegenerate in this new sense, then the stratum containing this point is maximal (in that connected component of $\mathbb{M}_k$) and has the expected dimension $k(n+1)$. We call this new nondegeneracy condition *unmarked nondegeneracy*, and the former one simply nondegeneracy, as before. We shall see that nondegenerate points in $\mathcal{M}_\Lambda$ are a fortiori nondegenerate in the unmarked moduli space $\mathbb{M}_k$. In particular, since the dipole metrics are nondegenerate in $\mathcal{M}_\Lambda$, any component of $\mathbb{M}_k$, $k = 2N$, containing a dipole metric has top stratum of the predicted dimension. Furthermore, all other points in this top-dimensional stratum, including ones where the singular points are no longer in the dipole configurations, are then nondegenerate in $\mathbb{M}_k$. We shall show, finally, that most of these are nondegenerate in their respective moduli spaces $\mathcal{M}_\Lambda$.

Before we can give the precise statements and proofs, we digress briefly to discuss some additional facts about Jacobi fields for Delaunay metrics which we need. We also quote several results and the (adaptations of the) relevant lemmas from [MPU] which are required.

We have already used the two temperate Jacobi fields $\phi_0^\pm$ on D_ε, one periodic and the other linearly growing, which correspond to infinitesimal translations and changes of Delaunay parameter. There is another set of Jacobi fields, $\phi_j^\pm$, $j = 1, \dots, n$, on D_ε which can be written explicitly. Using cylindrical coordinates, and writing the Delaunay metric $g_\varepsilon = u_\varepsilon^{4/(n-2)}(dt^2 + d\theta^2)$, then

$$\phi_j^\pm = e^{\pm t}\left(\frac{n-2}{2}u_\varepsilon \pm \frac{du_\varepsilon}{dt}\right)\psi_j(\theta), \tag{4.4}$$

where $\psi_j(\theta)$ are the eigenfunctions for Δ_θ with eigenvalue $n-1$ on S^{n-1}, are also solutions of $L_\varepsilon \phi = 0$. Like the $\phi_0^\pm$, but unlike all the other Jacobi fields for g_ε, these arise as derivatives of explicit one-parameter families of solutions. In fact, regarding Delaunay solutions as metrics on S^n with two singular points, we can pull these metrics back by any conformal transformation of S^n to obtain new solution metrics, also in $\mathbb{M}_2$. Families of solutions are obtained by pulling back by families of conformal transformations. If we differentiate the family obtained from the one-parameter family of conformal transformations fixing both singular points, we obtain ϕ_0^+. If we differentiate, on the other hand, the families of parabolic conformal transformations fixing one or the other of the singular points of the original g_ε, we obtain these other $\phi_j^\pm$. These parabolic transformations correspond to the translations on the images, $\mathbb{R}^n$, of the stereographic projection from S^n with one of the singular points removed. The parabolic transformations moving one singular point lead to Jacobi fields blowing up exponentially at that singular point and decaying exponentially at the other. Their exact decay rate is given by (4.4). Combining this with the earlier discussion of the Fredholm properties of L_ε on weighted Sobolev spaces, we see that L_ε is not Fredholm on the space $H^s_{\delta_1}$, with exponential weight $\delta_1 = P_\varepsilon$. Actually, as proved in [MPU], L_ε is Fredholm on all other weighted spaces H^s_δ for $0 < \delta < \delta_1 + \eta$, $\delta \neq \delta_1$, where η is some small positive number depending on ε.

If g is any CPSC metric on $S^n \setminus \Lambda$, with Λ finite, then in a neighborhood of each $x_j \in \Lambda$, g is asymptotically equivalent to a Delaunay metric. Approximate Jacobi fields may be constructed by transplanting cutoffs of the Jacobi fields $\phi_j^\pm$, $j = 0, \ldots, n$, from the appropriate D_ε to a neighborhood of each x_j. This, together with the approximate Jacobi fields corresponding to $\phi_0^\pm$, yields a $2k(n+1)$-dimensional family of approximate Jacobi fields, which we denote by $\mathcal{W}$; it substitutes for the deficiency space W in the analysis of the unmarked moduli space. Note that $\mathcal{W} \subset H^s_{\delta_1+\delta}$ for any $\delta > 0$. Since there are good parametrices for L_g (constructed in [MPU]), and since the elements of $\mathcal{W}$, in particular those of W, correspond to one-parameter families of solutions, the marked nondegeneracy conditions (2.13) and (2.15)–(2.17) reduce simply to the hypothesis that L_g is injective on $H^s_{-\delta_1-\delta}$ for any $\delta > 0$. By contrast, we make the

(4.5) DEFINITION. The solution $(g, \Lambda) \in \mathbb{M}_k$ is called *unmarked nondegenerate* if the linearized operator L_g has no nullspace in $H^s_{-\delta_1-\delta}$ for any $\delta > 0$.

By duality, g is unmarked nondegenerate if and only if

$$L_g : H^{s+2}_{\delta_1+\delta} \to H^s_{\delta_1+\delta}$$

is surjective for $\delta > 0$ sufficiently small. Using the parametrix construction and the proof of the linear decomposition lemma (4.18) from [MPU] we obtain

(4.6) LEMMA. *Suppose that $(g, \Lambda) \in \mathbb{M}_k$ is unmarked nondegenerate. Then there exists a bounded map G from $H^s_{\delta_1+\delta}$ to $H^{s+2}_{\delta_1+\delta}$ such that $L_g G = I$. If we restrict the domain of G to $H^s_{-\delta_1-\delta}$, then its range is $H^{s+2}_{-\delta_1-\delta} \oplus \mathcal{W}$. In particular,* (4.7) $$L_g : H^{s+2}_{-\delta_1-\delta} \oplus \mathcal{W} \to H^s_{-\delta_1-\delta}$$

is surjective.

The dimension of the space $\mathcal{W}$ is $2k(n+1)$ and its elements are the possible parameters for deformations of $(g, \Lambda) \in \mathbb{M}_k$. Which deformations actually occur is a difficult question; however, by a relative index theorem as in [MPU] we have

(4.8) LEMMA. *The kernel of the map* (4.7) *has dimension* $k(n+1)$.

We also require an analogue of (2.17), that the elements of $\mathcal{W}$ (which are sufficiently small in norm) can be "exponentiated" to one-parameter families of metrics which are locally of CPSC near the singular points. Thus let $w \in \mathcal{W}$; we can write it as a sum

$$w = \sum_{\ell=1}^{k} \sum_{j=0}^{n} a^{(\ell)}_{j,\pm} \chi^{(\ell)} \phi^{\pm}_j$$

near each singular point x_ℓ. Here $\chi^{(\ell)}$ are fixed cutoff functions equaling one in a small ball around x_ℓ and vanishing outside a slightly larger ball. We identify w with the collection of coefficients $\mathbf{a} = \{a^{(\ell)}_{j,\pm}\} \in \mathbb{R}^{2k(n+1)}$. Then by a procedure very similar to the one in §2, we define a metric $g_{\mathbf{a}}$ by altering g in the neighborhoods $B_\sigma(x_\ell)$ according to the amounts specified by the coefficients $\mathbf{a}$. The first step is to identify the Delaunay solution D_{ε_ℓ} to which g is asymptotic in $B_\sigma(x_\ell)$. In this ball we can write $g = (1+v)^{4/(n-2)} g_\varepsilon$, where $v = O(r)$. We first conformally deform v to equal zero in $B_{\sigma/2}(x_\ell)$. The new metric $\widetilde{g}$ is exactly Delaunay in these smaller balls, and is unchanged outside the larger balls. Now we can alter $\widetilde{g}$ in this smaller ball in a manner specified by the parameters $\mathbf{a} \in \mathbb{R}^{2k(n+1)}$; the new metric we obtain is denoted $g_{\mathbf{a}}$. We demonstrated earlier how to do this for the coefficients $a_{0,\pm} = \{a, b\}$ for the Delaunay solutions, and since this change was supported in the half-cylinder that discussion obviously localizes to these half-Delaunay solutions. The analogous procedure for the other coefficients is defined exactly as before. Note that the singular points x_ℓ will be moved in this procedure if any one of the $a^{(\ell)}_{j,+}$ is nonzero.

Now we may define the nonlinear operator

$$N : H^{s+2}_{-\delta_1-\delta} \oplus \mathcal{W} \to H^s_{-\delta_1-\delta}$$

for small elements $w \in \mathcal{W}$ at the metric $g \in \mathbb{M}_k$ by

$$N(v, w) = \Delta_{g_{\mathbf{a}}}(1+v) - \frac{n-2}{4(n-1)} R(g_{\mathbf{a}})(1+v) + \frac{n(n-2)}{4}(1+v)^{(n+2)/(n-2)},$$

where $w \in \mathcal{W}$ determines the coefficients $\mathbf{a}$. This is clearly a real analytic mapping. A neighborhood $\mathcal{V}$ of g in $\mathbb{M}_k$ is given as the zero set of this map N in $H^{s+2}_{-\delta_1-\delta} \oplus \mathcal{W}$. Its linearization coincides with the linearization L_g acting on $H^{s+2}_{-\delta_1-\delta} \oplus \mathcal{W}$. The analytic implicit function theorem and the relative index theorem of [MPU] now give

(4.9) PROPOSITION. *If $g \in \mathbb{M}_k$ is unmarked nondegenerate, so that*

$$L_g : H^{s+2}_{-\delta_1-\delta} \oplus \mathcal{W} \to H^s_{-\delta_1-\delta}$$

is surjective, then there is a neighborhood $\mathcal{V}$ of $g \in \mathbb{M}_k$ which is a real analytic manifold of dimension $k(n+1)$.

When g is unmarked degenerate a somewhat weaker statement is true:

(4.10) PROPOSITION. *If $g \in \mathbb{M}_k$ is unmarked degenerate, then there is a neighborhood $\mathcal{X}$ of $(0,0) \in H^{s+2}_{-\delta_1-\delta} \oplus \mathcal{W}$ and a real analytic diffeomorphism Ψ from this neighborhood to itself such that $\Psi(N^{-1}(0) \cap \mathcal{X})$ lies in a neighborhood of zero $\mathcal{Y}$ in a finite-dimensional subspace P and coincides with the zero set of a real analytic mapping from $\mathcal{Y}$ to $\mathbb{C}^N$, where N is the dimension of the kernel of L_g on $H^s_{-\delta_1-\delta}$.*

The proof of this final proposition is identical to the one in [KMP], and thus we give only the briefest sketch and refer there for the details. We use what is often called the Lyapunov–Schmidt or Kuranishi method. Let K denote the nullspace of L on $H^s_{-\delta_1-\delta}$, which is nontrivial since g is unmarked degenerate. Then

$$\mathcal{L} : H^{s+2}_{-\delta_1-\delta} \oplus \mathcal{W} \oplus K \to H^s_{-\delta_1-\delta}$$

defined by $\mathcal{L}(v, w, k) = L_g(v + w) + k$ is surjective, by construction. We can similarly define $\mathcal{N}(v, w, k) = N(v, w) + k$, so that $\mathcal{L}$ is the (surjective) linearization of $\mathcal{N}$. The zero set of N is identified with the set $\{(v, w, k) : \mathcal{N}(v, w, k) = k\}$. Using the implicit function theorem for $\mathcal{N}$, we can show that this set is real analytic.

It is clear that the projection $\pi : \mathbb{M}_k \to \mathcal{C}_k$ is real analytic.

We now combine these last two results with the results of the last subsection.

(4.11) COROLLARY. *Suppose that $k = 2N$ and that some component of $\mathbb{M}_k$ contains an element (g, Λ), where Λ is a dipole configuration and g a dipole metric. Then (g, Λ) is in the principal stratum of that component, and this principal stratum is nonsingular and has dimension $2N(n+1)$. Let $\widetilde{\pi}$ denote the restriction of the projection π to this component of $\mathbb{M}_k$ and let $\mathcal{C}$ be the image of $\widetilde{\pi}$ in $\mathcal{C}_k$. Then $\mathcal{C}$ is a subanalytic set, with principal stratum of dimension $kn = 2Nn$, and the preimage $\mathcal{M}_{\Lambda'}$ of any configuration Λ' in this principal stratum is a moduli space with a nonsingular $k = 2N$-dimensional principal stratum composed of nondegenerate elements.*

As a final remark we note that the singular strata for these moduli spaces may in fact be trivial. In [MPU] we show that they are trivial for generic conformal classes. In fact, we do not know whether degenerate solutions exist in any case. Either a construction of degenerate solutions, or a geometric criterion for their presence or absence, would be quite interesting.

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RAFE MAZZEO
Department of Mathematics
Stanford University
Stanford, CA 94305, USA

E-mail address: mazzeo@math.stanford.edu

DANIEL POLLACK
Department of Mathematics
University of Chicago
Chicago, IL 60637, USA

E-mail address: pollack@math.uchicago.edu

KAREN UHLENBECK
Department of Mathematics
University of Texas
Austin, TX 78712, USA

E-mail address: uhlen@math.utexas.edu