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PERIODIC SOLUTIONS OF A SECOND ORDER DIFFERENTIAL EQUATION WITH DISCONTINUITIES IN THE SPATIAL VARIABLE

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Dedicated to Louis Nirenberg on the occasion of his 70th birthday

Introduction. In this paper we prove an existence result for the second order periodic boundary value problem (PBVP)

$$x'' = f(t, x, x'), \quad x \in \mathbb{R}, \tag{1}$$

$$x(0) = x(1), \quad x'(0) = x'(1). \tag{2}$$

We assume f to be measurable, but make no continuity requirements on f. We use Filippov's definition of a solution (see [4]).

Many results for boundary value problems (BVP's) with discontinuities only in the time variable were proved by using Carathéodory's definition of a solution. Filippov's definition of a solution is more general than that of Carathéodory and it includes it as a special case. A standard approach to boundary value problems with discontinuities in the spatial variable is to solve the problem on each side of the discontinuity separately and then try to match the resulting solutions. A totally different approach is used in [9]. Using Filippov's theory, the BVP's are reformulated as differential inclusions and then the existence principles proved in [7] are applied to obtain existence results for the periodic problem and for the Dirichlet problem. The results are further used to establish the existence of

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periodic solutions to the dry friction equation

$$u'' + bu' + cu + k \operatorname{sgn} u' = e(t), \tag{3}$$

where $b, c, k \in \mathbb{R}$, $b, c, k > 0$, and e is a measurable 1-periodic function.

In this paper we first prove an existence theorem for periodic solutions of a second order differential inclusion and then we use the same approach as in [9]. The results here are applicable to the dry friction equation with the following nonlinearities:

$$u'' + b(u') + c(u) + k \operatorname{sgn} u' = e(t), \tag{4}$$

or

$$u'' + u'd(u) + c(u) + k \operatorname{sgn} u' = e(t), \tag{5}$$

where $k \in \mathbb{R}$, $k > 0$, b, c and g are nonlinear functions, and e is a measurable 1-periodic function.

First we recall Filippov's definition. Consider the Lebesgue measure μ and an initial value or a boundary value problem for the vector differential equation $x' = f(t,x)$, where $f(t,\cdot)$ may be discontinuous. Based on the idea that $N \subset \mathbb{R}^n$ with $\mu(N) = 0$ should play no role Filippov defined solutions of

$$x' = f(t,x), \quad x \in \mathbb{R}^n, \tag{6}$$

$$x(0) = x(1), \tag{7}$$

where $f = (f_1, \dots, f_n)$ and $x = (x_1, \dots, x_n)$, as solutions of the differential inclusion constructed as a convexification of f with respect to $x \in \mathbb{R}^n$ in the following way:

$$x'(t) \in \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \operatorname{conv} f(t, U(x(t),\delta) - N)$$

for almost every t, where $U(x,\delta) = \{y : |x-y| < \delta\}$ and $\operatorname{conv} Y$ is the closed convex hull of Y. We make no continuity requirements on f, but we assume f to be measurable.

Definition 1. Let x be absolutely continuous on $[0,1]$. If x satisfies (7) and

$$x'(t) \in \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \operatorname{conv} f(t, U(x(t),\delta) - N) = K\{f(t,x)\} = k_t(x)$$

for almost every $t \in (0,1)$, we say x is a *solution* of (6), (7).

In the next definition, which is equivalent to Definition 1, we use the following notation:

$$\operatorname*{ess\,max}_{x \in E} f(t,x) = \inf_{\mu(N)=0} \sup_{x \in E-N} f(t,x),$$

where f is scalar-valued. The essential upper bound of the function $f(t,x)$ at the point x is defined by

$$M_x\{f(t,x)\} = \lim_{\delta\to 0} \operatorname*{ess\,max}_{y\in U(x,\delta)} f(t,y).$$

Analogously the essential lower bound of $f(t,x)$ at x is denoted by $m_x\{f(t,x)\}$. The proof of the equivalence of Definition 1 and the following definition can be found in [4], p. 203.

DEFINITION 2. Let x be absolutely continuous on $[0,1]$ and let x satisfy (2). If

$$m_i(t,x) = m_x\{f_i(t,x_1,\dots,x_n)\} \le x_i' \le M_x\{f_i(t,x_1,\dots,x_n)\} = M_i(t,x)$$

for $i = 1,\dots,n$ and for almost every $t \in (0,1)$, then x is a *solution* of (6), (7).

REMARK 1. Let $k_t(x) = (k_t^1(x),\dots,k_t^n(x))$. Then

$$k_t^i(x) \subset [m_i(t,x), M_i(t,x)]$$

for almost every $t \in (0,1)$, or if we define $\mathcal{M}_i(x,t) = \max\{|m_i(t,x)|, |M_i(t,x)|\}$, then

$$k_t^i(x) \subset [-\mathcal{M}_i(t,x), \mathcal{M}_i(t,x)]$$

for almost every $t \in (0,1)$. Using vector notation and the symbol $\mathrm{cl}(Y)$ for the closure of Y we get

$$k_t(x) \subset \mathrm{cl}(U(0, |\mathcal{M}(x,t)|)), \tag{8}$$

which we write as

$$|k_t(x)| \le |\mathcal{M}(x,t)|$$

for almost every $t \in (0,1)$, where $\mathcal{M}(x,t) = (\mathcal{M}_1(x,t),\dots,\mathcal{M}_n(x,t))$.

DEFINITION 3. We say x_1 is a *solution* of (1), (2) if (x_1,x_2) is a Filippov solution of

$$\begin{aligned} &x_1' = x_2, \quad x_2' = f(t,x_1,x_2),\\ &x_1(0) = x_1(1), \quad x_2(0) = x_2(1). \end{aligned}$$

The following existence principle will be used to prove an existence theorem for periodic solutions of a second order differential inclusion. First we need to define an L^p-Carathéodory function.

Definition 4. Let $p \geq 1$. A set-valued function $F : [0,1] \times \mathbb{R}^{kn} \to \mathrm{Kv}(\mathbb{R}^n)$, where $\mathrm{Kv}(\mathbb{R}^n)$ is the family of all compact, convex, nonempty subsets of $\mathbb{R}^{kn}$, is an *L^p-Carathéodory function* provided:

(a) the map $x \to F(t,x)$ is upper semicontinuous for all $t \in [0,1]$;
(b) the map $t \to F(t,x)$ is measurable for all $x \in \mathbb{R}^{kn}$;
(c) for each $r > 0$ there exists $h_r \in L^p[0,1]$ such that $|x| \leq r$ implies $|F(t,x)| \leq h_r(t)$ for almost all $t \in [0,1]$.

Theorem 1 (Existence principle). *Consider an L^p-Carathéodory function F and the family of problems*

$$x'' - \alpha x \in \lambda[F(t,x,x') - \alpha x], \quad x(0) = x(1), \quad x'(0) = x'(1), \tag{9}$$

for $\lambda \in [0,1]$, where $\alpha \neq 0$ is fixed and is not an eigenvalue of

$$L(x) = x'', \tag{10}$$

where the domain of L is $\{x \in C^2(0,1) : x(0) = x(1),\ x'(0) = x'(1)\}$. Let U be a bounded, open set in $\{x \in C^1(0,1) : x(0) = x(1),\ x'(0) = x'(1)\}$ with $0 \in U$. Then either

$$x'' \in F(t,x,x'), \quad x(0) = x(1), \quad x'(0) = x'(1), \tag{11}$$

has a solution in $\mathrm{cl}(U)$, *or* (7) *has a solution on ∂U for some $\lambda \in (0,1)$.*

Proof. The theorem follows from the general existence principles proved in [7]. It is a particular case of Theorem 2(A) of [7].

We use this existence principle to prove an existence theorem for (11).

Theorem 2 (Periodic problem for inclusions). *Let F be an L^p-Carathéodory function such that*

$$|F(t,y,z) - \alpha y - zg(y)| < C \tag{12}$$

for all $y, z \in \mathbb{R}$, where $\alpha, C \in \mathbb{R}$, $C > 0$, $|\alpha| < \pi^2$ and $g \in C^0(\mathbb{R})$. Then (11) *has at least one solution.*

Proof. We need to establish *a priori* bounds independent of λ for solutions of (9) and their derivatives. Let x be a solution of (9), where $0 < \lambda < 1$. Then

$$x'' = \lambda(w - \alpha x) + \alpha x, \quad x(0) = x(1), \quad x'(0) = x'(1), \tag{13}$$

where $w(t) \in F(t,x,x')$ for almost every $t \in (0,1)$ and w is an integrable selection of $F(t,x,x')$ (see [5]). We integrate (13) from 0 to 1 to get

$$0 = \alpha \int_0^1 x\,dt + \lambda \int_0^1 (w - \alpha x)\,dt = \alpha \int_0^1 x\,dt + \lambda \int_0^1 (w - \alpha x - x'g(x))\,dt.$$

This implies that

$$\left| \int_0^1 x\,dt \right| \le \frac{\lambda C}{|\alpha|} \le \frac{C}{|\alpha|},$$

and for some $t_0 \in (0,1)$ we have

$$|x(t_0)| = \left| \int_0^1 x\,dt \right| \le \frac{C}{|\alpha|}.$$

Further, by the Hölder inequality we have

$$|x(t)| \le |x(t_0)| + \int_{t_0}^t |x'(s)|\,ds \le \frac{C}{|\alpha|} + \left(\int_0^1 |x'(s)|^2\,ds \right)^{1/2} \tag{14}$$

for $t \in [0,1]$. Now we will find an estimate for the L^2 norm of x'. We multiply (13) by x and integrate from 0 to 1 to get

$$-\int_0^1 (x')^2\,dt = \alpha \int_0^1 x^2\,dt + \lambda \int_0^1 (wx - \alpha x^2 - g(x)x'x)\,dt. \tag{15}$$

Here we integrated by parts on the left hand side of the equation, and on the right hand side we used the identity

$$\int_0^1 g(x)x'x\,dt = x(1)G(x(1)) - x(0)G(x(0)) - \int_0^1 G(x)x'\,dt = 0,$$

where $G' = g$. From (14), (15) and the Wirtinger inequality for periodic functions, we obtain

$$\begin{aligned} \int_0^1 |x'|^2\,dt &\le |\alpha| \int_0^1 x^2\,dt + \lambda C \int_0^1 |x|\,dt \\ &\le \frac{|\alpha|}{\pi^2} \int_0^1 |x'|^2\,dt + \lambda \frac{C^2}{|\alpha|} + \lambda C \left(\int_0^1 |x'|^2 dt \right)^{1/2}. \end{aligned} \tag{16}$$

If $\|x'\|_{L^2[0,1]} < 1$, then we have the desired estimate, so we assume $\|x'\|_{L^2[0,1]} \ge 1$. After a simple manipulation with (16) we get

$$\|x'\|_{L^2[0,1]} \le K,$$

where

$$K = \frac{C(C + |\alpha|)\pi^2}{|\alpha|(\pi^2 - |\alpha|)} > 0.$$

Here we used the assumption $|\alpha| < \pi^2$. From (14) it follows that

$$|x(t)| \le \frac{C}{|\alpha|} + K \qquad \text{for } t \in [0,1].$$

Finally, we need an estimate on x'. We multiply (13) by x'' and integrate from 0 to 1 to get

$$\begin{aligned}\int_0^1 (x'')^2\,dt &= \alpha\int_0^1 xx''\,dt + \lambda\int_0^1 (wx'' - \alpha xx'')\,dt \\ &= -\alpha\int_0^1 (x')^2\,dt + \lambda\int_0^1 (wx'' - \alpha xx'')\,dt.\end{aligned} \tag{17}$$

From (12), (17) and the Hölder inequality, it follows that

$$\begin{aligned}\int_0^1 (x'')^2\,dt &\le |\alpha|\int_0^1 |x'|^2\,dt + \lambda C\int_0^1 |x''|\,dt + \lambda\int_0^1 x''x'g(x)\,dt \\ &\le \alpha K^2 + \lambda C\int_0^1 |x''|\,dt + \lambda M\int_0^1 |x''x'|\,dt \\ &\le \alpha K^2 + C\Big(\int_0^1 |x''|^2\,dt\Big)^{1/2} + KM\Big(\int_0^1 |x''|^2 dt\Big)^{1/2},\end{aligned}$$

where $M = \max\{g(x) : |x| \le C/|\alpha| + K\}$. If $\|x''\|_{L^2[0,1]} < 1$, then we have the desired estimate, so we assume $\|x''\|_{L^2[0,1]} \ge 1$ to obtain

$$\|x''\|^2_{L^2[0,1]} \le (\alpha K^2 + C + KM)\|x''\|_{L^2[0,1]}$$

and so

$$\|x''\|_{L^2[0,1]} \le L,$$

where $L = \alpha K^2 + C + KM$. Since there exists $t_1 \in (0,1)$ such that $x'(t_1) = 0$, the fundamental theorem of calculus and Hölder inequality yield

$$|x'(t)| \le \int_{t_1}^t |x''|\,ds \le \|x''\|_{L^2[0,1]} \le L \qquad \text{for } t \in [0,1],$$

and the required *a priori* bounds are established. This proves the theorem.

Now we use Theorem 2 to prove an existence result for the PBVP, where we consider Filippov solutions.

Theorem 3 (Periodic problem for Filippov solutions). *Suppose $f(t,y,z)$ is measurable in $[0,1]\times\mathbb{R}^2$ and for any bounded, closed domain $D \subset [0,1]\times\mathbb{R}^n$, there exists an integrable function $B(t)$, which may depend on D, such that*

$$|f(t,y,z)| \le B(t) \tag{18}$$

almost everywhere in D. Moreover, assume that for all $(\overline{t},\overline{y},\overline{z}) \in [0,1]\times\mathbb{R}^2$, there exist $\delta_1, \delta_2 > 0$ and a function $C : [\overline{t}-\delta_1, \overline{t}+\delta_1] \to \mathbb{R}$ such that

$$|f(t,y,z)| \le C(t) \tag{19}$$

for $(t,y,z) \in [\bar{t}-\delta_1, \bar{t}+\delta_1] \times \operatorname{cl}(U((\bar{y},\bar{z}),\delta_2))$, *where at the endpoints, we consider appropriate one-sided neighborhoods. Finally, suppose that for each* $(y,z) \in \mathbb{R}^2$ *there exists an* $\varepsilon > 0$ *such that*

$$\sup_{(x,y)\in\mathbb{R}^2} |f(t,\tilde{y},\tilde{z}) - \alpha y - zg(y)| < C \tag{20}$$

for $(\tilde{y},\tilde{z}) \in U((y,z),\varepsilon) - N$, *where* $\mu(N) = 0$ *and* α, C *and* g *are as in Theorem* 2. *Then the PBVP has a solution.*

Proof. We apply Theorem 2. Based on the results of [8], it is shown in [9], Theorem 3, that

$$K\{f(t,x_1,x_2)\} = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \operatorname{conv} f(t, U((x_1,x_2),\delta) - N),$$

where f satisfies (18), (19), is an L^p-Carathéodory function. Let $h = \operatorname{col}(x_2, f(t,x_1,x_2))$, where $\operatorname{col}(x_2, f(t,x_1,x_2))$ is the column vector with components x_2 and $f(t,x_1,x_2)$. Then

$$\begin{aligned} K\{h(t,x_1,x_2)\} &= \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \operatorname{conv} h(t, U((x_1,x_2),\delta) - N) \\ &= \operatorname{col}(x_2, K\{f(t,x_1,x_2\}). \end{aligned}$$

So (1), (2) is equivalent to

$$x'' \in K\{f(t,x,x')\}, \quad (2).$$

From (20) it follows that $k_f(t,x,y)$ satisfies (12), so Theorem 2 applies and this completes the proof.

Example (Dry friction). Consider the equation

$$x'' + b(x') + c(x) + k \operatorname{sgn} x' = e(t),$$

where b, c are continuous functions and $e(t)$ is a 1-periodic measurable function with

$$\sup_{t\in\mathbb{R}} |e(t)| = M.$$

Condition (20) becomes

$$\begin{aligned} &|-b(z) - c(y) \pm k \operatorname{sgn} z + e(t) - \alpha y - zg(y)| \\ &\qquad \le |-b(z) - zg(y)| + |-c(y) - \alpha y| + |e(t) + k| \le C. \end{aligned}$$

If there exist $\beta, \gamma, C_1, C_2 \in \mathbb{R}$, $|\gamma| \le \pi^2$, $\gamma \ne 0$, such that

$$|b(z) - \beta z| \le C_1, \qquad |c(y) - \gamma y| \le C_2,$$

for all $y, z \in \mathbb{R}$, then we can take $C = C_1 + C_2 + M + k$ and condition (20) is satisfied. Consider $g(y) = -\beta$ and $\alpha = -\gamma$. If we replace $b(x')$ by $x'd(x)$ then we need an estimate on

$$|-zd(y) - zg(y)|,$$

and this vanishes if we take $g = d$. Thus, we have established the existence of periodic solutions to equations (4) and (5).

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