Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 6, 1995, 175–188

ON THE EXISTENCE OF TWO SOLUTIONS WITH A PRESCRIBED NUMBER OF ZEROS FOR A SUPERLINEAR TWO-POINT BOUNDARY VALUE PROBLEM

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Dedicated to Louis Nirenberg

1. Introduction

In a series of recent articles (cf. [1]–[4]) we considered an approach based on the Leray–Schauder degree for the solvability of some boundary value problems where there are no a priori bounds for the solutions. In particular, in [1] we studied a Sturm–Liouville problem for the second order nonlinear scalar equation

$$u'' + f(u) = p(t, u, u'), \quad a \le t \le b,$$

where the function f has superlinear growth at infinity and p grows at most linearly in u and u'. Our method is based on a continuation theorem for a coincidence equation of the form $Lu = N(u, \lambda)$, with the parameter λ varying in the unit interval I. In [1] and the preceding papers [4], [9, §5.5] we introduced a continuous functional $\varphi(u, \lambda)$ which takes integer values on large solutions and, under certain conditions, we were able to show that there are solutions of $Lu = N(u, 1)$ with $\varphi(u, 1)$ equal to some positive integer. In the applications to scalar ordinary differential equations φ was related to the number of zeros of the solutions and thus we could prove that the boundary value problems we considered have at least *one* solution with k zeros in $[a, b]$ for each sufficiently large k.

1991 *Mathematics Subject Classification.* 34B15, 47H15.

The aim of this note is to show that it is possible to refine further the abstract continuation theorem in [1], [4], [9, §5.2] as well as its applications in order to prove that for each k large there are at least *two* solutions with prescribed nodal properties. In this manner, we obtain a result which is sharp with respect to the number of zeros of the solutions, as shown for instance by the analysis of some elementary examples (see Example 1 in Section 3), and get more precise information also with respect to analogous theorems which are obtained using different approaches [12]. For the reader's convenience, we have tried to make this article as much selfcontained as possible. For this reason, there is some overlapping between the introductory part of Section 2 below and the corresponding Section 2 in [1], where all the notations needed for the abstract setting of the problem are introduced. We also point out that, in order to make more transparent the discussion for the part which is "new" (with respect to [1]), where we show how to obtain two solutions instead of one, we have confined ourselves to an application concerning the two-point boundary condition $u(a) = A$, $u(b) = B$. We stress that similar work could be done with respect to other boundary conditions.

2. A continuation theorem

Let X, Z be real Banach spaces, $L : X \supset D(L) \to Z$ a linear Fredholm mapping of index zero, $I = [0,1]$ and $N : X \times I \to Z$ an L-completely continuous operator (see [8] for the corresponding definitions). We consider the equation

$$Lu = N(u,\lambda), \qquad u \in D(L),\ \lambda \in I. \tag{1}$$

Let

$$\Sigma^* = \{(u,\lambda) \in D(L) \times I : Lu = N(u,\lambda)\}.$$

For any set $B \subset X \times I$ and any $\lambda \in I$ we denote by B_λ the section $\{u \in X : (u,\lambda) \in B\}$. Subsequently, we use the following conventions. For $\mathcal{O} \subset X \times I$, we denote by $\overline{\mathcal{O}}$ and $\partial\mathcal{O}$ its closure and boundary in $X \times I$ respectively. Similar notation is used for closure and boundary in X. If ω is an open subset of X (possibly not bounded) such that $S = \Sigma^*_\lambda \cap \overline{\omega}$ is compact and $S \subset \omega$ (i.e. there is no solution on $\partial\omega$), there exists $\mathcal{U}$ open bounded such that $S \subset \mathcal{U} \subset \overline{\mathcal{U}} \subset \omega$. For all such $\mathcal{U}$, the coincidence degree $D_L(L - N(\,\cdot\,,\lambda),\mathcal{U})$ is the same, by the excision property. We will denote it as

$$D_L(L - N(\,\cdot\,,\lambda),\omega).$$

Let $\mathcal{O} \subset X \times I$ be open in $X \times I$. Let us denote by Σ the set of solutions (u,λ) of (1) which belong to $\overline{\mathcal{O}}$, i.e.

$$\Sigma = \{(u,\lambda) \in \overline{\mathcal{O}} \cap (D(L) \times I) : Lu = N(u,\lambda)\},$$

and suppose that

(i$_1$) Σ_0 *is bounded in* X *and* $\Sigma_0 \subset \mathcal{O}_0$

and

(i$_2$) $D_L(L - N(\,\cdot\,,0), \mathcal{O}_0) \neq 0$,

so that $\Sigma_0 \neq \emptyset$. Further, we introduce a functional $\varphi : X \times I \to \mathbb{R}$ and suppose that

(i$_3$) φ *is continuous on* $X \times I$ *and proper on* Σ.

Consequently, the constants

$$\varphi_- = \min\{\varphi(u,0) : u \in \Sigma_0\}, \qquad \varphi_+ = \max\{\varphi(u,0) : u \in \Sigma_0\}$$

exist. The following result is proved in [1] (see also [4], [9]) using a theorem on the existence of connected branches of solutions of (1) emanating from Σ_0 (cf. [5]). In the sequel, we denote by $\mathbb{N}$ the set of natural numbers (0 included) and by $\mathbb{Z}^+ = \mathbb{N} \setminus \{0\}$ the set of positive integers.

Lemma 1. *Assume that conditions* (i$_1$), (i$_2$) *and* (i$_3$) *hold and that there exist constants* c_-, c_+ *with*

$$c_- < \varphi_- \leq \varphi_+ < c_+,$$

such that

$$\varphi(u,\lambda) \notin \{c_-, c_+\} \qquad \textit{whenever } (u,\lambda) \in (D(L) \times]0,1[\,) \cap \mathcal{O} \cap \Sigma,$$

and

$$\varphi(u,\lambda) \notin [c_-, c_+] \qquad \textit{whenever } (u,\lambda) \in (D(L) \times]0,1[\,) \cap \partial\mathcal{O} \cap \Sigma.$$

Then the equation

$$Lu = N(u,1) \tag{2}$$

has at least one solution in $D(L) \cap (\overline{\mathcal{O}})_1$.

Let us now consider a consequence of Lemma 1. Assume that $\varphi : X \times I \to \mathbb{R}$ is continuous and $(c_k)_{k\in\mathbb{Z}}$ is an increasing doubly infinite sequence with $c_k < 0$ for $k < 0$, $c_k > 0$ for $k > 0$ and $\lim_{k\to\pm\infty} c_k = \pm\infty$, that satisfies the following conditions:

(i$_4$) *There exists* $R > 0$ *such that* $\varphi(u,\lambda) \neq c_k$ *for all* $k \in \mathbb{Z}$ *and* $(u,\lambda) \in \Sigma^*$ *with* $\|u\| \geq R$.

(i$_5$) $\varphi^{-1}(\,]c_{-n}, c_n[\,) \cap \Sigma^*$ *is bounded for each* $n \in \mathbb{Z}^+$.

Let k_0 be a positive integer such that

$$\min\{-c_{-k_0}, c_{k_0}\} > \sup\{|\varphi(u,\lambda)| : (u,\lambda) \in \Sigma^*, \|u\| \le R\}. \tag{3}$$

For $k \in \mathbb{Z}$ with $|k| > k_0$, let

$$\mathcal{O}^k = \begin{cases} \varphi^{-1}(\,]c_k, c_{k+1}[\,) & \text{if } k > 0, \\ \varphi^{-1}(\,]c_{k-1}, c_k[\,) & \text{if } k < 0, \end{cases} \tag{4}$$

and $\Sigma^k = \overline{\mathcal{O}^k} \cap \Sigma^*$. By (i_5), $(\Sigma^k)_0$ is bounded and hence compact. But, by (i_4), $\varphi(x,\lambda) \ne c_k$ and $\varphi(x,\lambda) \ne c_{k+1}$ (if $k > 0$) or $\varphi(x,\lambda) \ne c_{k-1}$ (if $k < 0$), for all $(x,\lambda) \in \Sigma^k$, so that $\Sigma^k \subset \mathcal{O}^k$ and also $(\Sigma^k)_0 \subset (\mathcal{O}^k)_0$. Thus we have proved the condition (i_1).

We now prove that φ is proper on Σ^k. Let K be a compact subset of $\mathbb{R}$. Then $\varphi^{-1}(K) \cap \Sigma^k$ is closed and included in Σ^k which is compact, so it is also compact.

Let us assume that

(i_6) $D_L(L - N(\,\cdot\,,0), (\mathcal{O}^k)_0) \ne 0$.

Thus, all conditions of Lemma 1 with $\Sigma = \Sigma^k$, $\mathcal{O} = \mathcal{O}^k$ and $(c_-, c_+) = (c_k, c_{k+1})$ for $k > 0$ or $(c_-, c_+) = (c_{k-1}, c_k)$ for $k < 0$ are satisfied and equation (2) will have at least one solution $u \in D(L) \cap (\overline{\mathcal{O}^k})_1$. We therefore have the following result.

Lemma 2. *Assume that conditions* (i_4) *and* (i_5) *hold and that there is* $k_0 \in \mathbb{Z}^+$ *satisfying* (3) *such that* (i_6) *holds for some integer* k *with* $|k| > k_0$. *Then there is at least one solution* $\widetilde{u}$ *for* (2) *with* $\varphi(\widetilde{u},1) \in\,]c_k, c_{k+1}[$ *if* $k > 0$ *and* $\varphi(\widetilde{u},1) \in\,]c_{k-1}, c_k[$ *if* $k < 0$. *In particular, if* (i_6) *holds for every* $k \in \mathbb{Z}$ *with* $|k| > k_0$, *then, for each* $n \in \mathbb{N}$ *with* $n > k_0$, *equation* (2) *has at least two solutions* u_n *and* w_n *such that* $\varphi(u_n,1) \in\,]c_n, c_{n+1}[$ *and* $\varphi(w_n,1) \in\,]c_{-(n+1)}, c_{-n}[$. *Moreover,* $\lim_{n\to\infty} \|u_n\| = \lim_{n\to\infty} \|w_n\| = \infty$.

Proof. Only the last assertion is still to be proved. If it is not true, we can find a bounded subsequence $(u_{k_j})_j$ of solutions of (2) with $\varphi(u_{k_j}) \in\,]c_{k_j}, c_{k_j+1}[$. So $\varphi(u_{k_j}) \to \infty$ as $j \to \infty$. Thus we get a contradiction, as the sequence $(u_{k_j})_j$ is precompact. Similarly, one proves the claim for $(w_n)_n$. □

3. Superlinear second order equations with nonhomogeneous Dirichlet conditions

We now want to apply the abstract theory to the problem

$$u''(t) + f(u(t)) = p(t, u(t), u'(t)), \qquad u(a) = A,\ u(b) = B, \tag{5}$$

where $A, B \in \mathbb{R}$, f has superlinear growth at infinity, i.e.,

$$\lim_{|x|\to\infty} f(x)/x = \infty, \tag{6}$$

and $p : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and has at most a linear growth with respect to the last two variables, that is,

$$\exists K > 0 : \quad |p(t, x, y)| \le K(1 + |x| + |y|), \quad \forall t \in [a, b],\ (x, y) \in \mathbb{R}^2. \tag{7}$$

Without loss of generality we can assume that $f(u)u > 0$ for $u \ne 0$, moving if necessary a bounded term from f to p.

Problem (5) will be solved via the continuation principle described in Lemma 2. To this end, we consider the homotopy

$$\begin{cases} u''(t) + h(u(t), \lambda) = \lambda p(t, u(t), u'(t)), \\ u(a) = \lambda A, \quad u(b) = \lambda B, \quad \lambda \in I, \end{cases} \tag{8}$$

where

$$h(x, \lambda) = \lambda f(x) + (1 - \lambda) g(x), \qquad \lambda \in I, \tag{9}$$

and $g : \mathbb{R} \to \mathbb{R}$ is a smooth function which is also odd and satisfies the following conditions:

$$g(x) > 0 \quad \text{for } x > 0, \quad \lim_{x\to\infty} \frac{g(x)}{x} = \infty,$$
$$\frac{d}{dx}\left(\frac{g(x)}{x}\right) > 0 \qquad \text{for } x > 0.$$

An adequate choice for this auxiliary function is $g(x) = x^3$.

Clearly, the parametrized problem (8) joins (5) to the autonomous ordinary differential equation with the homogeneous Dirichlet boundary conditions

$$u'' + g(u) = 0, \qquad u(a) = 0 = u(b). \tag{10}$$

In what follows, we have to consider the time-map $\tau(s)$, which is the time needed for a solution (u, v) of the planar system $u' = v$, $v' = -g(u)$ to move from the point $(0, s)$ to the point $(0, -s)$ crossing once the half-plane $u > 0$. The maximum value $m(s) > 0$ reached by such a solution u is such that $2G(m(s)) = s^2$ and, by the oddness of g, the time $\tau(s)$ is the same as that taken by a solution to move from $(0, -s)$ to $(0, s)$ across the half-plane $u < 0$, with u reaching its minimum value $-m(s)$. From the energy integral associated with (10), we get

$$\tau(s) = 2 \int_0^{m(s)} \frac{dx}{\sqrt{s^2 - 2G(x)}} \qquad \text{for } s > 0,$$

where $G(x) = \int_0^x g(\xi)\, d\xi$. In [10, Th. 8] (see also [11, Th. 1.3.2]) it is proved that the assumption $g(x)/x$ increasing implies that $s \mapsto \tau(s)$ is decreasing, and using also the superlinear growth condition for g, we find that the continuous map $s \mapsto \tau(s)$ satisfies $\lim_{s\to\infty} \tau(s) = 0$. Hence, an elementary analysis shows that there exists $n_0 \in \mathbb{Z}^+$ such that problem (10) has two sequences of solutions $(\widetilde{u}_n)_n$ and $(-\widetilde{u}_n)_n$ with $n \ge n_0$ with $\widetilde{u}_n$ and $-\widetilde{u}_n$ each having $n - 1$ zeros in $]a, b[$

and such that $\widetilde{u}_n'(a) = \widetilde{s}_n > 0$, with $\widetilde{s}_n \to \infty$ as $n \to \infty$. Actually, to find $\widetilde{u}_n$, we only have to find $\widetilde{s}_n$ such that $\tau(\widetilde{s}_n) = (b-a)/n$ and then define $\widetilde{u}_n$ as the solution of $u'' + g(u) = 0$ with $u(a) = 0$ and $u'(a) = \widetilde{s}_n$.

Our goal now is to prove that a similar result can be derived for problem (5) using Lemma 2. To this end, let us consider the space $C^1([a,b])$ with the usual C^1 norm $|\cdot|_{1,\infty}$, and its subspace $C^1_0([a,b]) = \{u \in C^1([a,b]) : u(a) = 0 = u(b)\}$. In the sequel we identify $C^1([a,b])$ with $C^1_0([a,b]) \times \mathbb{R}^2$ and $C^1_0([a,b])$ with $C^1_0([a,b]) \times \{(0,0)\}$.

Problem (8) can easily be written as an abstract equation of type (1) for $X = C^1([a,b])$ and $Z = C([a,b]) \times \mathbb{R}^2$, by defining

$$Lu = (-u'', u(a), u(b)), \qquad N(u,\lambda) = (\ell(\,\cdot\,, u, u', \lambda), \lambda A, \lambda B)$$

where we have set

$$\ell(t,x,y,\lambda) = h(x,\lambda) - \lambda p(t,x,y).$$

Note that when the coincidence degree is defined with respect to some open set $\Omega \subset X$, we have

$$D_L(L - N(\,\cdot\,,\lambda), \Omega) = \deg_{\mathrm{LS}}(I_X - L^{-1}N(\,\cdot\,,\lambda), \Omega, 0),$$

where $\deg_{\mathrm{LS}}$ is the Leray–Schauder degree [7], [8] and I_X is the identity operator in X. In this setting, for $\lambda = 0$, we have $L^{-1}N(\,\cdot\,,0) : X \to Y = C^1_0([a,b])$ and therefore the "reduction property" of the Leray–Schauder degree implies

$$\deg_{\mathrm{LS}}(I_X - L^{-1}N(\,\cdot\,,0), \Omega, 0) = \deg_{\mathrm{LS}}(I_Y - L^{-1}N(\,\cdot\,,0)|_Y, \Omega \cap Y, 0).$$

From this, we can easily conclude that, provided that the degree is defined,

$$D_L(L - N(\,\cdot\,,0), \Omega) = D_L(L_0 - N_0, \Omega \cap Y), \tag{11}$$

where we have set $L_0 u = -u''$ and $N_0(u) = g(u(\,\cdot\,))$ for $u \in C^1_0([a,b])$ (cf. [8]).

Consistently with the notations of Section 2, we denote here by $\Sigma^* \subset C^1([a,b]) \times I$ the set of the solutions (u,λ) of the boundary value problem (8), which, from now on, will be regarded as a translation of the operator equation (1) into the space $X = C^1([a,b])$. With a view to using Lemma 2, we introduce the following functional φ which is a slight modification of that considered in [4].

Let

$$\begin{aligned} \delta : \mathbb{R}^2 \to \mathbb{R}, &\qquad (x,y) \mapsto \min\left\{1, \frac{1}{x^2+y^2}\right\}, \\ \eta : \mathbb{R} \to \mathbb{R}, &\qquad x \mapsto \min\{1, \max\{-1, x\}\}. \end{aligned}$$

Then we define the continuous functional φ on $C^1([0,1]) \times I$ by

$$\varphi(u,\lambda) = \eta(u'(a))\left|\frac{1}{\pi}\int_0^1 [u'(t)^2 + u(t)\ell(t,u(t),u'(t),\lambda)]\delta(u(t),u'(t))\,dt\right|. \tag{12}$$

To describe the meaning of $\varphi(u,\lambda)$, suppose that (u,λ) is a solution of (8) such that

$$u(t)^2 + u'(t)^2 \geq R^2 \geq 1 + A^2 \qquad \text{for all } t \in [a,b]. \tag{13}$$

In this case, we get, letting $v(t) = u'(t)$,

$$\begin{aligned}\varphi(u,\lambda) &= \operatorname{sgn}(u'(a))\left|\frac{1}{\pi}\int_a^b \frac{v(t)u'(t)-u(t)v'(t)}{u(t)^2+v(t)^2}\,dt\right| \\ &= \operatorname{sgn}(u'(a))\left|\frac{1}{\pi}\int_a^b \left(-\frac{d}{dt}\arctan\frac{v(t)}{u(t)}\right)dt\right| \\ &= \operatorname{sgn}(u'(a))\left|\frac{1}{\pi}\int_a^b \left(\frac{d}{dt}\arctan\frac{u(t)}{v(t)}\right)dt\right|\end{aligned}$$

By the assumptions on f, g and p, it follows that $y^2 + \ell(t,x,y,\lambda)x \to \infty$ as $x^2+y^2 \to \infty$, uniformly with respect to $t \in [a,b]$ and $\lambda \in [0,1]$. Therefore, there is

$$d > \max\{|A|,|B|\}$$

such that $y^2 + \ell(t,x,y,\lambda)x > 0$ for all (x,y) such that $x^2+y^2 \geq d^2$ and each $t \in [a,b]$, $\lambda \in [0,1]$. Thus, if $R \geq (1+d^2)^{1/2}$ in (13), we obtain $v(t)u'(t) - u(t)v'(t) > 0$ for all $t \in [a,b]$. Hence assuming (13) satisfied for such an R, we see that the above integrands are positive. We now evaluate $|\varphi(u,\lambda)|$ for $(u,\lambda) \in \Sigma^*$ and (u,u') satisfying (13) with R sufficiently large. To this end, it is useful to distinguish some cases as follows.

Let t_1 and t_2 in $[a,b]$ be two consecutive zeros of the solution u. Since $v(t) = u'(t) \neq 0$ for any t such that $u(t) = 0$, we deduce that $u(t) > 0$ in $]t_1,t_2[$ implies $u'(t_1) > 0 > u'(t_2)$ and, respectively, $u(t) < 0$ in $]t_1,t_2[$ implies $u'(t_1) < 0 < u'(t_2)$. Thus we obtain

$$\frac{1}{\pi}\int_{t_1}^{t_2} \frac{v(t)u'(t)-u(t)v'(t)}{u(t)^2+v(t)^2}\,dt = 1. \tag{14}$$

On the other hand, if s_1 and s_2 in $[a,b]$, with $s_1 < s_2$, are such that $|u(t)| \leq d$ for all $t \in [s_1,s_2]$, then it follows from (13) that $|u'(t)| \geq (R^2-d^2)^{1/2}$ for all $t \in [s_1,s_2]$, and hence, with easy computations, we obtain

$$\left|\frac{1}{\pi}\int_{s_1}^{s_2} \frac{v(t)u'(t)-u(t)v'(t)}{u(t)^2+v(t)^2}\,dt\right| \leq \frac{2}{\pi}\arctan(d/(R^2-d^2)^{1/2}).$$

Finally, if σ_1 and σ_2 in $[a,b]$, with $\sigma_1 < \sigma_2$, are such that either $u(\sigma_1) = d$, $u'(\sigma_1) > 0$, $u(\sigma_2) = d$, $u'(\sigma_2) < 0$ and $u(t) > d$ for all $t \in]\sigma_1,\sigma_2[$, or, respectively, $u(\sigma_1) = -d$, $u'(\sigma_1) < 0$, $u(\sigma_2) = -d$, $u'(\sigma_2) > 0$ and $u(t) < -d$ for all $t \in]\sigma_1,\sigma_2[$, then

$$\frac{1}{\pi}\int_{\sigma_1}^{\sigma_2} \frac{v(t)u'(t)-u(t)v'(t)}{u(t)^2+v(t)^2}\,dt = 1 - \frac{2}{\pi}\arctan(d/(R^2-d^2)^{1/2}).$$

Now, set

$$R_0 = (1 + d^2(1 + (\tan(\pi/16))^{-2}))^{1/2} \tag{15}$$

and suppose that (13) holds with $R \geq R_0$. We distinguish the following three possibilities:

Case 1. *$|u(t)| \leq d$ for all $t \in [a, b]$.* In this case, we have

$$|\varphi(u, \lambda)| \leq \frac{1}{8} < \frac{1}{4}.$$

Case 2. *$|u|_\infty > d$ and the set $\{t \in [a, b] : |u(t)| = d\}$ has exactly two points.* In this case, if we call these points σ_1 and σ_2, with $a < \sigma_1 < \sigma_2 < b$, then it follows that $\operatorname{sgn}(u'(a)) = \operatorname{sgn}(u'(\sigma_1)) = -\operatorname{sgn}(u'(\sigma_2)) = -\operatorname{sgn}(u'(b))$ and $|u(t)| \leq d$ for all $t \in [a, \sigma_1] \cup [\sigma_2, b]$. We also point out that there are at most two points, one between a and σ_1 and the other between σ_2 and b, where $u(\,\cdot\,)$ may vanish. Hence we have

$$|\,|\varphi(u, \lambda)| - 1| \leq \frac{1}{4}.$$

Case 3. *$|u|_\infty > d$ and the set $\{t \in [a, b] : |u(t)| = d\}$ has more than two points.* In this case, by taking into account that $u'(t) \neq 0$ when $|u(t)| \leq d$, as well as $|u(t)| > d$ when $u'(t) = 0$, it follows that there is a first zero of u in $]a, b[$ where u' has a sign opposite to that of $u'(a)$ and a last zero of u in $]a, b[$ where u' has a sign opposite to that of $u'(b)$. Call these two zeros a_1 and b_1, respectively, observe that $a_1 \leq b_1$ and consider the set of zeros of u which are between a_1 and b_1. Since the number of zeros of u is finite (as they are all simple), we can enumerate them as $a_1 = t_1 < \ldots < t_m = b_1$ with the convention that we assume $m = 1$ when $a_1 = b_1$. Note that $u'(a)u'(a_1) < 0$, $u'(b_1)u'(b) < 0$ and, if $m \geq 2$, $u'(t_{i-1})u'(t_i) < 0$ for $i = 2, \ldots, m$.

Now we have

$$\frac{1}{\pi}\int_{t_1}^{t_m} \frac{v(t)u'(t) - u(t)v'(t)}{u(t)^2 + v(t)^2}\,dt = m - 1,$$

$$1 - \frac{1}{8} \leq \frac{1}{\pi}\int_{a}^{a_1} \frac{v(t)u'(t) - u(t)v'(t)}{u(t)^2 + v(t)^2}\,dt \leq 1 + \frac{1}{8}$$

and

$$1 - \frac{1}{8} \leq \frac{1}{\pi}\int_{b_1}^{b} \frac{v(t)u'(t) - u(t)v'(t)}{u(t)^2 + v(t)^2}\,dt \leq 1 + \frac{1}{8}.$$

Thus we conclude that

$$|\,|\varphi(u, \lambda)| - (m + 1)| \leq \frac{1}{4}.$$

At this point, we just recall some properties already proved in [1], [2], [4] and [9] for homogeneous boundary conditions. The change to be done in order to adapt these results for the nonhomogeneous two-point boundary condition considered here is an easy exercise.

Lemma 3 (The "Elastic Property", see [1, Lemma 3]). *For each* $R_1 > 0$ *there is* $R_2 \geq R_1$ *such that for each* $(u, \lambda) \in \Sigma^*$,

$$|u|_{1,\infty} \geq R_2 \Rightarrow |u(t)|^2 + |u'(t)|^2 \geq R_1^2, \ \forall t \in [a, b].$$

Lemma 4 (Fast oscillations of large solutions, see [1, Lemma 6]). *For each* $N > 0$ *there is* $R_1(N) > 0$ *such that for all* $(u, \lambda) \in \Sigma^*$,

$$\min_{t \in [a,b]} (|u(t)|^2 + |u'(t)|^2) \geq R_1(N)^2 \Rightarrow |\varphi(u, \lambda)| \geq N.$$

From Lemmas 3 and 4, using the above properties of the function φ we obtain the following.

Proposition 1. *The functional* φ *defined in* (12) *satisfies* (i_4) *and* (i_5) *of Lemma* 2 *with respect to the doubly infinite sequence* $(c_k)_{k \in \mathbb{Z}}$ *with* $c_0 = 3/8$ *and* $c_k = (|k| - (1/2)) \operatorname{sgn}(k)$ *for all* $k \neq 0$.

Proof. Use the elastic property in order to find $r_0 \geq R_0$ with R_0 defined in (15) such that if $(u, \lambda) \in \Sigma^*$ with $|u|_{1,\infty} \geq r_0$, then $|u(t)|^2 + |u'(t)|^2 \geq R_0^2$ for all $t \in [a, b]$. Hence we are in one of the three cases considered above and therefore $|\,|\varphi(u, \lambda)| - n| \leq 1/4$ for some $n \in \mathbb{N}$ (respectively $n = 0$ in Case 1, $n = 1$ in Case 2 and $n > 1$ in Case 3). Then $|\varphi(u, \lambda)| \neq |k| - 1/2$ for all $k \in \mathbb{Z} \setminus \{0\}$ and also $\varphi(u, \lambda) \neq 3/8$. Thus we have ($\mathrm{i}_4$) for $R = r_0$.

Now, take any $n \in \mathbb{Z}^+$ and consider the number $N = n$. By Lemmas 3 and 4, there is $R_2(N) = R_2(R_1(N))$ such that if $(u, \lambda) \in \Sigma^*$ and $|u|_{1,\infty} \geq R_2(N)$, then $|\varphi(u, \lambda)| \geq N = n > n - 1/2$. Hence, $\varphi^{-1}(\,]c_{-n}, c_n[\,) \cap \Sigma^* = \varphi^{-1}(\,]-n + 1/2, n - 1/2[\,) \cap \Sigma^*$ is contained in the ball of center 0 and radius $R_2(N)$ of the space $C^1([a, b])$ and thus (i_5) is proved. □

For r_0 given in the proof of Proposition 1, we define

$$\varrho_0 = \sup\{|\varphi(u, \lambda)| : (u, \lambda) \in \Sigma^*, |u|_{1,\infty} \leq r_0\}$$

and denote by $[\varrho_0]$ its integer part. Note that even if the definition of ϱ_0 appears fairly "abstract", it is not difficult to provide concrete upper estimates for this constant, if necessary, in terms of the function ℓ and the constant d.

Consider now the autonomous equation (10) and let $\widetilde{u}$ be a solution of it with $|\widetilde{u}|_{1,\infty} > r_0$. Let $\widetilde{k} \in \mathbb{N}$ be the number of zeros of $\widetilde{u}$ in $]a, b[$ (possibly $\widetilde{k} = 0$). At this point, we are in a position to define a positive integer k_0 for which (i_6) will be satisfied. Namely, we set

$$k_0 = 1 + \widetilde{k} + [\varrho_0] \tag{16}$$

and so we have:

Proposition 2. *For any $k \in \mathbb{Z}$ with $|k| > k_0$,*

$$|D_L(L - N(\,\cdot\,, 0), (\mathcal{O}^k)_0)| = 1.$$

Proof. First of all, we observe that the constant k_0 defined in (16) satisfies condition (3) of Section 2.

Let $|k| > k_0$ and consider the open set $\mathcal{O}^k$ defined in (4). Let also $\Sigma^k = \overline{\mathcal{O}^k} \cap \Sigma^* = \mathcal{O}^k \cap \Sigma^*$ (by (i_4)). By definition, $(\Sigma^k)_0$ is the set of solutions u of (10) such that $k - 1/2 < \varphi(u, 0) < k + 1/2$. Hence from $|\eta(u'(a))| = 1$ and (14) it follows that $\varphi(u, 0) = k$ and this means that u is a solution of (10) with $\mathrm{sgn}(u'(a)) = \mathrm{sgn}(k)$ and u having exactly $|k| - 1$ zeros in $]a, b[$. By the previous discussion about the strict monotonicity of the time-map, we know that there is only one solution with this property, and this solution, in the notations introduced above, is $\mathrm{sgn}(k)\widetilde{u}_{|k|}$. Thus we see that

$$(\Sigma^k)_0 = \{\widetilde{u}_k\} \qquad \text{for each } k \in \mathbb{Z} \text{ with } |k| > k_0,$$

where we have set

$$\widetilde{u}_k = \mathrm{sgn}(k)\widetilde{u}_{|k|} = -\widetilde{u}_{|k|} \qquad \text{for } k < 0.$$

Furthermore, we observe that from (11) we have

$$D_L(L - N(\,\cdot\,, 0), (\mathcal{O}^k)_0) = D_L(L_0 - N_0, (\mathcal{O}^k)_0 \cap Y)$$

with $Y = C^1_0([a, b])$. Hence, in order to prove our result, it is sufficient that we find an open bounded set Ω in $C^1_0([a, b])$ with $\widetilde{u}_k \in \Omega \subset (\mathcal{O}^k)_0 \cap Y$ where we can show that $D_L(L_0 - N_0, \Omega) \neq 0$. To this end, we proceed as follows.

For $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta$, set

$$\Omega^\beta_\alpha(+) = \{u \in C^1_0([a, b]) : \alpha^2 < u'(t)^2 + 2G(u(t)) < \beta^2,\ u'(a) > 0\}$$

and

$$\Omega^\beta_\alpha(-) = \{u \in C^1_0([a, b]) : \alpha^2 < u'(t)^2 + 2G(u(t)) < \beta^2,\ u'(a) < 0\}.$$

Note that $\Omega^\beta_\alpha(+)$ and $\Omega^\beta_\alpha(-)$ are open bounded subsets of $C^1_0([a, b])$ with

$$\Omega^\beta_\alpha(+) \cup \Omega^\beta_\alpha(-) = \{u \in C^1_0([a, b]) : \alpha^2 < u'(t)^2 + 2G(u(t)) < \beta^2\}.$$

Now, for each $k \in \mathbb{Z}$ with $|k| > k_0$, consider $\widetilde{s}_k = \widetilde{u}'_k(a)$ and observe that $|\widetilde{s}_k| > |\widetilde{u}'(a)|$, as a consequence of (16) and the strict monotonicity of $\tau(\,\cdot\,)$. From the above, we know that $\widetilde{s}_{-k} = -\widetilde{s}_k$,

$$\varphi(\widetilde{u}_k, 0) = k \quad \text{and} \quad \tau(|s_k|) = \frac{b - a}{|k|}.$$

Then there is $\varepsilon_{|k|} > 0$ sufficiently small such that, if we set $\alpha_{|k|} = \widetilde{s}_{|k|} - \varepsilon_{|k|} > 0$ and $\beta_{|k|} = \widetilde{s}_{|k|} + \varepsilon_{|k|}$, then the open bounded set

$$\mathcal{V}^k = \begin{cases} \Omega_{\alpha_{|k|}}^{\beta_{|k|}}(+) & \text{if } k > 0, \\ \Omega_{\alpha_{|k|}}^{\beta_{|k|}}(-) & \text{if } k < 0, \end{cases}$$

satisfies $\widetilde{u}_k \in \mathcal{V}^k \subset \overline{\mathcal{V}^k} \subset (\mathcal{O}^k)_0 \cap C_0^1([a,b])$.

Now, for $s \in \mathbb{R}$, let $u(\,\cdot\,,s)$ be the solution of $u'' + g(u) = 0$ with $u(a) = 0$ and $u'(a) = s$ and denote by $U : s \mapsto u(b,s)$ the shooting map. By a result proved in [4], [9] (see also [1]) and based on a duality theorem from [6, Th. 29.4]), we know that $\mathcal{V}^k \subset C_0^1([a,b])$ and $]\widetilde{s}_k - \varepsilon_{|k|}, \widetilde{s}_k + \varepsilon_{|k|}[\, \subset \mathbb{R}$ have a common core (cf. [6]) with respect to problem (10) and therefore,

$$D_L(L_0 - N_0, \mathcal{V}^k) = \deg_{\mathrm{B}}(U,]\widetilde{s}_k - \varepsilon_{|k|}, \widetilde{s}_k + \varepsilon_{|k|}[, 0),$$

where $\deg_{\mathrm{B}}$ denotes the Brouwer degree.

On the other hand, since we know that $\tau(\alpha_{|k|}) > \tau(|\widetilde{s}_k|) = \tau(\widetilde{s}_{|k|}) > \tau(\beta_{|k|})$ and $|\,|\varphi(u(\,\cdot\,,\pm\alpha_{|k|}),0)| - |k|\,| < 1/2$, as well as $|\,|\varphi(u(\,\cdot\,,\pm\beta_{|k|}),0)| - |k|\,| < 1/2$, for $\overline{\mathcal{V}^k} \subset (\mathcal{O}^k)_0 \cap C_0^1([a,b])$, we arrive at the following description of the behaviour of the shooting map:

- *For* $k < 0$: The function $q(t) := u(t, \widetilde{s}_k - \varepsilon_{|k|}) = u(t, -\beta_{|k|})$ is such that $q(a) = 0$ and $q'(a) = \widetilde{s}_k - \varepsilon_{|k|} < \widetilde{s}_k$. Also, $q(t)$ has exactly $|k|$ zeros in $]a,b]$ and $q(b) \neq 0$. Moreover, $q(b) > 0$ or $q(b) < 0$ according as k is odd or even. The function $r(t) := u(t, \widetilde{s}_k + \varepsilon_{|k|}) = u(t, -\alpha_{|k|})$ is such that $r(a) = 0$ and $r'(a) = \widetilde{s}_k + \varepsilon_{|k|} > \widetilde{s}_k$. Also, $r(t)$ has exactly $|k| - 1$ zeros in $]a,b]$ and $r(b) \neq 0$. Moreover, $r(b) < 0$ or $r(b) > 0$ according as k is odd or even.

Thus we conclude that

$$\deg_{\mathrm{B}}(U,]\widetilde{s}_k - \varepsilon_{|k|}, \widetilde{s}_k + \varepsilon_{|k|}[, 0) = (-1)^k.$$

- *For* $k > 0$: The function $z(t) := u(t, \widetilde{s}_k - \varepsilon_k) = u(t, \alpha_k)$ is such that $z(a) = 0$, $z'(a) = \widetilde{s}_k - \varepsilon_k < \widetilde{s}_k$. Also, $z(t)$ has exactly $k - 1$ zeros in $]a,b]$ and $z(b) \neq 0$. Moreover, $z(b) > 0$ or $z(b) < 0$ according as k is odd or even. The function $y(t) := u(t, \widetilde{s}_k + \varepsilon_k) = u(t, \beta_k)$ is such that $y(a) = 0$, $y'(a) = \widetilde{s}_k + \varepsilon_k > \widetilde{s}_k$. Also, $y(t)$ has exactly k zeros in $]a,b]$ and $y(b) \neq 0$. Moreover, $y(b) < 0$ or $y(b) > 0$ according as k is odd or even.

Thus we conclude that

$$\deg_{\mathrm{B}}(U,]\widetilde{s}_k - \varepsilon_{|k|}, \widetilde{s}_k + \varepsilon_{|k|}[, 0) = (-1)^k.$$

Finally, we put together all the previous relations about the coincidence degree, the Leray–Schauder degree and the Brouwer degree to obtain

$$D_L(L - N(\,\cdot\,,0), (\mathcal{O}^k)_0) = (-1)^k, \qquad \forall k \in \mathbb{Z} \text{ with } |k| > k_0. \tag{17}$$

In this manner Proposition 2 is proved. □

Remark 1. In previous articles (see [4] and [9]) the functional considered was of the form $|\varphi(u,0)|$ on large solutions of equation (10). Thus, in those papers it was possible to consider $\mathcal{O}^k$ only for $k > k_0$. Correspondingly, the sets $\mathcal{O}^k$ (which were "larger" than the sets considered in the present article) contained both $\widetilde{u}_k$ and $-\widetilde{u}_k$. Hence, if we apply formula (17) to the situation considered in [4] and [9], using the additivity property of the coincidence degree (cf. [8]), we obtain $D_L = 2(-1)^k$. This is precisely the result found in [4] and [9].

At this point we have checked that all the conditions of Lemma 2 are satisfied and we can apply it to equation (5) in order to obtain the following result.

Theorem 1. *Let f and p satisfy* (6) *and* (7), *respectively. Then there is $k_0 \in \mathbb{Z}^+$ such that for each $n > k_0$, the boundary value problem* (5) *has at least two solutions u_n and w_n with $u_n'(a) > 0$ and $w_n'(a) < 0$, such that*

$$\lim_{n\to\infty} \min_{t\in[a,b]} (|u_n(t)| + |u_n'(t)|) = \lim_{n\to\infty} \min_{t\in[a,b]} (|w_n(t)| + |w_n'(t)|) = \infty. \tag{18}$$

These solutions have the following nodal properties:

- *For n odd: $u_n'(b) < 0$ and, moreover, u_n has exactly $n+1$ zeros in $[a,b]$ if $A \le 0$ and $B \le 0$; u_n has exactly n zeros in $[a,b]$ if $A \le 0$ and $B > 0$ or if $A > 0$ and $B \le 0$; u_n has exactly $n-1$ zeros in $[a,b]$ if $A > 0$ and $B > 0$.*
- *For n even: $u_n'(b) > 0$ and, moreover, u_n has exactly $n+1$ zeros in $[a,b]$ if $A \le 0$ and $B \ge 0$; u_n has exactly n zeros in $[a,b]$ if $A \le 0$ and $B < 0$ or if $A > 0$ and $B \ge 0$; u_n has exactly $n-1$ zeros in $[a,b]$ if $A > 0$ and $B < 0$.*
- *For n odd: $w_n'(b) > 0$ and, moreover, w_n has exactly $n+1$ zeros in $[a,b]$ if $A \ge 0$ and $B \ge 0$; w_n has exactly n zeros in $[a,b]$ if $A \ge 0$ and $B < 0$ or if $A < 0$ and $B \ge 0$; u_n has exactly $n-1$ zeros in $[a,b]$ if $A < 0$ and $B < 0$.*
- *For n even: $w_n'(b) < 0$ and, moreover, w_n has exactly $n+1$ zeros in $[a,b]$ if $A \ge 0$ and $B \le 0$; w_n has exactly n zeros in $[a,b]$ if $A \ge 0$ and $B > 0$ or if $A < 0$ and $B \le 0$; u_n has exactly $n-1$ zeros in $[a,b]$ if $A < 0$ and $B > 0$.*

All the zeros of u_n and w_n are simple and all the local maxima or minima of u_n and w_n are strict. Between any two consecutive zeros of a solution, as well as between a and the first zero or between the last zero and b, there is only one critical point of the solution.

Proof. The existence of sequences $(u_n)_n$ and $(w_n)_n$ of solutions of (5) with

$$\lim_{n\to\infty} |u_n|_{1,\infty} = \lim_{n\to\infty} |w_n|_{1,\infty} = \infty \tag{19}$$

and with $n-1/2 < \varphi(u_n,1) < n+1/2$ and $-n-1/2 < \varphi(w_n,1) < -n+1/2$ follows from Lemma 2. Then the elastic property yields (18) from (19). Concerning the nodal properties of the solutions, we only discuss the case of the u_n's, that of the w_n's being completely symmetric.

Let $n > k_0$ with k_0 given in (16). Since $n \geq 2$, we must necessarily be in Case 3 analysed before. This means that $n = m+1$, where $m \geq 1$ is the number of the zeros of u_n between a_1 and b_1 (counting a_1 and b_1 as well), recalling that a_1 is the first zero of u_n in $]a,b[$ where u' has sign opposite to $\mathrm{sgn}(u'(a))$ and b_1 is the last zero of u_n in $]a,b[$ where u' has sign opposite to $\mathrm{sgn}(u'(b))$. By the definition of φ we know that $u_n'(a) > 0$. Hence if $u_n(a) \leq 0$, there is another zero of u_n before a_1, while if $u_n(a) > 0$, there are no other zeros of u_n before a_1. Thus, in the former case, u_n has n zeros in $[a,b_1]$, and in the latter, u_n has $n-1$ zeros in $[a,b_1]$. On the other hand, if $u_n'(b) > 0$ and $u(b) \geq 0$ or if $u_n'(b) < 0$ and $u(b) \leq 0$, there must be just another zero of u_n after b_1, while in all the other cases (i.e. for $u_n'(b) > 0$ and $u(b) < 0$ or $u_n'(b) < 0$ and $u(b) > 0$) no other zeros appear. Finally, we have to decide whether $u_n'(b) > 0$ or $u_n'(b) < 0$ knowing that $u_n'(a) > 0$. By the above discussion, it is clear that $u_n'(a)u_n'(b) < 0$ or $u_n'(a)u_n'(b) > 0$ according as n is odd or even. Putting together all these remarks, we are able to describe precisely the nodal properties of the solutions (the reader can help him/her-self drawing here a picture corresponding to the possible cases).

The last assertions about the local maxima and minima follow from the fact that if $|u(t)| \geq d$ is large and $u'(t) = 0$, with $u(\cdot)$ any solution of (5), then $u(t)u''(t) < 0$ (see the above discussion after the definition of φ for the meaning of the functional we have introduced). □

Example 1. An elementary analysis of the boundary value problem

$$u'' + u\max\{K^2, u^2\} = 0, \qquad u(0) = A,\ u(\pi) = B, \tag{20}$$

shows that Theorem 1 is sharp with respect to all its conclusions. Namely, we cannot hope to have nontrivial solutions for all $n \geq 1$, but only for n sufficiently large, say $n > k_0$. To see this, just take $A = B = 0$ and choose $K \in \mathbb{Z}^+$ large enough; it is then straightforward to check that there are no solutions of (20) with $n < K$ zeros in $[0,\pi[$. Moreover, we cannot hope to have more than two solutions with n zeros in $[0,\pi[$ when $A = B = 0$ and, finally, varying suitably A and B, we easily find all the possible nodal properties described above.

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Manuscript received May 23, 1995

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