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MINKOWSKI PROBLEMS FOR COMPLETE NONCOMPACT CONVEX HYPERSURFACES

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Dedicated to Professor L. Nirenberg on the occasion of his 70th birthday

We present some sufficient conditions on the existence of a complete non-compact convex hypersurface whose Gauss curvature is equal to a prescribed function on the unit sphere. By the Legendre transform this problem is reduced to the solvability of the Monge–Ampère equation subject to certain boundary conditions.

Introduction

Let X be a compact, strictly convex C^2-hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. The Gauss map of X maps the hypersurface one-to-one and onto the unit n-sphere S^n. One may parametrize X by the inverse of the Gauss map. Consequently, the Gauss curvature can be regarded as a function on S^n. The classical Minkowski problem asks conversely when a positive function K on S^n is the Gauss curvature of a compact convex hypersurface. It turns out that a necessary and sufficient condition is

$$\int_{S^n} \frac{x_i}{K(x)}\,dx = 0, \qquad i = 1, \ldots, n+1. \tag{1}$$

Furthermore, convex hypersurfaces with the same K (as functions of the outer normal) are identical up to translations. For a detailed discussion on this problem one may consult, for instance, [CY] and [P1].

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The same problem makes perfectly sense for complete, noncompact, convex hypersurfaces. Now the spherical image of such a hypersurface is an open convex subset of S^n contained in some hemisphere. We may ask: Given an open convex proper subset D of S^n and a positive function K in D, when can we find a complete convex hypersurface with spherical image D and Gauss curvature K? And when is it unique? In this paper we give some sufficient conditions for this to hold. Roughly speaking, the integrability condition (1) which ensures the closedness of X will be replaced by a certain decay condition on K at the boundary of D.

To proceed further we need to write down an equation for our problem. More precisely, let X be a complete, noncompact, strictly convex C^2-hypersurface in $\mathbb{R}^{n+1}$ and let D be its spherical image. By suitably rotating axes we may assume D satisfies one and exactly one of the following conditions:

(I) D is strictly contained in $S^n_- = \{x \in S^n : x_{n+1} < 0\}$,
(II) $D = S^n_-$,
(III) D is a proper subset of S^n_- and it is not strictly contained in any hemisphere.

We shall say X is of *type I, II, or III* according to whether (I), (II), or (III) holds. Notice that by our choice of coordinates, X is the graph of a convex function over a convex domain in the $(x_1, \dots, x_n)$-space.

The *support function* of X is defined by

$$H(x) = \sup_{p \in X} \langle p, x \rangle, \qquad x \in D,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^{n+1}$. It is well known that X can be recovered from H and

$$\det(\nabla_{ij} H + \delta_{ij} H) = 1/K \qquad \text{on } D$$

(∇ is the covariant differentiation on S^n). If we extend H to be a 1-homogeneous function over the cone $\{\lambda x : x \in D,\ \lambda > 0\}$, then for $\Omega = \{\lambda x : x \in D, \lambda > 0\} \cap \{x : x_{n+1} = -1\}$ and $u(x_1, \dots, x_n) = H(x_1, \dots, x_n, -1)$, this equation becomes

$$\det \nabla^2 u(x) = (1 + |x|^2)^{-(n+2)/2} K^{-1} \left(\frac{x, -1}{\sqrt{1 + |x|^2}} \right), \qquad x \in \Omega. \tag{2}$$

Whenever a convex solution of (2) is given, it determines X in the following way (see [CY] or [P1]): Let $\Omega^* = \nabla u(\Omega)$ and

$$u^*(\xi) = \sup\{\langle \xi, x \rangle - u(x) : x \in \Omega\}, \qquad \xi \in \Omega^*.$$

Then X is the graph $\{(\xi, u^*(\xi)) : \xi \in \Omega^*\}$, and its Gauss curvature is equal to K. Thus to solve the Minkowski problem we must solve (2). However, when is X complete?

When D is strictly contained in S^n_-, Ω is a bounded convex domain in $\mathbb{R}^n$. It is clear that X is complete if and only if $\Omega^* = \mathbb{R}^n$. We pose the following Minkowski problem for type I hypersurfaces: For given D and K, solve (2) subject to

$$|\nabla u(x)| \to \infty \qquad \text{as } x \to \partial\Omega, \tag{3}$$

$$u(x) = \phi(x), \qquad x \in \partial\Omega, \tag{4}$$

where ϕ is prescribed. Our first result is

Theorem A. *Let D be a uniformly convex C^2-domain strictly contained in S^n_-, K a positive function in $C^\alpha(D)$, $\alpha \in (0,1)$, and $\phi \in C^2(\partial D)$. Suppose there exist two positive functions h and g defined in $(0, r_0]$, $r_0 > 0$, satisfying*

(a) $\int_0^{r_0} h(t)\,dt = \infty$,
(b) $\int_0^{r_0} (\int_s^{r_0} g(t)\,dt)^{1/n}\,ds < \infty$,

so that

$$h(\operatorname{dist}(x, \partial D)) \le K^{-1}(x) \le g(\operatorname{dist}(x, \partial D))$$

near ∂D. Then the Minkowski problem (2)–(4) has a unique solution in $C^{2,\alpha}(D) \cap C(\overline{D})$.

The Minkowski problem for type I hypersurfaces was first studied by Pogorelov [P3]. He proved that if (i) $K(x)/\operatorname{dist}(x, \partial D)$ is bounded above and (ii) $K(x)/\operatorname{dist}(x, \partial D)^{n+1-\gamma}$, $\gamma > 0$, is bounded below, then the Minkowski problem has a unique solution. One can easily check that his result is contained in Theorem A.

Next we consider type II hypersurfaces. By integrating (2) we find that

$$|\Omega^*| = \int_D \frac{|x_{n+1}|}{K(x)}\,d\sigma. \tag{5}$$

We distinguish two cases: $|\Omega^*| < \infty$ and $|\Omega^*| = \infty$. The first case is easier. Since now Ω is the entire space, it is not appropriate to prescribe the boundary value. Instead we prescribe Ω^*.

Theorem B. *Let K be a positive function in $C^\alpha_{\mathrm{loc}}(S^n_-)$ and Ω^* a bounded uniformly convex $C^{2,\alpha}$-domain in $\mathbb{R}^n$ satisfying (5). Then there exists a convex hypersurface X, which is a graph over Ω^*, admitting K as its Gauss curvature. X is unique up to translation along the x_{n+1}-axis. Furthermore, X is complete if*

$$K(x) \le C x_{n+1}, \qquad -1 < x_{n+1} < 0. \tag{6}$$

Theorems A and B are nearly optimal when one restricts to spherically symmetric hypersurfaces. However, when it comes to type II hypersurfaces satisfying

$|\Omega^*| = \infty$, our result is less general. In [CW] we show that the equation

$$\det \nabla^2 u(x) = f(x), \qquad 0 < C_0 \le f \le C_1,\ x \in \mathbb{R}^n,$$

has infinitely many solutions subjecting to the normalization condition $u(0) = \nabla u(0) = 0$. From this one deduces

Theorem C. *Let K be a positive function in $C^{\alpha}_{\rm loc}(S^n_-)$ satisfying*

$$0 < C_0 \le K(x) x_{n+1}^{-n-2} \le C_1, \qquad x \in S^n_-.$$

Then there are infinitely many type II hypersurfaces whose Gauss curvature functions are equal to K.

Recently we have generalized this theorem by relaxing the condition on K to $0 < C_0 \le K(x) x_{n+1}^{-n-\gamma} \le C_1$ for $\gamma > 0$. The proof will be published elsewhere.

1. Type I hypersurfaces

In this section we prove Theorem A. First we point out that it suffices to produce a generalized solution u of (2) in $C(\overline{\Omega})$ satisfying (3) and (4). Its uniqueness follows from the comparison principle. Moreover, by [C1] and (3), u must be strictly convex and hence belongs to $C^{2,\alpha}(\Omega)$ according to [C2].

To simplify notation write

$$R(x) = (1+|x|^2)^{-(n+2)/2} K^{-1}\left(\frac{x, -1}{\sqrt{1+|x|^2}}\right), \qquad x \in \Omega,$$

and assume the boundary value ϕ belongs to $C^2(\overline{\Omega})$ and is convex. Let $\Omega(r) = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r)$. For $r_0 > 0$ small depending on the geometry of Ω, $\Omega(r)$ is still uniformly convex for $r \in (0, 2r_0)$. For $x \in \Omega_{r_0}$, x can be represented uniquely by $x_b + dn(x_b)$, where $x_b \in \partial\Omega$, $d = \operatorname{dist}(x, \partial D)$, and $n(x_b)$ is the unit inner normal at x_b. For a function f defined near $\partial\Omega$ we write $f(x) = f(x_b, d)$. The proof of Theorem A relies on the following two lemmas.

Lemma 1. *Suppose there exists a positive function g satisfying* (b) *of Theorem* A *such that $R(x_b, d) \le g(d)$. Then* (2) *admits a unique generalized solution u in $C(\Omega)$ and $u = \phi$ on $\partial\Omega$.*

Proof. For $x = x_b + dn(x_b)$ in $\Omega \backslash \Omega_{r_0}$ we define

$$v(x) = \varrho(d) = -\int_0^d \left(\int_s^{r_0} g(t)\,dt\right)^{1/n} ds.$$

We claim

$$\det \nabla^2 v(x) = \prod_{i=1}^{n-1} \frac{k_i(x_b)}{1 - k_i(x_b) d} (-\varrho'(d))^{n-1} \varrho''(d) \tag{7}$$

in $\Omega\backslash\Omega_{r_0}$, where $k_i(x_b)$ $(i = 1, \dots, n-1)$ are the principal curvatures of $\partial\Omega$ at x_b.

To prove (7) we first observe that the Monge–Ampère operator is invariant under a rigid motion. So we can assume x_b is the origin, the positive x_n-axis lies in the inner normal direction, and moreover, the x_i-axes $(i = 1, \dots, n-1)$ are the principal directions of $\partial\Omega$ at the origin. For the function $v(x) = \varrho(d)$,

$$\frac{\partial^2 v}{\partial x_i \partial x_j} = \varrho'' \frac{\partial d}{\partial x_i}\frac{\partial d}{\partial x_j} + \varrho' \frac{\partial^2 d}{\partial x_i \partial x_j}.$$

By a direct computation,

$$\frac{\partial d}{\partial x_i} = 0, \quad i = 1, \dots, n-1, \quad \frac{\partial d}{\partial x_n} = 1,$$
$$\frac{\partial^2 d}{\partial x_i \partial x_j} = \frac{-k_i(0)}{1 - k_i(0)d}\delta_{ij}, \quad 1 \le i, j \le n-1, \quad \frac{\partial^2 d}{\partial x_i \partial x_n} = 0, \quad i = 1, \dots, n,$$

at $x = (0, \dots, 0, d)$. Hence (7) follows.

Now observe that $v(x) = \text{const}$ on $\partial\Omega_r$ for $r \in [0, r_0]$. We can extend v to Ω_{r_0} so that $\det \nabla^2 v = \varepsilon > 0$ in Ω_{r_0}. For ε small, v is uniformly convex in Ω. Let $w(x) = \phi(x) + Av(x)$. For A large enough we have

$$\det \nabla^2 w(x) \ge A^n \det \nabla^2 v(x) \ge g$$

in Ω. This means that w is a subsolution of (2) and (4).

Denote Φ by the set of all subsolutions of (2) and (4), and let $u(x) = \sup\{\widetilde{u}(x) : \widetilde{u} \in \Phi\}$. One can easily verify that u is a generalized solution of (2). Since $w \in \Phi$ we conclude that $u = \phi$ on $\partial\Omega$. $\square$

Lemma 2. *Suppose that there exists a positive h satisfying* (a) *of Theorem A such that $h(d) \le R(x_b, d)$. Then the solution produced by Lemma* 1 *satisfies* (3).

Proof. For any boundary point x_0 we shall assume x_0 is the origin and the positive x_n-axis is in the inner normal direction. Since $\phi \in C^2(\overline{\Omega})$ and Ω is uniformly convex, by adding a linear function to ϕ we may also assume $\phi(x_0) = \max_{\overline{\Omega}} u = 0$. To prove $|\nabla u(x)| \to \infty$ as $x \to x_0$, we introduce a function $v(x)$ as in the proof of Lemma 1 by

$$v(x) = \varrho(d) = -\int_0^d \left(\int_s^{r_0} h(t)\,dt\right)^{1/n} ds, \qquad x \in \Omega\backslash\Omega_{r_0}.$$

Then $|\nabla v(x)| \to \infty$ as $x \to \partial\Omega$. Extend $v(x)$ to Ω_{r_0} as in the proof of Lemma 1 and then modify $v(x)$ to get a uniformly convex function $\widetilde{v}(x) \in C^2(\Omega)$ so that $\widetilde{v}(x) = v(x)$ in $\Omega\backslash\Omega_{r_0/2}$. Then for $\varepsilon > 0$ small enough, we have $\det \nabla^2(\varepsilon v) \le f(x)$ and $\varepsilon v = 0 \ge u$ on $\partial\Omega$. By the comparison principle, $\varepsilon v(x) \ge u(x)$ in Ω. Hence $|\nabla u(x)| \to \infty$ as $x \to x_0$. $\square$

We remark that the same line of proof yields a similar result for type III hypersurfaces. To formulate it we observe that for a type III hypersurface, Ω is of the form $\omega \times \mathbb{R}^m$, where ω is a bounded convex domain in $\mathbb{R}^{n-m}$. Near $\partial\omega$ we may write $\widetilde{x} = (x_1, \dots, x_{n-m}) = \widetilde{x}_b + dn(\widetilde{x}_b)$ as before. Then we have

Theorem D. *Let $\Omega = \omega \times \mathbb{R}^m$, where ω is a uniformly convex C^2-domain in $\mathbb{R}^{n-m}$. Suppose that ϕ can be extended to Ω so that $\{\nabla^2\phi(x)\} \geq \delta_0 I$ for some positive constant δ_0, where I is the identity matrix. Suppose moreover there exist two positive functions h and g satisfying*

(a) $\int_0^{r_0} h(t)\,dt = \infty$,
(b) $\int_0^{r_0} (\int_s^{r_0} g(t)\,dt)^{1/(n-m)}\,ds < \infty$,

such that

$$h(\operatorname{dist}(\widetilde{x}, \partial\omega)) \leq f(x) \leq g(\operatorname{dist}(\widetilde{x}, \partial\omega)), \qquad \widetilde{x} = \widetilde{x}_b + dn(\widetilde{x}_b),$$

near $\partial\Omega$. Then there exists a unique solution u of (2)–(4) in $C(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$.

2. An auxiliary proposition

Proposition. *Let Ω and Ω^* be two bounded convex domains in $\mathbb{R}^n$. Suppose $f(x)$ and $g(p)$ are two positively pinched functions which satisfy*

$$\int_\Omega f(x)\,dx = \int_{\Omega^*} g(p)\,dp < \infty. \tag{8}$$

Then there exists a solution, unique up to an additive constant, to the problem

$$g(\nabla u)\det\nabla^2 u = f(x) \quad \text{in } \Omega, \tag{9}$$

$$\nabla u \text{ maps } \Omega \text{ bijectively onto } \Omega^*. \tag{10}$$

Proof. In the following proof we shall assume additionally that Ω is a uniformly convex C^2-domain. This extra condition can be removed by an approximation argument.

Let $f_\varepsilon(x,u) = e^{\varepsilon u} f(x)$, where $\varepsilon > 0$ is a positive constant. We consider the approximation problem

$$g(\nabla u)\det\nabla^2 u = f_\varepsilon(x,u) \qquad \text{in } \Omega. \tag{9_ε}$$

For any $\varepsilon > 0$, let Ψ_ε be the set of all subsolutions u of $(9)_\varepsilon$ so that

$$N_u(\Omega) \subset \overline{\Omega^*}, \tag{11}$$

where $N_u(\Omega)$ is the normal image of u over Ω. It is easy to see that Ψ_ε is not empty. For any $u \in \Psi_\varepsilon$, we have

$$|\Omega^*| \geq |N_u(\Omega)| \geq \int_\Omega e^{\varepsilon u} f(x)\,dx,$$

which, together with (8), implies $\inf_\Omega u(x) \le 0$. From (11) it follows that $u(x) \le C^*\mathrm{diam}(\Omega)$, where $C^* = \sup\{|x| : x \in \Omega^*\}$. Let $u_\varepsilon(x) = \sup\{u(x) : u \in \Psi_\varepsilon\}$. One easily verifies that u_ε is a generalized solution of $(9)_\varepsilon$. Since Ω^* is convex, by (11) we have $N_{u_\varepsilon}(\Omega) \subset \overline{\Omega^*}$. Extend u_ε to $\mathbb{R}^n$ by

$$u_\varepsilon(x) = \sup\{\langle \xi, x - y\rangle + u(y) : y \in \Omega,\ \xi \in N_u(y)\}$$

so that N_{u_ε} is well defined on $\partial\Omega$. We claim that $N_{u_\varepsilon}(\overline{\Omega}) = \overline{\Omega^*}$. Indeed, suppose there exists $x_0 \in \partial\Omega$ so that there is a point $p_0 \in N_{u_\varepsilon}(x_0)$ which lies in the interior of Ω^*. We will construct a subsolution $u \in \Psi_\varepsilon$ so that $u(x_0) > u_\varepsilon(x_0)$, which contradicts the definition of $u_\varepsilon(x)$.

Without loss of generality we may suppose x_0 is the origin and Ω is contained in $\{x_n > 0\}$. Let

$$f_0 = \sup\{f(x)e^{\varepsilon u}/g(p) : x \in \Omega,\ u \le C^*\mathrm{diam}(\Omega),\ p \in \Omega^*\},$$

and let $\ell(x) = u_\varepsilon(0) + p_0 \cdot x$ be a supporting hyperplane of $u_\varepsilon(x)$ at the origin. Let

$$w(x) = \ell(x) - \frac{1}{2} f_0 M^{n-1}\delta^2 + f_0 M^{n-1}(x_n - \delta)^2 + \frac{1}{M}\sum_{i=1}^{n-1} x_i^2,$$

where $\delta > 0$ and $M > 1$ are respectively small and large constants to be specified below. Then $\det \nabla^2 w = 2^n f_0$ in Ω. In $\Omega \cap \{x_n < \delta\}$ we have

$$|\nabla w - \nabla \ell| \le 2\delta f_0 M^{n-1} + \frac{2}{M}\sum |x_i| \le C(\delta^{1/2} + \delta M^{n-1}).$$

For δM^{n-1} small enough, we have $\nabla w(x) \subset \Omega^*$ for $x \in \Omega \cap \{x_n < \delta\}$. On $\{x_n = \delta\} \cap \Omega$ we have

$$w(x) \le \ell(x) - \frac{1}{2} f_0 M^{n-1}\delta^2 + \frac{C\delta}{M},$$

where C depends on the lower bound of the principal curvatures of $\partial\Omega$. Let M be large enough so that $M^n\delta > 2C/f_0$. We obtain

$$w(x) < \ell(x) \le u_\varepsilon(x) \qquad \text{on } \{x_n = \delta\} \cap \Omega. \tag{12}$$

On the other hand,

$$w(0) = \ell(0) + \frac{1}{2} f_0 M^{n-1}\delta^2 > u_\varepsilon(0).$$

Let ω be the component of $\{x \in \Omega : w(x) > u(x)\}$ containing the origin. By (12) we have $\omega \subset \{x_n < \delta\} \cap \Omega$. Set

$$u(x) = \begin{cases} w(x) & \text{if } x \in \omega, \\ u_\varepsilon(x) & \text{else.} \end{cases}$$

Then $u(x)$ is a subsolution of $(9)_\varepsilon$ with $u(0) > u_\varepsilon(0)$ so that $N_u(\Omega) \subset \overline{\Omega^*}$. However, this is impossible by the definition of u_ε. We have proved that $(9)_\varepsilon$, (10) admits a solution u_ε. Integrating $(9)_\varepsilon$ over Ω gives

$$\int_{\Omega^*} g(p)\,dp = \int_\Omega e^{\varepsilon u} f(x)\,dx.$$

By virtue of (8), we have $\inf u_\varepsilon \le 0$ and $\sup u_\varepsilon \ge 0$. Hence there exists a subsequence of u_ε which converges uniformly to some function u_0 in Ω. And u_0 is a generalized solution of (9) with $N_{u_0}(\Omega) \subset \overline{\Omega^*}$. We claim $\overline{N_{u_0}(\Omega)} \supset \Omega^*$. Indeed, let $\Omega_1^* = \Omega^* \backslash \overline{N_{u_0}(\Omega)}$. By the definition of a generalized solution we have

$$\int_\Omega f(x)\,dx = \int_{\Omega^* \backslash \Omega_1^*} g(p)\,dp,$$

which by (8) implies $\Omega_1^* = \emptyset$. Thus u_0 is a generalized solution of (9), (10).

If the graph of u_0 contains a line segment, the endpoints of the segment lie on the boundary of the graph according to [C1]. In this case the normal image of u_0 cannot be a convex domain. Hence u_0 must be strictly convex. By [C3], it follows that $u_0 \in C^{1+\alpha}(\Omega)$ and so ∇u_0 maps Ω bijectively onto Ω^*.

To show uniqueness suppose there are two solutions u_1 and u_2 so that $u_1 = u_2$ at some point and $\Omega_1 = \{x \in \Omega : u_1(x) > u_2(x)\}$ is nonempty. By adding a constant we may suppose the measure of $N_{u_1}(\Omega_1) \backslash N_{u_2}(\Omega_1)$ is positive. On the other hand, by the definition of generalized solution,

$$\int_{N_{u_1}(\Omega_1)} g(p)\,dp = \int_{\Omega_1} f(x)\,dx = \int_{N_{u_2}(\Omega_1)} g(p)\,dp.$$

This is impossible. □

It was Pogorelov [P3] who first proved the existence of a generalized solution for the above problem. His proof is based on the approximation by polyhedra. A more recent alternate proof was provided by Caffarelli [C4]. Here we have presented an elementary proof. The advantage of our proof is that it can also be used to treat the existence of solutions to the oblique derivative problem for the Monge–Ampère equations in all dimensions. Let us consider

$$\det \nabla^2 u = f(x) \quad \text{in } \Omega, \tag{13}$$

$$\nabla_\beta u = \lim_{t \to 0} \frac{u(x+\beta t) - u(x)}{t} = \phi(x,u) \quad \text{on } \partial\Omega, \tag{14}$$

where Ω is a uniformly convex domain in $\mathbb{R}^n$, β, ϕ and f are continuous functions of their arguments with ϕ nonincreasing in u, $f \ge C_0 > 0$, and

(H1) $\beta(x) \cdot \nu(x) \ge \beta_0 > 0$ on $\partial\Omega$, where ν is the unit outer normal to $\partial\Omega$,
(H2) $\lim_{|u| \to +\infty} \operatorname{sign}(u)\phi(x,u) = -\infty$ uniformly for $x \in \partial\Omega$.

We say a convex function u is a generalized solution (subsolution, resp.) of (13), (14) if

$$\det \nabla^2 u = (\geq, \text{resp.})\ f(x)$$

in Alexandrov's sense, and

$$\nabla_\beta u(x) = (\leq, \text{resp.})\ \phi(x,u) \qquad \text{on } \partial\Omega.$$

Let Φ be the set of all subsolutions to the problem (13), (14). Let $u(x) = \sup\{w(x) : w \in \Phi\}$. Following the proof above one sees that u is a generalized solution of (13), (14). Note that for a Lipschitz convex function, (14) is well defined.

It is worthwhile to point out that even when f, ϕ, β and $\partial\Omega$ are C^∞ smooth, solutions of (13), (14) may fail to be C^2 (see [U2]).

3. Type II hypersurfaces over bounded domains

In this section we prove Theorem B.

Lemma 3. *Suppose that R is locally bounded and Ω^* is a convex domain satisfying*

$$|\Omega^*| = \int_{\mathbb{R}^n} R(x)\, dx < \infty. \tag{15}$$

Then there exists a unique solution u, up to an additive constant, of

$$\det \nabla^2 u(x) = R(x), \qquad x \in \mathbb{R}^n, \tag{16}$$

so that ∇u maps $\mathbb{R}^n$ bijectively onto Ω^*.

Proof. Let $\Omega^*_\delta = \{x \in \Omega^* : \text{dist}(x, \partial\Omega^*) > \delta\}$. For $k \geq 1$, let $\Omega_k = B_{2^k}(0)$ and $\delta_k > 0$ be so that $|\Omega^*_{\delta_k}| = \int_{\Omega_k} R(x)\, dx$. By the above proposition there exists a solution u_k to the problem

$$\begin{cases} \det \nabla^2 u = R(x) & \text{in } \Omega_k, \\ \nabla u \text{ maps } \Omega_k \text{ onto } \Omega^*_{\delta_k}. \end{cases}$$

Since the Lipschitz constant of u_k depends only on Ω^*, there exists a subsequence of $w_k = u_k - u_k(0)$ which converges to a function u_0 in $\mathbb{R}^n$. Obviously u_0 satisfies (16) and $N_{u_0}(x) \subset \overline{\Omega^*}$ for any $x \in \mathbb{R}^n$. By (8) we have $N_{u_0}(\mathbb{R}^n) = \Omega^*$. As before by [C1, C3], u_0 is strictly convex and belongs to $C^{1+\alpha}_{\text{loc}}(\mathbb{R}^n)$. Similar to the proof of the proposition above we have the uniqueness. □

Notice that when $K \in C^\alpha_{\text{loc}}(S^n_-)$, the hypersurface X determined by u is in $C^{2,\alpha}$ ([C2]). It is a graph over Ω^* and $|\nabla u^*(x)| \to \infty$ as $x \to \partial\Omega^*$. But X may be bounded (such hypersurfaces have been studied by Urbas [U1]). To guarantee completeness we need (6), which is equivalent to

$$R(x) \geq C(1 + |x|^2)^{-(n+1)/2}. \tag{17}$$

Lemma 4. *Suppose that $R(x) \in C^{\alpha}_{\mathrm{loc}}(S^n_-)$ satisfies* (15) *and* (17) *and Ω^* is a uniformly convex C^2-domain. Then the hypersurface obtained by Lemma* 3 *is complete.*

Proof. For a convex function u we define

$$S^0_{h,x_0}(u) = \{x : u(x) < h + u(x_0) + \nabla u(x_0) \cdot (x - x_0)\},$$

and $S_{h,x_0}(u) = \partial S^0_{h,x_0}(u)$. The subscript x_0 will be omitted if u attains its minimum at x_0.

We may suppose $u(0) = 0$ and $\nabla u(0) = 0$. To prove Lemma 4 it suffices to show that for any $x \neq 0$,

$$tx \cdot \nabla u(tx) - u(tx) \to \infty \qquad \text{as } t \to \infty.$$

We argue by contradiction. Suppose

$$m = \inf_{x \in S^{n-1}} \lim_{t\to\infty} \{tx \cdot \nabla u(tx) - u(tx)\} < \infty. \tag{18}$$

Let G be the graph of u. Let $\widetilde{\Phi}$ be the cone consisting of the rays $\{\lambda x : \lambda \geq 0\}$ which are contained in the convex body bounded by G. Let Φ be the boundary of $\widetilde{\Phi}$. Then Φ can be represented as the graph of a convex affine function ϕ over $\mathbb{R}^n$. By definition, we have

$$S^0_h(\phi) \subset S^0_h(u), \qquad S_{\lambda h}(\phi) = \lambda S_h(\phi),$$

and for any $h > 0$, $S^0_h(\phi)$ is a uniformly convex domain with C^2-boundary.

Since u is convex and $\nabla u(x)$ is bounded, it is easy to see that (18) is equivalent to

$$\widetilde{m} = \lim_{h\to\infty} \operatorname{dist}(S_h(\phi), S_h(u)) < \infty. \tag{19}$$

For any $h > 0$, let $x_h \in S_h(u)$ and $y_h \in S_h(\phi)$ so that

$$\operatorname{dist}(x_h, y_h) = \operatorname{dist}(S_h(u), S_h(\phi)).$$

We may suppose $x_h/|x_h| \to x_0$ as $h \to \infty$. Then $y_h/|y_h| \to x_0$ too. Suppose for simplicity that $x_h/|x_h| = (1, 0, \dots, 0)$. Let $\ell_h(x)$ be the tangent hyperplane of G at x_h, and let

$$D_{\varepsilon,h} = \{x \in \mathbb{R}^n : u(x) < \ell_h(x) + \varepsilon\},$$

where $\varepsilon > 0$ is chosen as the largest number so that $D_{\varepsilon,h} \subset \mathbb{R}^n \cap \{h/2 < x_1 < 2h\}$. By (19) we have $\varepsilon \to 0$ as $h \to \infty$. Let $w_{\varepsilon,h}(x) = u(x) - \ell_h(x) - \varepsilon$. We have

$$\det \nabla^2 w_{\varepsilon,h} = R(x) \quad \text{in } D_{\varepsilon,h}, \qquad w_{\varepsilon,h} = 0 \quad \text{on } \partial D_{\varepsilon,h}$$

and $\inf w_{\varepsilon,h} = -\varepsilon$.

Let F be the minimum ellipsoid in $D_{\varepsilon,h}$ so that $\frac{1}{n}(F-\widetilde{x}) \subset D_{\varepsilon,h} - \widetilde{x} \subset F - \widetilde{x}$, where $\widetilde{x}$ is the center of F. Let $T = A(x-\widetilde{x})$ be an affine transformation with $\det A = 1$ so that $T(F) = B_\delta(0)$ for some $\delta > 0$. By the invariance of the Monge–Ampère operator under affine transformations and the comparison principle it follows that

$$\varepsilon^n \geq C \inf\{R(x) : x \in D_{\varepsilon,h}\} \cdot |D_{\varepsilon,h}|^2.$$

Let $E_{\varepsilon,h} = D_{\varepsilon,h} \cap \{x_1 = h\}$. By the choice of ε and the convexity of $D_{\varepsilon,h}$ we have

$$|D_{\varepsilon,h}| \geq C\mathcal{H}^{n-1}(E_{\varepsilon,h})h,$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. We claim that

$$B_r(0) \subset E_{\varepsilon,h} \qquad \text{with } r \geq C(h\varepsilon)^{1/2}, \tag{20}$$

from which it follows that

$$\varepsilon^n \geq C \inf\{R(x) : x \in D_{\varepsilon,h}\} h^{n+1} \varepsilon^{n-1},$$

i.e., $\inf\{R(x) : x \in D_{\varepsilon,h}\} \leq C\varepsilon h^{-n-1}$. On the other hand, by the condition (17) we have $\inf\{R(x) : x \in D_{\varepsilon,h}\} \geq Ch^{-n-1}$. This is a contradiction for $\varepsilon > 0$ small.

Proof of (20). Suppose near the point x_h, $S_h(u)$ is represented by

$$x_1 = \varrho_u(\widetilde{x}), \qquad \widetilde{x} = (x_2, \ldots, x_n),$$

and near the point y_h, $S_h(\phi)$ is represented by

$$x_1 = \varrho_\phi(\widetilde{x}).$$

Then ϱ_u and ϱ_ϕ are concave functions with $\nabla\varrho_u(0) = \nabla\varrho_\phi(0) = 0$. By the choice of x_h and y_h we see that $\varrho_u(\widetilde{x}) - \varrho_\phi(\widetilde{x})$ attains its minimum at $\widetilde{x} = 0$. Hence $\{\widetilde{x} : \varrho_\phi(\widetilde{x}) > \varrho_\phi(0) - \delta\} \subset \{\widetilde{x} : \varrho_u(\widetilde{x}) > \varrho_u(0) - \delta\}$ for $\delta > 0$ small. Note that the principal radii of the level surface $S_h(\phi)$ are greater than Ch for some $C > 0$. We have

$$B_r(0) \subset \{\widetilde{x} : \varrho_\phi(\widetilde{x}) > \varrho_\phi(0) - \varepsilon\} \qquad \text{with } r \geq C(h\varepsilon)^{1/2}.$$

Notice that $\{\widetilde{x} : \varrho_u(\widetilde{x}) > \varrho_u(0) - \varepsilon/C\} \subset E_{\varepsilon,h}$ for some $C > 0$ depending on $\inf\{|\nabla u(x)| : x \in S_h(u)\}$. Hence (20) holds. □

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References

[C1] L. A. Caffarelli, *A localization property of viscosity solutions to the Monge–Ampère equation and their strict convexity*, Ann. of Math. **131** (1990), 129–134.

[C2] ______, *Interior $W^{2,p}$ estimates for solutions of the Monge–Ampère equation*, Ann. of Math. **131** (1990), 135–150.

[C3] ______, *Some regularity properties of solutions of Monge Ampère equation*, Comm. Pure Appl. Math. **44** (1991), 965–969.

[C4] ______, *The regularity of mappings with a convex potential*, J. Amer. Math. Soc. **5** (1992), 99–104.

[CY] S. Y. Cheng and S. T. Yau, *On the regularity of the solution of the n-dimensional Minkowski problem*, Comm. Pure Appl. Math **29** (1976), 495–516.

[CW] K. S. Chou and X. J. Wang, *Entire solutions of Monge–Ampère equation*, Comm. Pure Appl. Math. (to appear).

[P1] A. Pogorelov, *The Minkowski Multidimensional Problem*, Wiley, New York, 1978.

[P2] ______, *An analog of Minkowski's problem for infinite complete convex hypersurfaces*, Soviet Math. Dokl. **21** (1980), 138–140.

[P3] ______, *Monge–Ampère Equations of Elliptic Type*, Noordhoff, Groningen, 1964.

[U1] J. I. E. Urbas, *The equation of prescribed Gauss curvature without boundary conditions*, J. Differential Geom. **20** (1984), 311–327.

[U2] ______, *Nonlinear oblique boundary value problems for Hessian equations in two dimensions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **12** (1995), 507–575.

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