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AN ANALYTIC COMPUTATION OF $ko_{4\nu-1}(BQ_8)$

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Dedicated to Louis Nirenberg

The connective K-theory groups $ko_*(B\pi)$ of a group π appear in many contexts; for example, they are the building blocks for equivariant spin bordism at the prime 2. They also play an important role in the Gromov–Lawson–Rosenberg conjecture which was the starting point of our original investigation [5].

The second author first studied the eta invariant, which is an analytic invariant, whilst a graduate student under the direction of L. Nirenberg so this is perhaps a fitting subject for this volume. In this paper, we will use the eta invariant to determine the additive structure of $ko_{4\nu-1}(BQ_8)$, where

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

is the quaternion group of order 8. We refer to D. Bayen and R. Bruner [2] for an independent topological computation of these groups.

Theorem 1.
(a) $ko_{8\mu+3}(BQ_8) \cong (\mathbb{Z}/2^{3+4\mu}) \oplus (\mathbb{Z}/2^{2\mu}) \oplus (\mathbb{Z}/2^{2\mu+2}) \oplus (\mathbb{Z}/2^{2\mu+2})$.
(b) $ko_{8\mu+7}(BQ_8) \cong (\mathbb{Z}/2^{6+4\mu}) \oplus (\mathbb{Z}/2^{2\mu}) \oplus (\mathbb{Z}/2^{2\mu+2}) \oplus (\mathbb{Z}/2^{2\mu+2})$.

Remark. In fact, we not only determine the additive structure of these groups, our method can also be used to find explicit geometrical generators.

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Remark. The eta invariant is trivial on $ko_m(BQ_8)$ for $m \not\equiv 3 \pmod 4$ and gives no information in these dimensions. We refer to [2] for the calculation of $ko_m(BQ_8)$ for these values of m; there are no extension problems to be solved in these dimensions in contrast to the case $m \equiv 3 \pmod 4$.

We begin by reviewing some of the facts we shall need concerning the eta invariant and connective K-theory. Let D be an operator of Dirac type on a compact Riemannian manifold M. Let

$$\eta(D)(z) := \sum_{\lambda \neq 0} \operatorname{sign}(\lambda) \cdot \dim(\ker(D - \lambda I)) \cdot |\lambda|^{-z}$$

be the *eta function* of Atiyah, Patodi, and Singer [1]. This converges absolutely for $\mathrm{Re}(z) > 0$ and has a meromorphic extension to $\mathbb{C}$ which is regular at $z = 0$. The *eta invariant* is a measure of the spectral asymmetry of D defined by

$$\eta(D) := \tfrac{1}{2}\{\eta(D)(z) + \dim(\ker(D))\}|_{z=0} \in \mathbb{R}.$$

We refer to [1, 9] for the proof of the following result.

Lemma 2. *Let D_t be a smooth 1-parameter family of operators of Dirac type on a compact manifold M. The reduction of $\eta(D_t)$ to $\mathbb{R}/\mathbb{Z}$ is smooth and the derivative $\dot\eta(D_t)$ is given by integrating a local formula over M. If $\ker(D_t)$ is trivial, then $\eta(D_t)$ is smooth as a real-valued invariant.*

Remark. In general, $\eta(D_t)$ is not smooth in t; discontinuities occur when the eigenvalues cross or touch the origin; reduction mod $\mathbb{Z}$ eliminates these discontinuities.

Let π be a finite group. Let (M, g, s, σ) denote a closed manifold of dimension m with a Riemannian metric g, a spin structure s, and a π structure σ. If m is odd, let D_ϱ be the Dirac operator on M with coefficients in the flat bundle determined by a representation ϱ of π. Define

$$\eta(M)(\varrho) = \eta(M, g, s, \sigma)(\varrho) := \eta(D_\varrho) \in \mathbb{R}.$$

Let $R_0(\pi)$ be the augmentation ideal of all virtual representations of virtual dimension 0 in the group representation ring $R(\pi)$. It is clear that $\eta(\cdot)(\varrho)$ is additive in ϱ and hence extends to $R(\pi)$ and $R_0(\pi)$.

Let $\mathrm{MSpin}_m(B\pi)$ be the set of bordism classes of triples (M, s, σ), where M is a closed manifold of dimension m, where s is a spin structure on M, and where σ is a π structure on M.

Theorem 3. *Let $\varrho \in R_0(\pi)$ and let m be odd. Then the homomorphism*

$$\eta(\varrho) : \mathrm{MSpin}_m(B\pi) \to \mathbb{R}/\mathbb{Z}$$

which maps a class represented by (M, s, σ) *in dimension* m *to* $\eta(M, g, s, \sigma)(\varrho)$ *is well defined. Furthermore, if* ϱ *is of real type and* $m \equiv 3 \pmod 8$ *or if* ϱ *is of quaternion type and* $m \equiv 7 \pmod 8$, *we can replace the range of* $\eta(\varrho)$ *by* $\mathbb{R}/2\mathbb{Z}$.

Proof. We use the index theorem of Atiyah, Patodi, and Singer [1]. Let M be the boundary of a spin manifold N and suppose the π structure on M extends over N. To prove the first assertion, we must show $\eta(\varrho) \in \mathbb{Z}$. We extend the metric on M to a metric on N which is product near the boundary. Let P_ϱ be the operator of the spin complex over N with coefficients in the flat bundle V_ϱ determined by the virtual representation ϱ. We take suitable non-local boundary conditions for P_ϱ and apply the index theorem to see

$$\operatorname{index}(P_\varrho) = \int_N \widehat{A} \cdot \operatorname{ch}(V_\varrho) - \eta(\varrho),$$

where $\widehat{A}$ is the differential form on N whose representative in de Rham cohomology gives the $\widehat{A}$-genus. Since V_ϱ is a flat bundle of virtual dimension zero, $\operatorname{ch}(V_\varrho) = 0$ and we see $\eta(\varrho) \in \mathbb{Z}$ as desired. If $m \equiv 3 \pmod 8$, then the spin bundle on N admits a natural quaternion structure; if $m \equiv 7 \pmod 8$, then the spin bundle on N admits a natural real structure. Thus if ϱ is real if $m \equiv 3 \pmod 8$ or quaternion if $m \equiv 7 \pmod 8$, then the spin bundle with coefficients in ϱ on N has a natural quaternion structure and the eigenspaces of P_ϱ admit natural quaternion structures. Consequently, $\operatorname{index}(P_\varrho)$ is divisible by 2 in these cases. □

Remark. Invariants similar to those defined in Theorem 3 completely detect the K-theory of spherical space forms and the reduced equivariant unitary bordism of spherical space form groups; see [7, 8].

There is a geometric way to think of the connective K-theory groups $ko_n(B\pi)$ localized at the prime 2. Let $T_m(B\pi)$ be the subgroup of $\mathrm{MSpin}_m(B\pi)$ represented by pairs (E, α), where $\alpha : E \to B$ is a fiber bundle with fibre $\mathbb{HP}^2$, the quaternionic projective plane, and structure group the group of isometries of $\mathbb{HP}^2$. Stolz [11] showed the map

$$\mathrm{MSpin}_m(B\pi)/T_m(B\pi) \to ko_m(B\pi)$$

is an isomorphism when localized at the prime 2. We use the following theorem to extend the eta invariant to a map in K-theory.

Theorem 4. *Let* π *be a finite group, let* $\varrho \in R_0(\pi)$, *and let* m *be odd. Then the homomorphism*

$$\eta^{ko}(\varrho) : (ko_m(B\pi))_{(2)} \to (\mathbb{R}/\mathbb{Z})_{(2)}$$

which maps a class represented by (M, s, σ) *in dimension* m *to* $\eta(M, g, s, \sigma)(\varrho)$ *is well defined when localized at the prime* 2. *Furthermore, if* ϱ *is of real type and*

$m \equiv 3 \pmod 8$ *or if* ϱ *is of quaternion type and* $m \equiv 7 \pmod 8$, *we can replace the range of* $\eta(\varrho)$ *by* $(\mathbb{R}/2\mathbb{Z})_{(2)}$.

Proof. Let $\alpha : E \to B$ be a geometrical fiber bundle with fiber $\mathbb{HP}^2$. We must show $\eta(E)(\varrho) = 0$. Since $\mathbb{HP}^2$ is simply connected, the π structure on the total space E arises from a π structure on the base B. Let g^F be the standard Riemannian metric of positive scalar curvature on the fiber $F = \mathbb{HP}^2$ and let g^B be any Riemannian metric on the base B. Let F_x be the fiber of E over a point $x \in B$. There exists a metric g^E on the total space E so that the induced metric on each F_x is g^F, so that each F_x is totally geodesic, and so that the projection α is a Riemannian submersion [3, 9.59]. Let $\mathcal{V}$ and $\mathcal{H}$ be the vertical and horizontal distributions of the submersion. Define the canonical variation g_t^E of the metric by imposing the conditions

$$g_t^E|_{\mathcal{V}} = tg^F, \quad g_t^E|_{\mathcal{H}} = \alpha^*(g^B), \quad g_t^E(\mathcal{V}, \mathcal{H}) = 0.$$

Let τ^F and τ_t^E be the scalar curvature of the metrics on F and on E. Then

$$\tau_t^E = t^{-1}\tau^F + O(1);$$

see [3, 9.70]. In particular, $\tau_t^E \to \infty$ as $t \to 0$.

Let $\varrho \in R_0(\pi)$. We will show that there exists $t_0(\varrho)$ so that if $0 < t < t_0(\varrho)$,

$$(*) \qquad \eta(g_t^E)(\varrho) = 0 \qquad \text{in } \mathbb{R}.$$

Let δ be the right regular representation of π and let 1 be the trivial representation of π. Let $\chi := |\pi| \cdot 1 - \delta$. Then

$$\operatorname{Tr}(\chi(1)) = 0 \quad \text{and} \quad \operatorname{Tr}(\chi)(\lambda) = |\pi| \quad \text{for } \lambda \neq 1.$$

Thus if $\varrho \in R_0(\pi)$, then $|\pi|\varrho = \chi\varrho$. Since $\mathbb{R}$ is without torsion, we may replace ϱ by $\chi\varrho$ in proving equation $(*)$.

Let $\varrho = \mu_1 - \mu_2$, where the μ_i are actual representations of π of the same dimension. Let ζ_i^B be the corresponding flat bundles over the base B. Since these bundles admit flat connections and have the same dimension, the rational Chern classes of the difference $\zeta_1^B - \zeta_2^B$ vanish. Thus this virtual bundle is rationally trivial. Again, by replacing ϱ by a suitable integer multiple, we may assume $\zeta_1^B \cong \zeta_2^B \cong \zeta^B$.

Let ∇_i^B be the flat connections on ζ^B defined by the flat structures μ_i. Define a smooth 1-parameter family of connections with curvatures Ω_ε^B by defining

$$\nabla_\varepsilon^B := \varepsilon\nabla_1^B + (1-\varepsilon)\nabla_2^B.$$

Pull back these structures to define the corresponding structures over E. Since α is a Riemannian submersion for any t, the norm of the curvature tensor Ω_ε^E can be uniformly bounded with respect to the metric g_t^E for all $(\varepsilon, t) \in [0,1] \times \mathbb{R}$.

Let $D_{\varepsilon,t}$ be the Dirac operator with coefficients in E defined by the metric g_t^E and connection ∇_ε^E. We use the generalized Lichnerowicz formula to express the square of $D_{\varepsilon,t}$ in the form

$$(D_{\varepsilon,t})^2 = \nabla_{\varepsilon,t}^* \nabla_{\varepsilon,t} + \tau_t^E/4 + \Psi(\Omega_\varepsilon^E);$$

the error term $\Psi(\cdot)$ depends only on the Clifford module structure of the base B. Thus the pointwise operator norm of $\Psi(\cdot)$ is uniformly bounded in (ε, t). Since $\tau_t^E \to \infty$ as $t \to 0$, $\tau_t^E/4 + \Psi(\Omega_\varepsilon^E)$ is positive for t sufficiently small. Thus there are no twisted harmonic spinors.

Let $D_{\varepsilon,t,\chi}$ denote $D_{\varepsilon,t}$ with coefficients in the flat virtual bundle defined by χ. The same argument as that given above shows $\ker(D_{\varepsilon,t,\chi})$ is trivial for all (ε, t). Thus by Theorem 2, $\eta(D_{\varepsilon,t,\chi})$ is a smooth real-valued function of (ε, t). Furthermore, the derivative with respect to ε or t of the eta invariant is given by a local formula. Since χ has virtual dimension 0, the local formula vanishes and $\eta(D_{\varepsilon,t,\chi})$ is independent of (ε, t). This shows that

$$|\pi|\eta(g_t)(\varrho) = \eta(D_{\varepsilon,t,\chi}) - \eta(D_{\varepsilon,t,\chi}) = 0. \qquad \square$$

Remark. The adiabatic limit theorem of Bismut and Cheeger [4, (0.5)] can also be used to establish this result. However, the proof we have just given of Theorem 4 generalizes to the case of spin^c and pin^c structures, where the associated complex line bundle is flat.

Remark. We will show in Lemma 6 that $ko_{4\nu-1}(BQ_8)$ is a finite 2-group. Thus it is not necessary to localize at the prime 2 and $\eta(\varrho)$ defines a homomorphism from $ko_{4\nu-1}(BQ_8)$ to $\mathbb{R}/\mathbb{Z}$ or to $\mathbb{R}/2\mathbb{Z}$.

Spherical space forms play a crucial role in our analysis. Let $\tau : \pi \to SU(2\nu)$ be a fixed point free representation of π to the special unitary group. Let M be the quotient manifold $S^{4\nu-1}/\tau(\pi)$; M inherits a natural metric of constant sectional curvature $+1$ and is called a *spherical space form*. The isomorphism $\pi_1(M) \cong \pi$ defines a natural π structure on M. Let $T(M) \oplus 1$ be the stable tangent bundle of M. We identify $T(M) \oplus 1$ with the flat bundle over M defined by τ to define a natural $SU(2\nu)$ structure on $T(M) \oplus 1$. We use the lift of the special unitary group to the spinor group discussed by Hitchin [10] to give $T(M) \oplus 1$ and $T(M)$ natural spin structures. Donnelly [6] has generalized the Atiyah–Patodi–Singer theorem to the equivariant setting; the following theorem follows from his results.

Theorem 5. *Let $\varrho \in R_0(\pi)$ and let $\tau : \pi \to SU(2\nu)$ be a fixed point free representation. Let $M = S^{4\nu-1}/\tau(\pi)$ with the structures defined above. Then*

$$\eta(M)(\varrho) = |\pi|^{-1} \sum_{\lambda \in \pi, \lambda \neq 1} \mathrm{Tr}(\varrho(\lambda)) \det(I - \tau(\lambda))^{-1}.$$

We specialize henceforth to the group $\pi = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. Let

$$H_i := \langle i \rangle, \quad H_j := \langle j \rangle, \quad \text{and} \quad H_k := \langle k \rangle$$

be the 3 cyclic subgroups of Q_8 which have order 4. The group Q_8 has 4 inequivalent real linear representations defined by

$$\begin{aligned}
&\varrho_0(\pm 1) = 1, \quad \varrho_0(\pm i) = 1, \quad \varrho_0(\pm j) = 1, \quad \varrho_0(\pm k) = 1,\\
&\varrho_i(\pm 1) = 1, \quad \varrho_i(\pm i) = 1, \quad \varrho_i(\pm j) = -1, \quad \varrho_i(\pm k) = -1,\\
&\varrho_j(\pm 1) = 1, \quad \varrho_j(\pm i) = -1, \quad \varrho_j(\pm j) = 1, \quad \varrho_j(\pm k) = -1,\\
&\varrho_k(\pm 1) = 1, \quad \varrho_k(\pm i) = -1, \quad \varrho_k(\pm j) = -1, \quad \varrho_k(\pm k) = 1.
\end{aligned}$$

Let τ be the inclusion of Q_8 into $SU(2)$ which we identify with the set of unit quaternions; τ is of quaternion type. The representations ϱ_0, ϱ_i, ϱ_j, ϱ_k, and τ are the irreducible representations of Q_8 up to unitary equivalence. Let

$$\tau_\nu := \tau \oplus \ldots \oplus \tau$$

be the diagonal embedding of Q_8 into $SU(2\nu)$; τ_ν is fixed point free. Let

$$\begin{aligned}
&M_Q^{4\nu-1} := S^{4\nu-1}/\tau_\nu(Q_8), \quad M_i^{4\nu-1} := S^{4\nu-1}/\tau_\nu(H_i),\\
&M_j^{4\nu-1} := S^{4\nu-1}/\tau_\nu(H_j), \quad M_k^{4\nu-1} := S^{4\nu-1}/\tau(H_k),\\
&\vec{\eta}(\cdot) := (\eta(\cdot)(2-\tau), \eta(\cdot)(\varrho_0 - \varrho_i), \eta(\cdot)(\varrho_0 - \varrho_j), \eta(\cdot)(\varrho_0 - \varrho_k)),\\
&\mathcal{A}_{4\nu-1} := \operatorname{span}_{\mathbb{Z}}\{\vec{\eta}(M_Q^{4\nu-1}), \vec{\eta}(M_i^{4\nu-1}), \vec{\eta}(M_j^{4\nu-1}), \vec{\eta}(M_k^{4\nu-1})\},\\
&\mathcal{A}_{8\mu+3} \subset (\mathbb{R}/\mathbb{Z}) \oplus (\mathbb{R}/2\mathbb{Z})^3, \quad \text{and} \quad \mathcal{A}_{8\mu+7} \subset (\mathbb{R}/2\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z})^3.
\end{aligned}$$

Lemma 6.

(a) $\vec{\eta}(M_Q^{4\nu-1}) = (2^{-1-2\nu}(1 + 3 \cdot 2^\nu), 2^{-\nu}, 2^{-\nu}, 2^{-\nu})$.

(b) $\vec{\eta}(M_i^{4\nu-1}) = (2^{-2\nu}(1 + 2^\nu), 0, 2^{-\nu}, 2^{-\nu})$.

(c) $\vec{\eta}(M_j^{4\nu-1}) = (2^{-2\nu}(1 + 2^\nu), 2^{-\nu}, 0, 2^{-\nu})$.

(d) $\vec{\eta}(M_k^{4\nu-1}) = (2^{-2\nu}(1 + 2^\nu), 2^{-\nu}, 2^{-\nu}, 0)$.

(e) $\mathcal{A}_{8\mu+3} \cong (\mathbb{Z}/2^{3+4\mu}) \oplus (\mathbb{Z}/2^{2\mu}) \oplus (\mathbb{Z}/2^{2\mu+2}) \oplus (\mathbb{Z}/2^{2\mu+2})$.

(f) $\mathcal{A}_{8\mu+7} \cong (\mathbb{Z}/2^{6+4\mu}) \oplus (\mathbb{Z}/2^{2\mu}) \oplus (\mathbb{Z}/2^{2\mu+2}) \oplus (\mathbb{Z}/2^{2\mu+2})$.

(g) $ko_{4\nu-1}(BQ_8)$ *is a finite* 2*-group and* $|ko_{4\nu-1}(BQ_8)| \le |\mathcal{A}_{4\nu-1}|$.

Proof. Let $a, b \in \{\pm i, \pm j, \pm k\}$. Set $\varepsilon_{a,b} = 0$ if $a = b$ and $\varepsilon_{a,b} = 1$ if $a \ne b$. We prove the first 4 assertions of the lemma by computing:

$$\det((I_{2\nu}-\tau_\nu)(-1))=2^{2\nu},\quad \det((I_{2\nu}-\tau_\nu)(\pm a))=2^{\nu},$$
$$\mathrm{Tr}((2-\tau)(-1))=4,\quad \mathrm{Tr}((2-\tau)(\pm a))=2,$$
$$\mathrm{Tr}((\varrho_0-\varrho_a)(-1))=0,\quad \mathrm{Tr}((\varrho_0-\varrho_a)(\pm b))=2\varepsilon_{a,b},$$
$$\vec\eta(M_Q^{4\nu-1})(2-\tau)=2^{-3}\{\mathrm{Tr}((2-\tau)(-1))\cdot 2^{-2\nu}+6\mathrm{Tr}((2-\tau)(j))\cdot 2^{-\nu}\},$$
$$\vec\eta(M_Q^{4\nu-1})(\varrho_0-\varrho_a)=2^{-3}\{4\mathrm{Tr}((\varrho_0-\varrho_j)(i))\cdot 2^{-\nu}\},$$
$$\vec\eta(M_a^{4\nu-1})(2-\tau)=2^{-2}\{\mathrm{Tr}((2-\tau)(-1))\cdot 2^{-2\nu}+2\mathrm{Tr}((2-\tau)(j))\cdot 2^{-\nu}\},$$
$$\vec\eta(M_a^{4\nu-1})(\varrho_0-\varrho_b)=2^{-2}\{2\mathrm{Tr}((\varrho_0-\varrho_b)(a))2^{-\nu}\}=2^{-\nu}\varepsilon_{a,b}.$$

We use Gaussian elimination on the eta matrix to prove the next 2 assertions of the lemma. Let A_i denote suitably chosen integers. We subtract the second row from the third and fourth rows to obtain the matrix

$$\begin{pmatrix} 2^{-1-2\nu}(1+3\cdot 2^{\nu}) & 2^{-\nu} & 2^{-\nu} & 2^{-\nu}\\ 2^{-2\nu}(1+2^{\nu}) & 0 & 2^{-\nu} & 2^{-\nu}\\ 0 & 2^{-\nu} & -2^{-\nu} & 0\\ 0 & 2^{-\nu} & 0 & -2^{-\nu}\end{pmatrix}.$$

We add the third and fourth rows to the first and second rows to obtain the matrix

$$\begin{pmatrix} 2^{-1-2\nu}(1+3\cdot 2^{\nu}) & 3\cdot 2^{-\nu} & 0 & 0\\ 2^{-2\nu}(1+2^{\nu}) & 2^{1-\nu} & 0 & 0\\ 0 & 2^{-\nu} & -2^{-\nu} & 0\\ 0 & 2^{-\nu} & 0 & -2^{-\nu}\end{pmatrix}.$$

We add the third and fourth columns to the second column to obtain the matrix

$$\begin{pmatrix} 2^{-1-2\nu}(1+3\cdot 2^{\nu}) & 3\cdot 2^{-\nu} & 0 & 0\\ 2^{-2\nu}(1+2^{\nu}) & 2^{1-\nu} & 0 & 0\\ 0 & 0 & -2^{-\nu} & 0\\ 0 & 0 & 0 & -2^{-\nu}\end{pmatrix}.$$

We multiply the first column by $-3(1-3\cdot 2^{\nu})2^{\nu+1}$ and add it to the second column; since 2 divides $2^{\nu+1}$, this is permissible even if the first column is defined mod $\mathbb{Z}$ and the second column is defined mod $2\mathbb{Z}$. This yields the matrix

$$\begin{pmatrix} 2^{-1-2\nu}(1+3\cdot 2^{\nu}) & 2A_1 & 0 & 0\\ 2^{-2\nu}(1+2^{\nu}) & (1-3)2^{1-\nu}+2B_1 & 0 & 0\\ 0 & 0 & -2^{-\nu} & 0\\ 0 & 0 & 0 & -2^{-\nu}\end{pmatrix}.$$

We subtract an appropriate multiple of the first row from the second row to put the eta matrix in the form $\mathrm{diag}(A_3 2^{-1-2\nu}, A_4 2^{2-\nu}, -2^{-\nu}, -2^{-\nu})$ for A_3 and A_4 odd. If $m=8\mu+3$, then $\nu=2\mu+1$. The first column is defined mod $\mathbb{Z}$, the remaining columns are defined mod $2\mathbb{Z}$, and assertion (e) follows. If $m=8\mu+7$, then $\nu=2\mu+2$. The first column is defined mod $2\mathbb{Z}$, the remaining columns are defined mod $\mathbb{Z}$, and assertion (f) follows.

We use the Atiyah–Hirzebruch spectral sequence to obtain an upper bound for the order of the groups $ko_{4\nu-1}(B\mathbb{Z}_n)$ in order to prove the final assertion of the lemma. The E^2 term in the spectral sequence for $ko_m(BQ_8)$ is given by

$$E^2_{p,q} := \bigoplus_{p+q=m} H_p(BQ_8; ko_q).$$

Consequently, we may estimate

$$|ko_{4\nu-1}(B\mathbb{Z}_n)| \leq \left| \bigoplus_{p+q=4\nu-1} H_p(BQ_8; ko_q) \right|.$$

We recall $H_\nu(BQ_8;\mathbb{Z})$ is periodic with period 4 for $\nu > 0$ and ko_ν is periodic with period 8 for all ν. Let $2\mathbb{Z}_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Recall that

ν	0	1	2	3	4	5	6	7	8
$H_\nu(BQ_8;\mathbb{Z})$	$\mathbb{Z}$	$2\mathbb{Z}_2$	0	$\mathbb{Z}_8$	0	$2\mathbb{Z}_2$	0	$\mathbb{Z}_8$	0
bo_ν	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

We complete the proof of the lemma by checking that

$$\left| \bigoplus_{p+q=4\nu-1} H_p(BQ_8; ko_q) \right| = |\mathcal{A}_{4\nu-1}|. \qquad \square$$

Proof of Theorem 1. Since $\vec{\eta}$ extends to $ko_{4\nu-1}(BQ_8)$, since range($\vec{\eta}$) contains $\mathcal{A}_{4\nu-1}$ and since $|ko_{4\nu-1}(BQ_8)| \leq |\mathcal{A}_{4\nu-1}|$, $\vec{\eta}^{ko}$ is an isomorphism in these dimensions. $\square$

Remark. We have shown as a byproduct that the eta invariant completely detects $ko_{4\nu-1}(BQ_8)$. The corresponding assertion holds true for the higher quaternion spherical space form groups despite the fact that we do not know the explicit additive structure; we refer to [5, Corollary 2.13 for details].

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