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A NOTE ON THE RESONANCE SET FOR A SEMILINEAR ELLIPTIC EQUATION AND AN APPLICATION TO JUMPING NONLINEARITIES

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Dedicated to Louis Nirenberg on the occasion of his 70th birthday

Introduction

The research of the number of solutions for elliptic boundary problems with jumping nonlinearities is closely linked with the properties of the resonance set, that is,

$$\Sigma = \{(\alpha,\beta)\in\mathbb{R}^2 \mid \Delta u + \alpha u^+ - \beta u^- = 0 \text{ has a nontrivial solution in } H^1_0(\Omega)\},$$

where Ω is a bounded smooth domain, $u^+ = \max(u,0)$ and $u^- = -\min(u,0)$. The study of Σ turns out to be difficult except when Ω is an interval in $\mathbb{R}$. Therefore it is interesting to have some information about the resonance set, as precise as possible.

In [GK] the authors showed that if λ_k is a simple eigenvalue of $-\Delta$ then $\Sigma\cap]\lambda_{k-1},\lambda_{k+1}[^2$ coincides with two continuous curves through the point (λ_k,λ_k). In [DeFG] the authors characterized a curve γ through the point (λ_2,λ_2) which belongs to Σ such that $\Sigma\cap\{(\alpha,\beta)\in\mathbb{R}^2 \mid \lambda_1<\beta<\gamma(\alpha),\ \alpha>\lambda_1\}=\emptyset$. Finally, in [MMP] and [M] the following result was shown: if $k\geq 2$ is such that $\lambda_k<\lambda_{k+1}$ then there exist two continuous curves $(\alpha,\varphi_{k+1}(\alpha))$, through $(\lambda_{k+1},\lambda_{k+1})$, and $(\alpha,\psi_k(\alpha))$, through (λ_k,λ_k), which respectively lie in the sets $\Sigma\cap]\lambda_k,+\infty[^2$ and

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$\Sigma \cap]-\infty, \lambda_{k+1}[^2$, with the property

$$\Sigma \cap (\{(\alpha,\beta) \in \mathbb{R}^2 \mid \alpha > \lambda_k,\ \lambda_k < \beta < \varphi_{k+1}(\alpha)\} \cup \{(\alpha,\beta) \in \mathbb{R}^2 \mid \alpha < \lambda_{k+1},\ \psi_k(\alpha) < \beta < \lambda_{k+1}\}) = \emptyset.$$

Our goal in this paper is to show that also the sets

$$\{(\alpha,\lambda_k) \mid \lambda_k \le \alpha < \overline{\alpha}\} \qquad \text{with } \varphi_{k+1}(\overline{\alpha}) = \lambda_k$$

and

$$\{(\alpha,\lambda_{k+1}) \mid \underline{\alpha} < \alpha \le \lambda_{k+1}\} \qquad \text{with } \psi_k(\underline{\alpha}) = \lambda_{k+1}$$

do not intersect Σ. In order to prove that, we need to use a characterization of the curves φ_{k+1} and ψ_k different from both the one given in [M] and the one given in [MMP]. Finally, in §1 we obtain our main result (see (1.33)).

Theorem. *Let $k \ge 2$ with $\lambda_k < \lambda_{k+1}$. There exists an open connected set $\mathcal{S}_k$ such that*

$$\mathcal{S}_k \supset \{(\alpha,\beta) \in \mathbb{R}^2 \mid \lambda_k \le \alpha < \overline{\alpha},\ \lambda_k \le \beta < \varphi_{k+1}(\alpha)\} \cup \{(\alpha,\beta) \in \mathbb{R}^2 \mid \underline{\alpha} < \alpha \le \lambda_{k+1},\ \psi_k(\alpha) < \beta \le \lambda_{k+1}\}$$

(*where $\underline{\alpha}$ is the unique solution of $\psi_k(\underline{\alpha}) = \lambda_{k+1}$ and $\overline{\alpha}$ is the unique solution of $\varphi_{k+1}(\overline{\alpha}) = \lambda_k$*) *with the property $\mathcal{S}_k \cap \Sigma = \emptyset$* (see Fig. 1).

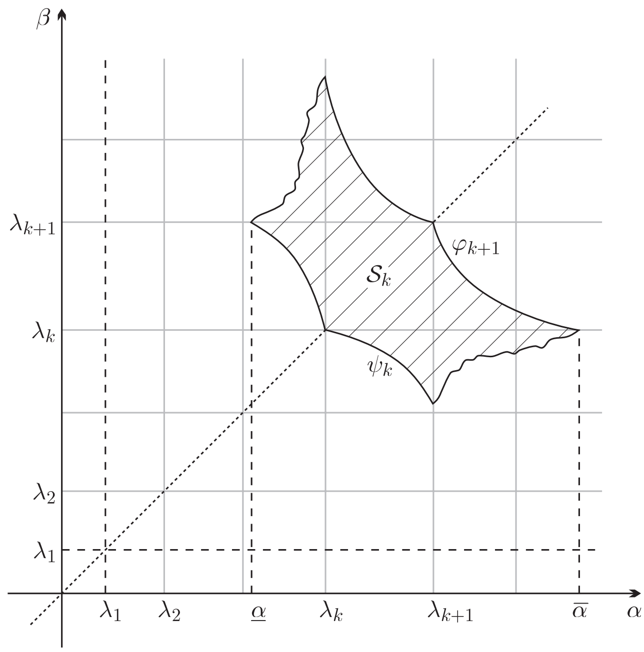

Figure 1

Moreover, in §2 we use the above statement to prove (see Theorem (2.1)) the existence of three solutions of a jumping problem in the region $\mathcal{S}_k \cap \{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha > \lambda_{k+1} \text{ or } \alpha < \lambda_k\}$.

1. The statement

We recall some basic definitions and set up some terminology.

(1.1) Definition. Let $(\lambda_n)_{n\geq 1}$ be the sequence of eigenvalues of the problem $\Delta u + \lambda u = 0$, $u \in H_0^1(\Omega)$. We recall that $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_i \leq \ldots$ and $\lim_n \lambda_n = +\infty$. Let e_n be an eigenfunction corresponding to λ_n, with $\|e_n\|_{L^2(\Omega)} = 1$. We can choose e_1 such that $e_1 > 0$ in Ω. Moreover, set $\mathrm{H}_i = \mathrm{span}(e_1, \ldots, e_i)$ and $\mathrm{H}_i^\perp = \{w \in H_0^1(\Omega) \mid (u, w) = 0\ \forall u \in \mathrm{H}_i\}$.

(1.2) Definition. If $(\alpha, \beta) \in \mathbb{R}^2$, define the functional $Q_{\alpha,\beta} : H_0^1(\Omega) \to \mathbb{R}$ by

$$Q_{\alpha,\beta}(u) = \int_\Omega (|\nabla u|^2 - \alpha(u^+)^2 - \beta(u^-)^2).$$

(1.3) Definition. If $i \geq 1$, define

$$\mathcal{M}_i(\alpha, \beta) = \{u \in H_0^1(\Omega) \mid Q'_{\alpha,\beta}(u)(v) = 0\ \forall v \in \mathrm{H}_i\}.$$

(1.4) Remark. It is well known that if $\alpha > \lambda_i$ and $\beta > \lambda_i$ then $\mathcal{M}_i(\alpha, \beta)$ is the graph of a positive homogeneous and Lipschitz continuous map $\gamma_i(\alpha, \beta) : \mathrm{H}_i^\perp \to \mathrm{H}_i$, which is characterized by the property

$$\forall w \in \mathrm{H}_i^\perp\ \exists_1 \gamma_i(\alpha, \beta)(w) \in \mathrm{H}_i \text{ such that } Q_{\alpha,\beta}(\gamma_i(\alpha, \beta)(w) + w) = \max_{v \in \mathrm{H}_i} Q_{\alpha,\beta}(v + w).$$

First of all we extend the above statement to the case when either $\alpha = \lambda_i$ or $\beta = \lambda_i$.

(1.5) Proposition. *Let $i \geq 2$. If either $\alpha > \lambda_i$ and $\beta = \lambda_i$ or $\alpha = \lambda_i$ and $\beta > \lambda_i$, then $\mathcal{M}_i(\alpha, \beta)$ is the graph of a positive homogeneous and continuous map $\gamma_i(\alpha, \beta) : \mathrm{H}_i^\perp \to \mathrm{H}_i$.*

Proof. To fix ideas, we assume $\alpha > \lambda_i$ and $\beta = \lambda_i$.

Step 1. $\forall w \in \mathrm{H}_i^\perp\ \exists \overline{v} \in \mathrm{H}_i$ *such that* $Q_{\alpha,\lambda_i}(\overline{v} + w) = \max_{v \in \mathrm{H}_i} Q_{\alpha,\lambda_i}(v + w)$.

It is enough to observe that for fixed $w \in \mathrm{H}_i^\perp$,

$$\lim_{\substack{v \in \mathrm{H}_i \\ \|v\| \to +\infty}} Q_{\alpha,\lambda_i}(v + w) = -\infty. \tag{1.6}$$

Let $(v_n)_{n\in\mathbb{N}}$ in H_i be such that $\lim_n \|v_n\| = +\infty$. We can assume that, up to a subsequence, $\lim_n v_n/\|v_n\| = v$ strongly in $H_0^1(\Omega)$. In particular, $v \in \mathrm{H}_i$ and

$\|v\| = 1$. Therefore we get

$$\lim_n \frac{Q_{\alpha,\lambda_i}(v_n + w)}{\|v_n\|^2} = 1 - \alpha\int_\Omega (v^+)^2 - \lambda_i\int_\Omega (v^-)^2 = Q_{\alpha,\lambda_i}(v).$$

We obtain (1.6) by using the following property:

$$\max_{\substack{v\in \mathrm{H}_i\\ \|v\|=1}} Q_{\alpha,\lambda_i}(v) < 0. \tag{1.7}$$

Let us prove (1.7). First of all, since $v \in \mathrm{H}_i$, we have $\int_\Omega |\nabla v|^2 \le \lambda_i \int_\Omega v^2$ and so $Q_{\alpha,\lambda_i}(v) \le (\lambda_i - \alpha)\int_\Omega (v^+)^2 \le 0$, because $\alpha > \lambda_i$. Secondly, arguing by contradiction, if $Q_{\alpha,\lambda_i}(v) = 0$ then $v^+ = 0$; so $0 = Q_{\alpha,\lambda_i}(v) = \int_\Omega |\nabla v|^2 - \lambda_i \int_\Omega v^2$. This implies $v \in \mathrm{Ker}(\Delta - \lambda_i I)$; so v changes sign in Ω, because $i \ge 2$. Finally, since $v^+ = 0$, we have $v = 0$, which contradicts the fact that $\|v\| = 1$.

Step 2. $\forall w \in \mathrm{H}_i^\perp\ \exists_1 \overline{v} \in \mathrm{H}_i$ *such that* $Q'_{\alpha,\lambda_i}(\overline{v} + w)(v) = 0\ \forall v \in \mathrm{H}_i$.

Arguing by contradiction, suppose that there exist $v_1 \in \mathrm{H}_i$ and $v_2 \in \mathrm{H}_i$ such that $v_1 \ne v_2$ and $Q'_{\alpha,\lambda_i}(v_1 + w)(v) = 0$ and $Q'_{\alpha,\lambda_i}(v_2 + w)(v) = 0$ for all $v \in \mathrm{H}_i$. In particular, if $v = v_1 - v_2$ we obtain

$$\int_\Omega \nabla v_1 \nabla(v_1 - v_2) - \alpha(w + v_1)^+(v_1 - v_2) + \lambda_i(w + v_1)^-(v_1 - v_2) = 0$$

and

$$\int_\Omega \nabla v_2 \nabla(v_1 - v_2) - \alpha(w + v_2)^+(v_1 - v_2) + \lambda_i(w + v_2)^-(v_1 - v_2) = 0.$$

Therefore

$$\begin{aligned} &\int_\Omega |\nabla(v_1 - v_2)|^2 \\ &\quad = \int_\Omega \{\alpha[(w + v_1)^+ - (w + v_2)^+] - \lambda_i[(w + v_1)^- - (w + v_2)^-]\}(v_1 - v_2). \end{aligned} \tag{1.8}$$

First of all, observe that

$$\lambda_i(t - s)^2 \le (\alpha(t^+ - s^+) - \lambda_i(t^- - s^-))(t - s) \le \alpha(t - s)^2 \quad \forall t, s \in \mathbb{R}. \tag{1.9}$$

By (1.9) and (1.8) we get $\lambda_i \int_\Omega (v_1 - v_2)^2 \le \|v_1 - v_2\|^2 \le \alpha\int_\Omega (v_1 - v_2)^2$. However, since $v_1 - v_2 \in \mathrm{H}_i$, we also have

$$\|v_1 - v_2\|^2 = \lambda_i \int_\Omega (v_1 - v_2)^2. \tag{1.10}$$

In particular, we deduce $v_1 - v_2 \in \mathrm{Ker}(\Delta - \lambda_i I) \setminus \{0\}$ and so, since $i \ge 2$, we get

$$\mathrm{meas}\{x \in \Omega \mid v_1(x) = v_2(x)\} = 0. \tag{1.11}$$

On the other hand, if we consider again the expression (1.8), by using (1.10), we deduce

$$0=\int_\Omega\{\alpha[(w+v_1)^+-(w+v_2)^+]-\lambda_i[(w+v_1)^--(w+v_2)^-]\}(v_1-v_2)-\lambda_i(v_1-v_2)^2;$$

by taking into account that the integrand is positive in Ω in view of (1.9), we also get

$$\{\alpha[(w+v_1)^+-(w+v_2)^+]-\lambda_i[(w+v_1)^--(w+v_2)^-]\}(v_1-v_2) \\ =\lambda_i(v_1-v_2)^2 \qquad \text{a.e. in } \Omega. \tag{1.12}$$

Finally, by (1.12) and (1.11) we deduce that $(w+v_1)(x)\le 0$ and $(w+v_2)(x)\le 0$ a.e. in Ω. In fact, if $(w+v_1)(x)>0$ and $(w+v_2)(x)>0$ on a set of positive measure, then by (1.12) we get $(\alpha-\lambda_i)(v_1(x)-v_2(x))^2=0$ and so $v_1(x)=v_2(x)$ on such a set, which is absurd; on the other hand, if $(w+v_1)(x)>0$ and $(w+v_2)(x)\le 0$ on a set of positive measure, then by (1.12) we get again $(\alpha-\lambda_i)(w(x)+v_1(x))(v_1(x)-v_2(x))=0$ and so $v_1(x)=v_2(x)$ (similarly if $(w+v_1)(x)\le 0$ and $(w+v_2)(x)>0$).

We will get a final contradiction by showing that the functions $w+v_1\ne 0$ and $w+v_2\ne 0$ have to change sign in Ω. In fact, since $Q'_{\alpha,\lambda_i}(v_1+w)(v)=0$ for all $v\in \mathrm{H}_i$, we have $\Delta(v_1+w)+\alpha(v_1+w)^+-\lambda_i(v_1+w)^-\in \mathrm{H}_i^\perp$. If $(v_1+w)^+=0$ then either $v_1+w\in \mathrm{Ker}(\Delta-\lambda_i I)$ or $v_1+w\in \mathrm{H}_i^\perp$; so it follows that $v_1+w=0$ a.e. in Ω, which is absurd. On the other hand, if $(v_1+w)^-=0$ then $v_1+w\in \mathrm{H}_i^\perp$, and so we have again a contradiction.

STEP 3. *The function* $\gamma_i(\alpha,\lambda_i):\mathrm{H}_i^\perp\to \mathrm{H}_i$ *defined by*

$$Q_{\alpha,\lambda_i}(\gamma_i(\alpha,\lambda_i)(w)+w)=\max_{v\in \mathrm{H}_i} Q_{\alpha,\lambda_i}(v+w) \tag{1.13}$$

is positive homogeneous and continuous from $\mathrm{H}_i^\perp$ *equipped with the weak topology.*

It is easy to verify that $\gamma_i(\alpha,\lambda_i)$ is positive homogeneous, that is, $\gamma_i(\alpha,\lambda_i)(tw)=t\gamma_i(\alpha,\lambda_i)(w)$ for all $w\in \mathrm{H}_i^\perp$ and for all $t\ge 0$.

Let us prove the continuity of $\gamma_i(\alpha,\lambda_i)$. Let $(w_n)_{n\in\mathbb{N}}$ and w in $\mathrm{H}_i^\perp$ be such that $\lim_n w_n=w$ weakly in $\mathrm{H}_i^\perp$. If $v_n=\gamma_i(\alpha,\lambda_i)(w_n)$ by (1.13) we get

$$v_n-\mathrm{P}_{\mathrm{H}_i} i^*(\alpha(v_n+w_n)^+-\lambda_i(v_n+w_n)^-)=0, \tag{1.14}$$

where $\mathrm{P}_{\mathrm{H}_i}:H_0^1(\Omega)\to \mathrm{H}_i$ denotes the orthogonal projection and i^* is the adjoint operator of the Sobolev imbedding $i:H_0^1(\Omega)\to L^2(\Omega)$.

First of all we observe that the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded. In fact, arguing by contradiction, we can assume that, up to a subsequence, $\lim_n v_n/\|v_n\|=v$ strongly in $H_0^1(\Omega)$. In particular, $v\in \mathrm{H}_i$ and $\|v\|=1$. As a result, if we multiply (1.14) by $v_n/\|v_n\|^2$ and pass to the limit, we get $0=1-\alpha\int_\Omega(v^+)^2-\lambda_i\int_\Omega(v^-)^2=Q_{\alpha,\lambda_i}(v)$, which is absurd in virtue of (1.7).

Therefore, we can assume that $\lim_n v_n = v$ strongly in $H_0^1(\Omega)$; finally, by (1.14) we obtain $v - \mathrm{P}_{\mathrm{H}_i} i^*(\alpha(v+w)^+ - \lambda_i(v+w)^-) = 0$, and then $v = \gamma_i(\alpha, \lambda_i)(w)$, by uniqueness (see Step 2). □

(1.15) Definition. Let $i \geq 2$. If $\alpha \geq \lambda_i$ and $\beta \geq \lambda_i$ with $(\alpha,\beta) \neq (\lambda_i, \lambda_i)$, set

$$m_i(\alpha,\beta) = \inf_{\substack{w \in \mathrm{H}_i^\perp \\ \|w\|=1}} Q_{\alpha,\beta}(\gamma_i(\alpha,\beta)(w) + w).$$

(1.16) Remark. We point out that if $m_i(\alpha,\beta) > 0$ then $(\alpha,\beta) \notin \Sigma$. In fact, if $(\alpha,\beta) \in \Sigma$ then there exists $u \in H_0^1(\Omega)$, $u \neq 0$, such that $\Delta u + \alpha u^+ - \beta u^- = 0$. Therefore $u \in \mathcal{M}_i(\alpha,\beta)$ and $Q_{\alpha,\beta}(u) = 0$. It follows that $m_i(\alpha,\beta) \leq 0$.

At this stage, by the properties of m_i, we will find a region in the (α,β) plane where $m_i(\alpha,\beta) > 0$ and give a characterization of the number $\overline{\alpha} = \sup\{\alpha > \lambda_i \mid (\alpha, \lambda_i) \notin \Sigma\}$.

(1.17) Lemma. *Let $i \geq 2$ be such that $\lambda_i < \lambda_{i+1}$. If $\alpha \geq \lambda_i$ and $\beta \geq \lambda_i$ with $(\alpha,\beta) \neq (\lambda_i,\lambda_i)$, then the function m_i has the following properties:*

(a) $m_i(\alpha,\beta) = m_i(\beta,\alpha)$;
(b) m_i *is continuous with respect to* (α,β);
(c) m_i *is strictly decreasing with respect to both* α *and* β;
(d) $m_i(\lambda_{i+1}, \lambda_{i+1}) = 0$;
(e) $\alpha > \lambda_{i+1},\ \beta > \lambda_{i+1} \Rightarrow m_i(\alpha,\beta) < 0$;
(f) $\alpha < \lambda_{i+1},\ \beta < \lambda_{i+1} \Rightarrow m_i(\alpha,\beta) > 0$;
(g) $\alpha \geq \lambda_i \Rightarrow \lim_{\beta \to +\infty} m_i(\alpha,\beta) = -\infty$.

Proof. (a) This is an immediate consequence of the property $\gamma_i(\alpha,\beta)(-w) = -\gamma_i(\beta,\alpha)(w)$ for all $w \in \mathrm{H}_i^\perp$.

(b) Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ be such that $\lim_n \alpha_n = \alpha > \lambda_i$, $\lim_n \beta_n = \beta \geq \lambda_i$ and $\alpha \geq \beta$. We show that $\lim_n m_i(\alpha_n,\beta_n) = m_i(\alpha,\beta)$.

By the definition of m_i, for $\varepsilon > 0$ there exists $(w_n)_{n\in\mathbb{N}}$ in $\mathrm{H}_i^\perp$ with $\|w_n\| = 1$ such that $\lim_n w_n = w$ weakly in $\mathrm{H}_i^\perp$ and

$$m_i(\alpha_n,\beta_n) \leq Q_{\alpha_n,\beta_n}(\gamma_i(\alpha_n,\beta_n)(w_n) + w_n) \leq m_i(\alpha_n,\beta_n) + \varepsilon. \tag{1.18}$$

Set $\gamma_i(\alpha_n,\beta_n)(w_n) = v_n$. We also recall that

$$v_n - \mathrm{P}_{\mathrm{H}_i} i^*(\alpha_n(v_n + w_n)^+ - \beta_n(v_n + w_n)^-) = 0, \tag{1.19}$$

where $\mathrm{P}_{\mathrm{H}_i} : H_0^1(\Omega) \to \mathrm{H}_i$ denotes the orthogonal projection and i^* is the adjoint operator of the Sobolev imbedding $i : H_0^1(\Omega) \to L^2(\Omega)$.

Observe that the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded. In fact, arguing by contradiction, we can assume that, up to a subsequence, $\lim_n v_n/\|v_n\| = v$ strongly in $H_0^1(\Omega)$. In particular, $v \in \mathrm{H}_i$ and $\|v\| = 1$. As a result, if we divide (1.19) by $\|v_n\|$

and pass to the limit, we get $v - \mathrm{P}_{\mathrm{H}_i} i^*(\alpha(v+w)^+ - \lambda_i(v+w)^-) = 0$, which implies $Q_{\alpha,\beta}(v) = 0$. On the other hand, since $\beta \le \lambda_i$, we have $Q_{\alpha,\beta}(v) \le Q_{\alpha,\lambda_i}(v) < 0$, by (1.7). Thus a contradiction arises.

That is why we can assume that, up to a subsequence, $\lim_n v_n = v$ strongly in $H_0^1(\Omega)$; finally, by passing to the limit in (1.19) we obtain

$$v - \mathrm{P}_{\mathrm{H}_i} i^*(\alpha(v+w)^+ - \lambda_i(v+w)^-) = 0,$$

and then $v = \gamma_i(\alpha, \lambda_i)(w)$ by uniqueness (see Step 2 in the proof of Proposition (1.5)).

Moreover, by (1.18), ε being arbitrary, we obtain

$$\lim_n m_i(\alpha_n, \beta_n) = 1 + \int_\Omega |\nabla v|^2 - \alpha \int_\Omega ((v+w)^+)^2 - \beta \int_\Omega ((v+w)^-)^2.$$

Now we claim that

$$\lim_n m_i(\alpha_n, \beta_n) \le m_i(\alpha, \beta). \tag{1.20}$$

In fact, by the second inequality of (1.18) and by the definition (1.15), it follows that for all $\overline{w} \in \mathrm{H}_i^\perp$ with $\|\overline{w}\| = 1$,

$$Q_{\alpha_n,\beta_n}(v_n + w_n) \le Q_{\alpha_n,\beta_n}(\gamma_i(\alpha_n, \beta_n)(\overline{w}) + \overline{w}) + \varepsilon$$

and, by passing to the limit,

$$1 + \int_\Omega |\nabla v|^2 - \alpha \int_\Omega ((v+w)^+)^2 - \beta \int_\Omega ((v+w)^-)^2 \le Q_{\alpha,\beta}(\gamma_i(\alpha,\beta)(\overline{w}) + \overline{w}) + \varepsilon;$$

so (1.20) follows.

Finally, we show that,

$$\lim_n m_i(\alpha_n, \beta_n) \ge m_i(\alpha, \beta). \tag{1.21}$$

First, if $w = 0$ then also $v = 0$; so $\lim_n m_i(\alpha_n, \beta_n) = 1$. On the other hand, for all $\overline{w} \in \mathrm{H}_i^\perp$ with $\|\overline{w}\| = 1$,

$$\begin{aligned} m_i(\alpha,\beta) &\le Q_{\alpha,\beta}(\gamma_i(\alpha,\beta)(\overline{w}) + \overline{w}) \\ &= 1 + \int_\Omega |\nabla \gamma_i(\alpha,\beta)(\overline{w})|^2 - \alpha \int_\Omega ((\gamma_i(\alpha,\beta)(\overline{w}) + \overline{w})^+)^2 \\ &\quad - \beta \int_\Omega ((\gamma_i(\alpha,\beta)(\overline{w}) + \overline{w})^-)^2 \le 1, \end{aligned}$$

since $\alpha \ge \beta \ge \lambda_i$ and $\gamma_i(\alpha,\beta)(\overline{w}) \in \mathrm{H}_i$. Therefore (1.21) follows.

Next, if $w \neq 0$ then we put $w^* = w/\|w\|$ and so

$$\begin{aligned} m_i(\alpha,\beta) &\leq Q_{\alpha,\beta}(\gamma_i(\alpha,\beta)(w^*) + w^*) \\ &= 1 + \frac{1}{\|w\|}\bigg(\int_\Omega |\nabla\gamma_i(\alpha,\beta)(w)|^2 - \alpha\int_\Omega ((\gamma_i(\alpha,\beta)(w)+w)^+)^2 \\ &\quad - \beta\int_\Omega ((\gamma_i(\alpha,\beta)(w)+w)^-)^2\bigg) \\ &\leq 1 + \int_\Omega |\nabla\gamma_i(\alpha,\beta)(w)|^2 - \alpha\int_\Omega ((\gamma_i(\alpha,\beta)(w)+w)^+)^2 \\ &\quad - \beta\int_\Omega ((\gamma_i(\alpha,\beta)(w)+w)^-)^2 \\ &= \lim_n m_i(\alpha_n,\beta_n). \end{aligned}$$

Therefore (1.21) also holds in this case.

(c) Let $\alpha > \lambda_i$ and $\beta' > \beta \geq \lambda_i$. We will show that $m_i(\alpha,\beta) > m_i(\alpha,\beta')$. By the definition of $\gamma_i(\alpha,\beta)$ we get, for any $w \in \mathrm{H}_i^\perp$,

$$\begin{aligned} Q_{\alpha,\beta}(\gamma_i(\alpha,\beta)(w) + w) &\geq Q_{\alpha,\beta}(\gamma_i(\alpha,\beta')(w) + w) \\ &= Q_{\alpha,\beta'}(\gamma_i(\alpha,\beta')(w) + w) \\ &\quad + (\beta-\beta')\int_\Omega ((\gamma_i(\alpha,\beta')(w)+w)^-)^2 \\ &\geq m_i(\alpha,\beta') + (\beta-\beta') \min_{\substack{w\in \mathrm{H}_i^\perp \\ \|w\|=1}} \int_\Omega ((\gamma_i(\alpha,\beta')(w)+w)^-)^2. \end{aligned}$$

As a result we obtain

$$m_i(\alpha,\beta) \geq m_i(\alpha,\beta') + (\beta-\beta') \min_{\substack{w\in \mathrm{H}_i^\perp \\ \|w\|=1}} \int_\Omega ((\gamma_i(\alpha,\beta')(w)+w)^-)^2.$$

In order to get our claim, it is enough to prove that for any $\alpha > \lambda_i$ and $\beta \geq \lambda_i$, if $u \in \mathcal{M}(\alpha,\beta)\setminus\{0\}$ then $u^- \neq 0$. In fact, if $u \in \mathcal{M}(\alpha,\beta)\setminus\{0\}$, then $u = \gamma_i(\alpha,\beta)(w) + w$ with $w \in \mathrm{H}_i^\perp$, $w \neq 0$. Suppose $u^- = 0$. If $\gamma_i(\alpha,\beta)(w) = 0$, then $u = w \in \mathrm{H}_i^\perp$ and so $u = 0$. On the other hand, if $\gamma_i(\alpha,\beta)(w) \neq 0$, then by the definition of $\gamma_i(\alpha,\beta)$ and by (1.7) we get

$$0 = Q'_{\alpha,\beta}(u)(\gamma_i(\alpha,\beta)(w)) = 2Q_{\alpha,\beta}(\gamma_i(\alpha,\beta)(w)) \leq 2Q_{\alpha,\lambda_i}(\gamma_i(\alpha,\beta)(w)) < 0,$$

which is absurd.

(d) First, if $w \in \mathrm{H}_i^\perp$, then $\gamma_i(\lambda_{i+1},\lambda_{i+1})(w) = 0$; in fact, by the definition of $\gamma_i(\lambda_{i+1},\lambda_{i+1})$ we have $\Delta\gamma_i(\lambda_{i+1},\lambda_{i+1})(w) - \lambda_{i+1}(\gamma_i(\lambda_{i+1},\lambda_{i+1})(w) + w) \in \mathrm{H}_i^\perp$, which implies $\gamma_i(\lambda_{i+1},\lambda_{i+1})(w) = 0$, since $\gamma_i(\lambda_{i+1},\lambda_{i+1})(w) \in \mathrm{H}_i$. Moreover, for any $w \in \mathrm{H}_i^\perp$,

$$Q_{\lambda_{i+1},\lambda_{i+1}}(\gamma_i(\lambda_{i+1},\lambda_{i+1})(w) + w) = \int_\Omega |\nabla w|^2 - \lambda_{i+1}\int_\Omega w^2 \geq 0.$$

Then if $w \in \operatorname{Ker}(\Delta - \lambda_{i+1} I)$, we get our claim.

(e) If $w \in \operatorname{Ker}(\Delta - \lambda_{i+1} I)$, $w \neq 0$, then for any $v \in \mathrm{H}_i$,

$$\begin{aligned} Q_{\alpha,\beta}(v+w) &= \int_\Omega |\nabla(v+w)|^2 - \alpha \int_\Omega ((v+w)^+)^2 - \beta \int_\Omega ((v+w)^-)^2 \\ &\leq (\lambda_{i+1} - \alpha) \int_\Omega ((v+w)^+)^2 + (\lambda_{i+1} - \beta) \int_\Omega ((v+w)^-)^2 < 0. \end{aligned}$$

(f) If $w \in \mathrm{H}_i^\perp$, $w \neq 0$, then

$$\begin{aligned} Q_{\alpha,\beta}(w) &= \int_\Omega |\nabla w|^2 - \alpha \int_\Omega (w^+)^2 - \beta \int_\Omega (w^-)^2 \\ &\geq (\lambda_{i+1} - \alpha) \int_\Omega (w^+)^2 + (\lambda_{i+1} - \beta) \int_\Omega (w^-)^2 > 0. \end{aligned}$$

(g) First, observe that there is $w^* \in \mathrm{H}_i^\perp$ with $\|w^*\| = 1$ such that $(w^* + \mathrm{H}_i) \cap \{u \in H_0^1(\Omega) \mid u \geq 0 \text{ a.e. in } \Omega\} = \emptyset$. In fact, if $n \geq 2$ we can choose $w_0 \in H_0^1(\Omega)$ with $\operatorname{ess\,inf} w_0 = -\infty$ and if $n = 1$ we can choose $w_0(x) = [\operatorname{dist}(x, \partial\Omega)]^\delta$ with $1/2 < \delta < 1$; so w^* denotes the component of w_0 on $\mathrm{H}_i^\perp$ normalized in $H_0^1(\Omega)$.

Therefore it is enough to prove that if $\alpha \geq \lambda_i$ then

$$\lim_{\beta \to +\infty} Q_{\alpha,\beta}(\gamma_i(\alpha,\beta)(w^*) + w^*) = -\infty.$$

Let $(\beta_n)_{n\in\mathbb{N}}$ be such that $\lim_n \beta_n = +\infty$ and set $v_n = \gamma_i(\alpha, \beta_n)(w^*)$. We have

$$\begin{aligned} Q_{\alpha,\beta_n}(v_n + w^*) = 1 + \|v_n\|^2 - \alpha \int_\Omega ((v_n + w^*)^+)^2 \\ - \beta_n \int_\Omega ((v_n + w^*)^-)^2. \end{aligned} \tag{1.22}$$

Now if $(v_n)_{n\in\mathbb{N}}$ is bounded then, up to a subsequence, $\lim_n v_n = v \in \mathrm{H}_i$ in $H_0^1(\Omega)$ and $(v + w^*)^- \neq 0$, by the property of w^*; so $\lim_n Q_{\alpha,\beta_n}(v_n + w^*) = -\infty$.

On the other hand, if $\lim_n \|v_n\| = +\infty$, we can suppose $\lim_n v_n/\|v_n\| = v \in \mathrm{H}_i$ in $H_0^1(\Omega)$, $\|v\| = 1$. If, by contradiction, $(Q_{\alpha,\beta_n}(v_n + w^*))_{n\in\mathbb{N}}$ is bounded from below, from (1.22) (dividing by $\|v_n\|^2$ and passing to the limit) we get $v \geq 0$ a.e. in Ω. Moreover, since

$$Q_{\alpha,\beta_n}(v_n + w^*) \leq \|v_n\|^2 - \alpha \int_\Omega ((v_n + w^*)^+)^2$$

we also obtain

$$0 \leq 1 - \alpha \int_\Omega v^2 \leq 1 - \frac{\alpha}{\lambda_i}.$$

Finally, if $\alpha > \lambda_i$ a contradiction arises immediately; if $\alpha = \lambda_i$ we get $v \in \operatorname{Ker}(\Delta - \lambda_i I) \setminus \{0\}$, which is absurd because $v \geq 0$ a.e. in Ω. □

From Lemma (1.17) we deduce immediately the following result.

(1.23) Proposition. *Let $i \geq 2$ be such that $\lambda_i < \lambda_{i+1}$. There exist a unique $\overline{\alpha} > \lambda_{i+1}$ and a continuous strictly decreasing map $\varphi_{i+1} : [\lambda_i, \overline{\alpha}] \to [\lambda_i, \overline{\alpha}]$ such that $\varphi_{i+1}(\lambda_{i+1}) = \lambda_{i+1}$, $\varphi_{i+1}(\overline{\alpha}) = \lambda_i$ and $\varphi_{i+1} \circ \varphi_{i+1} = I$, with the property*

$$\lambda_i \leq \beta < \varphi_{i+1}(\alpha) \Leftrightarrow m_i(\alpha, \beta) > 0.$$

(1.24) Remark. By (1.16) and (1.23), the number $\overline{\alpha} = \sup\{\alpha > \lambda_i \mid (\alpha, \lambda_i) \notin \Sigma\}$ satisfies $\overline{\alpha} > \lambda_{i+1}$ and $\varphi_{i+1}(\overline{\alpha}) = \lambda_i$. Moreover, $\varphi_{i+1}(\lambda_i) = \overline{\alpha} = \sup\{\beta > \lambda_i \mid (\lambda_i, \beta) \notin \Sigma\}$.

(1.25) Remark. It is easy to prove that the functions defined in (1.23) coincide with the functions μ_{i+1} introduced in [MMP] and the functions J_- introduced in [M].

Now we will give a characterization of $\inf\{\beta < \lambda_{k+1} \mid (\lambda_{k+1}, \beta) \notin \Sigma\}$ for $k \geq 1$. We are not able to proceed as in the previous case, since the set

$$\mathcal{N}_k(\alpha, \beta) = \{u \in H_0^1(\Omega) \mid Q'_{\alpha,\beta}(u)(w) = 0 \ \forall w \in \mathrm{H}_k^\perp\},$$

which is the graph of a suitable map when $\alpha < \lambda_{k+1}$ and $\beta < \lambda_{k+1}$, does not have this property when either $\alpha < \lambda_{k+1}$ and $\beta = \lambda_{k+1}$ or $\alpha = \lambda_{k+1}$ and $\beta < \lambda_{k+1}$. In fact, the following result holds.

(1.26) Remark. If $\beta \leq \lambda_{k+1}$, then there exist infinitely many $\overline{w} \in \mathrm{H}_k^\perp$ such that

$$Q_{\lambda_{k+1},\beta}(e_1 + \overline{w}) = \min_{w \in \mathrm{H}_k^\perp} Q_{\lambda_{k+1},\beta}(e_1 + w).$$

Indeed, since $w \in \mathrm{H}_k^\perp$ and $\beta < \lambda_{k+1}$, we have

$$\begin{aligned} Q_{\lambda_{k+1},\beta}(e_1 + w) &= Q_{\lambda_{k+1},\beta}(e_1) + \int_\Omega |\nabla w|^2 - \lambda_{k+1} \int_\Omega w^2 \\ &\quad + (\lambda_{k+1} - \beta) \int_\Omega ((e_1 + w)^-)^2 \\ &\geq Q_{\lambda_{k+1},\beta}(e_1). \end{aligned}$$

Moreover, there exists $\varrho > 0$ such that $e_1 + \varrho e > 0$ for all $e \in \mathrm{Ker}(\Delta - \lambda_{k+1} I)$ with $\|e\| = 1$. Hence

$$Q_{\lambda_{k+1},\beta}(e_1 + \varrho e) = Q_{\lambda_{k+1},\beta}(e_1) + \varrho^2 \left(\int_\Omega |\nabla e|^2 - \lambda_{k+1} \int_\Omega e^2 \right) = Q_{\lambda_{k+1},\beta}(e_1).$$

The previous remark suggests to proceed in the following different way.

(1.27) Definition. If $k \geq 2$ define

$$\mathcal{Z}_k(\alpha, \beta) = \{u \in H_0^1(\Omega) \mid Q'_{\alpha,\beta}(u)(z) = 0 \ \forall z \in \mathrm{H}_1 \oplus \mathrm{H}_k^\perp\}.$$

(1.27) Remark. It is well known that if $\lambda_1 < \alpha < \lambda_{k+1}$ and $\lambda_1 < \beta < \lambda_{k+1}$ then $\mathcal{Z}_k(\alpha,\beta)$ is the graph of a positive homogeneous and Lipschitz continuous map $\zeta_k(\alpha,\beta) : \mathrm{H}_k \cap \mathrm{H}_1^\perp \to \mathrm{H}_1 \oplus \mathrm{H}_k^\perp$, which is characterized by the property

$$\forall v \in \mathrm{H}_k \cap \mathrm{H}_1^\perp\ \exists_1 \zeta_k(\alpha,\beta)(v) \in \mathrm{H}_1 \oplus \mathrm{H}_k^\perp \text{ such that}$$
$$Q_{\alpha,\beta}(v + \zeta_k(\alpha,\beta)(v)) = \min_{w\in \mathrm{H}_k^\perp} \max_{s\in\mathbb{R}} Q_{\alpha,\beta}(se_1 + v + w).$$

We extend this to the case when either $\alpha = \lambda_{k+1}$ or $\beta = \lambda_{k+1}$.

(1.29) Proposition. *Let $k \ge 2$. If either $\alpha < \lambda_{k+1}$ and $\beta = \lambda_{k+1}$ or $\alpha = \lambda_{k+1}$ and $\beta < \lambda_{k+1}$, then the set $\mathcal{Z}_k(\alpha,\beta)$ is the graph of a positive homogeneous and continuous map $\zeta_k(\alpha,\beta) : \mathrm{H}_k \cap \mathrm{H}_1^\perp \to \mathrm{H}_1 \oplus \mathrm{H}_k^\perp$.*

Proof. The proof is similar to that of Proposition (1.5). We only point out the following properties. For simplicity we consider the case $\alpha = \lambda_{k+1}$ and $\beta < \lambda_{k+1}$.

$$\forall v \in \mathrm{H}_k \cap \mathrm{H}_1^\perp,\ \forall w \in \mathrm{H}_k^\perp, \qquad \lim_{|s|\to+\infty} Q_{\alpha,\beta}(se_1 + v + w) = -\infty,$$
$$\forall v \in \mathrm{H}_k \cap \mathrm{H}_1^\perp,\ \forall w \in \mathrm{H}_k^\perp, \qquad s \to Q_{\alpha,\beta}(se_1 + v + w) \text{ is strictly concave},$$
$$\forall v \in \mathrm{H}_k \cap \mathrm{H}_1^\perp,\ \forall s \in \mathbb{R}, \qquad \lim_{\substack{w\in \mathrm{H}_k^\perp \\ \|w\|\to+\infty}} Q_{\alpha,\beta}(se_1 + v + w) = +\infty,$$
$$\forall v \in \mathrm{H}_k \cap \mathrm{H}_1^\perp,\ \forall s \in \mathbb{R}, \qquad w \to Q_{\alpha,\beta}(se_1 + v + w) \text{ is weakly convex}.$$

As a result, in virtue of [Ro] and [EK], we deduce that

$$\forall v \in \mathrm{H}_k \cap \mathrm{H}_1^\perp\ \exists_1 \overline{s} \in \mathbb{R}\ \exists \overline{w} \in \mathrm{H}_k^\perp \text{ such that}$$
$$Q_{\alpha,\beta}(\overline{s}e_1 + v + \overline{w}) = \min_{w\in \mathrm{H}_k^\perp} \max_{s\in\mathbb{R}} Q_{\alpha,\beta}(se_1 + v + w).$$

Arguing as in the second step of the proof of (1.5), we can show the uniqueness of $\overline{w}$. □

Using a similar argument to the proof of Proposition (1.23), we obtain the following result.

(1.30) Proposition. *Let $k \ge 2$ be such that $\lambda_k < \lambda_{k+1}$. There exist a unique $\underline{\alpha} < \lambda_k$ and a continuous strictly decreasing map $\psi_k : [\underline{\alpha}, \lambda_{k+1}] \to [\underline{\alpha}, \lambda_{k+1}]$ such that $\psi_k(\lambda_k) = \lambda_k$, $\psi_k(\underline{\alpha}) = \lambda_{k+1}$ and $\psi_k \circ \psi_k = I$, with the property*

$$\psi_k(\alpha) < \beta \le \lambda_{k+1} \Leftrightarrow \inf_{\substack{v\in \mathrm{H}_k\cap \mathrm{H}_1^\perp \\ \|v\|=1}} Q_{\alpha,\beta}(v + \zeta_k(\alpha,\beta)(v)) < 0 \quad (\Rightarrow (\alpha,\beta) \notin \Sigma).$$

(1.31) Remark. As in (1.24), the number $\underline{\alpha} = \inf\{\beta < \lambda_{k+1} \mid (\lambda_{k+1},\beta) \notin \Sigma\}$ satisfies $\underline{\alpha} < \lambda_k$ and $\psi_k(\underline{\alpha}) = \lambda_{k+1}$. Moreover, $\psi_k(\lambda_{k+1}) = \underline{\alpha} = \inf\{\alpha < \lambda_{k+1} \mid (\alpha, \lambda_{k+1}) \notin \Sigma\}$.

(1.32) Remark. It is easy to prove that the functions defined in (1.30) coincide with the functions ν_k introduced in [MMP] and the functions J_+ introduced in [M].

Finally, we get our main result.

(1.33) Theorem. *Let $k \geq 2$ with $\lambda_k < \lambda_{k+1}$. There exists an open connected set $\mathcal{S}_k$ such that*

$$\mathcal{S}_k \supset \{(\alpha,\beta) \in \mathbb{R}^2 \mid \lambda_k \leq \alpha < \overline{\alpha},\ \lambda_k \leq \beta < \varphi_{k+1}(\alpha)\} \cup \{(\alpha,\beta) \in \mathbb{R}^2 \mid \underline{\alpha} < \alpha \leq \lambda_{k+1},\ \psi_k(\alpha) < \beta \leq \lambda_{k+1}\},$$

(*where $\underline{\alpha}$ is the unique solution of $\psi_k(\underline{\alpha}) = \lambda_{k+1}$* (see (1.30)) *and $\overline{\alpha}$ is the unique solution of $\varphi_{k+1}(\overline{\alpha}) = \lambda_k$* (see (1.23))), *with the property $\mathcal{S}_k \cap \Sigma = \emptyset$.*

Proof. It is well known that the resonance set Σ is closed in $\mathbb{R}^2$. Our claim follows by (1.16), (1.23), (1.24) and also (1.30), (1.31). □

2. An application

Let $\Omega \subset \mathbb{R}^N$ be an open bounded smooth domain and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a C^1 function, with $|\partial g(x,s)/\partial s| \leq c(1 + |u|^p)$, where $c \in \mathbb{R}$ and $p < 4/(N-2)$, such that

$$(g,\alpha,\beta) \quad \begin{cases} |g(x,s)| \leq a(x) + b|s| \text{ a.e. in } \Omega,\ \forall s \in \mathbb{R},\ \text{with } a \in L^2(\Omega),\ b \in \mathbb{R}; \\ \lim_{s\to+\infty} g(x,s)/s = \alpha \text{ and } \lim_{s\to-\infty} g(x,s)/s = \beta \text{ a.e. in } \Omega. \end{cases}$$

We are interested in the problem

$$(P_t) \qquad \begin{cases} \Delta u + g(x,u) = te_1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $t \in \mathbb{R}$ and e_1 is the positive eigenfunction, normalized in $L^2(\Omega)$, associated with the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$.

(2.1) Theorem. *Let $k \geq 2$ be such that $\lambda_k < \lambda_{k+1}$. Assume (g,α,β) with $(\alpha,\beta) \in \mathcal{S}_k$ and either $\alpha > \lambda_{k+1}$ or $\alpha < \lambda_k$. If the problem (P_t) admits only nondegenerate solutions for t positive and large enough, then (P_t) has at least three solutions for t positive and large enough.*

Proof. We consider the following functional $f_t : H_0^1(\Omega) \to \mathbb{R}$:

$$f_t(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \int_0^{u(x)} g(x,\sigma)\,d\sigma + te_1 u \right) dx,$$

whose critical points are (weak) solutions of (P_t).

Let $\lambda_k < \lambda_{k+1} = \ldots = \lambda_{k+\nu} < \lambda_{k+\nu+1}$. The assumption $(\alpha,\beta) \in \mathcal{S}_k$ and $\alpha > \lambda_{k+1}$ enable us to use the "links and bounds" theorem (see Th. (6.6)) of [MMP]. Therefore the functional f_t has two critical points u_1 and u_2 such that

$$\inf_{\Delta} f_t \le f_t(u_1) \le \sup_{\partial B} f_t < \inf_{\Sigma} f_t \le f_t(u_2) \le \sup_{B} f_t,$$

where

$$B = \left\{ \frac{t}{\alpha-\lambda_1} e_1 + v \,\middle|\, v \in \mathrm{H}_{k+\nu},\ \|v\| \le r \right\}$$

and $\partial B =$ the boundary of B in $\mathrm{H}_{k+\nu}$,

$$\Delta = \left\{ \frac{t}{\alpha-\lambda_1} e_1 + \sigma e + w \,\middle|\, \sigma \ge 0,\ w \in \mathrm{H}_{k+\nu}^{\perp},\ \|\sigma e + w\| \le \varrho \right\},$$

where $e \in \mathrm{H}_{k+\nu},\ e \ne 0$,

$\Sigma =$ the boundary of Δ in $\mathrm{H}_{k+\nu}^{\perp} \oplus \mathrm{span}(e)$ and $\varrho > r$.

By assumption u_1 and u_2 are nondegenerate, therefore we can evaluate their Leray–Schauder indices:

$$i(\nabla f_t, u_1) = (-1)^{k+\nu-1} \quad \text{and} \quad i(\nabla f_t, u_2) = (-1)^{k+\nu}.$$

On the other hand, there exists a path $\theta : [0,1] \to \mathbb{R}^2 \setminus \Sigma$ joining (α,β) to the set $\{(\lambda,\lambda) \mid \lambda \in \mathbb{R},\ \lambda \ne \lambda_i\}$, because $(\alpha,\beta) \in \mathcal{S}_k$. This property ensures (see Th. 6 of [D1]) that for R positive and large enough, $\deg(\nabla f, B_R(0), 0) = (-1)^k$. By the additive property of the degree, we get our claim. □

In [Ra1] a result of the same type was obtained.

(2.2) Remark. We point out that the assumption $g \in C^1(\Omega \times \mathbb{R})$ can be weakened. It is enough to assume that g is a Carathéodory function such that $(\nabla f_t)'(u) : H_0^1(\Omega) \to H_0^1(\Omega)$ is a continuous and symmetric operator for any critical point u of the functional f_t. In such a case u is a nondegenerate solution of (P_t) if $(\nabla f_t)'(u)$ is an isomorphism.

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