

**REGULARITY FOR VISCOSITY SOLUTIONS OF
FULLY NONLINEAR EQUATIONS $F(D^2u) = 0$**

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Dedicated to Louis Nirenberg

1. Introduction

In this paper we study Hölder regularity for the first and second derivatives of continuous viscosity solutions of fully nonlinear equations of the form

$$(1.1) \quad F(D^2u) = 0.$$

It is well known that viscosity solutions of (1.1) are $C^{1,\alpha}$ for some $0 < \alpha < 1$, and in the case that the functional F is convex, they are $C^{2,\alpha}$. In this paper we use the Krylov–Safonov Harnack inequality, Jensen’s approximate solutions and some basic lemmas of real analysis to give new and simpler proofs of these results.

In (1.1), u is a real function defined in a bounded domain Ω of \mathbb{R}^n and D^2u denotes the Hessian of u . F is a real-valued function defined on the space \mathcal{S} of real $n \times n$ symmetric matrices. We assume that F is a *uniformly elliptic operator*, that is, for any $M \in \mathcal{S}$ and any nonnegative definite symmetric matrix N ,

$$(1.2) \quad \lambda \|N\| \leq F(M + N) - F(M) \leq \Lambda \|N\|,$$

where $\lambda \leq \Lambda$ are two positive constants, which are called *ellipticity constants*, and $\|N\|$ denotes the maximum eigenvalue of N .

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Note that (1.2) implies that F is increasing and Lipschitz in $M \in \mathcal{S}$. We do not assume any further regularity on F , and hence we include important examples of equations such as Bellman's equations in optimal control theory (see [19], [20]) and Isaacs' equations in differential games (see [21]).

A continuous function u in Ω is said to be a *viscosity subsolution* (respectively, *viscosity supersolution*) of (1.1) in Ω when the following condition holds: for any $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local maximum at x_0 , we have

$$(1.3) \quad F(D^2\varphi(x_0)) \geq 0$$

(respectively, if $u - \varphi$ has a local minimum at x_0 then $F(D^2\varphi(x_0)) \leq 0$). We say that u is a *viscosity solution* of (1.1) when it is both a viscosity subsolution and supersolution.

It is easy to verify that a $C^2(\Omega)$ function u is a viscosity solution of (1.1) if and only if it is a classical solution. The concept of viscosity solution was introduced by Crandall and Lions [4] and Evans [5], [6]. Its idea is to put the derivatives on a test function via the maximum principle. It is very useful when proving existence of solutions by Perron's method.

Let us denote by $C^{k,\alpha}$ the space of k -times differentiable functions with k th derivatives Hölder continuous with exponent $\alpha \in (0, 1)$ (Lipschitz continuous if $\alpha = 1$). We say that a constant is *universal* if it depends only on n, λ and Λ . By a *ball* we always mean an open ball.

The following theorem states the $C^{1,\alpha}$ regularity of viscosity solutions of (1.1) for some universal $\alpha \in (0, 1)$; it already appears in [25].

THEOREM 1.1. *Let u be a viscosity solution of $F(D^2u) = 0$ in the unit ball B_1 . Then $u \in C^{1,\alpha}(\overline{B}_{1/2})$ and*

$$(1.4) \quad \|u\|_{C^{1,\alpha}(\overline{B}_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + |F(0)|),$$

where $0 < \alpha < 1$ and C are universal constants.

Further regularity can be obtained in the case of concave or convex functionals. We say that F is *concave* if it is a concave function on the space \mathcal{S} of symmetric matrices. Note that if F is concave then $G(M) = -F(-M)$ is convex and still uniformly elliptic. Bellman's equations are examples of concave functionals.

Evans [7] and Krylov [15], [16] independently discovered that classical solutions u of concave equations (1.1) satisfy an interior $C^{2,\alpha}$ *a priori* estimate in terms of $\|u\|_{L^\infty}$. In fact, viscosity solutions of concave equations (1.1) are $C^{2,\alpha}$.

THEOREM 1.2. *Let F be concave and u be a viscosity solution of $F(D^2u) = 0$ in B_1 . Then $u \in C^{2,\alpha}(\overline{B}_{1/2})$ and*

$$(1.5) \quad \|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + |F(0)|),$$

where $0 < \alpha < 1$ and C are universal constants.

In this paper we give new proofs of Theorems 1.1 and 1.2. The main tools that we use are the following. First, the Krylov–Safonov Harnack inequality [17], [18] and related estimates for solutions of second order elliptic equations in nondivergence form with bounded measurable coefficients; that is, equations of the form

$$(1.6) \quad a_{ij}(x)\partial_{ij}u(x) = f(x);$$

we use summation convention over repeated indices.

The other important tool is Jensen’s approximate solutions of equation (1.1), first introduced in [13] and further studied in [11], [14]. They were used in these articles to prove uniqueness of viscosity solution for the Dirichlet problem

$$(1.7) \quad \begin{cases} F(D^2u) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

To prove Theorem 1.1 we also use a basic lemma of real analysis about Hölder continuity of functions, Lemma 3.1 below.

In the case of concave equations, estimate (1.5)—proved for classical solutions of certain functionals of the form $F(D^2u, x)$ —was used in [7] to apply the method of continuity and prove existence of classical solution for the Dirichlet problem (1.7). This existence result and the uniqueness of continuous viscosity solution for (1.7) in [11] imply Theorem 1.2.

In this paper we present a new and more direct proof of Theorem 1.2. We do not need to use the method of continuity. We first prove $C^{1,1}$ regularity of viscosity solutions of concave equations (1.1). We then adapt the proof of the Evans–Krylov $C^{2,\alpha}$ estimate for C^2 functions to $C^{1,1}$ functions.

REMARK 1.3. It is not known if Theorem 1.2 and estimate (1.5) are true when F is neither concave nor convex. In this case it is not even known if viscosity solutions of (1.1) are $C^{1,1}$. The only results in this direction need to assume that D^2F is small depending on the ellipticity Λ/λ (see [12]), or consider equations $\mu u - F(D^2u) = 0$ with μ large enough depending on the size of D^2F (see [9]).

We conclude this introductory section with some references to related work. Solutions of concave equations $F(D^2u) = 0$ also satisfy a $C^{2,\alpha}$ *a priori* estimate up to the boundary depending only on ellipticity and the regularity of the boundary value. This estimate was independently discovered by Krylov [15], [16] and Caffarelli, Nirenberg and Spruck [3] (see also [2] and [10]).

Viscosity solutions of fully nonlinear equations of the form

$$(1.8) \quad F(D^2u, x) = f(x)$$

(that is, equations with variable coefficients) are Hölder continuous for some universal $0 < \alpha < 1$. This is a consequence of the Krylov–Safonov Harnack inequality (see Section 2).

The perturbation theory introduced in [1] for equations of the form $F(D^2u) = 0$, together with Theorems 1.1 and 1.2 above, gives regularity for viscosity solutions of $F(D^2u, x) = f(x)$. [1] contains $C^{1,\alpha}$ (for all F) and $C^{2,\alpha}$ and $W^{2,p}$ (for concave F) interior *a priori* estimates for solutions of $F(D^2u, x) = f(x)$ under appropriate regularity assumptions on the dependence of F and f on x . Related results, also for equations of the form $F(D^2u, Du, u, x) = 0$, were obtained by different means by Safonov [22], [23] and Trudinger [24], [25].

The plan of the paper is as follows. In Section 2 we state the auxiliary results that we use. Theorem 1.1 is proved in Section 3. Theorem 1.2 is proved in Sections 4 and 5. In Section 4 we prove the Evans–Krylov $C^{2,\alpha}$ estimate for classical solutions. Section 5 contains a new proof of the $C^{1,1}$ regularity of viscosity solutions of concave equations.

2. Preliminaries

In this section we define the class \mathcal{S} of “all weak solutions to all elliptic equations” with given ellipticity constants. For this, we introduce Pucci’s extremal operators. The idea is to substitute any particular equation by certain inequalities given by the ellipticity constants. We follow the terminology of [2], which contains detailed proofs of all the results in this section.

Let $0 < \lambda \leq \Lambda$. For $M \in \mathcal{S}$, we define *Pucci’s extremal operators* by

$$\begin{aligned} \mathcal{M}^-(M, \lambda, \Lambda) &= \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i, \\ \mathcal{M}^+(M, \lambda, \Lambda) &= \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i, \end{aligned}$$

where $e_i = e_i(M)$ are the eigenvalues of M .

Let now A be a symmetric matrix with eigenvalues in $[\lambda, \Lambda]$, that is $\lambda|\xi|^2 \leq A_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$ for any $\xi \in \mathbb{R}^n$. In this case we will write that $A \in \mathcal{A}_{\lambda,\Lambda}$. Define a linear functional L_A on \mathcal{S} by

$$L_A M = A_{ij} M_{ij} = \text{Trace}(AM), \quad M \in \mathcal{S}.$$

It is easy to see that

$$\mathcal{M}^-(M, \lambda, \Lambda) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} L_A M, \quad \mathcal{M}^+(M, \lambda, \Lambda) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} L_A M.$$

It follows that \mathcal{M}^- and \mathcal{M}^+ are uniformly elliptic with ellipticity constants $\lambda, n\Lambda$ (as defined in Section 1 above), \mathcal{M}^- is concave and \mathcal{M}^+ is convex. In particular, Theorem 1.2 applies to \mathcal{M}^- .

Given a continuous function f in Ω , we consider the equation

$$(2.1) \quad F(D^2u) = f(x).$$

The definition of viscosity subsolution (respectively, supersolution) of (2.1) is the same as the one given in Section 1 for equation (1.1), with condition (1.3) replaced by $F(D^2\varphi(x_0)) \geq f(x_0)$ (respectively, $F(D^2\varphi(x_0)) \leq f(x_0)$). If u is a viscosity subsolution (respectively, supersolution, solution) of (2.1), we say that $F(D^2u) \geq f(x)$ (respectively, $\leq, =$) *in the viscosity sense*.

We can now define the class \mathcal{S} corresponding to solutions of elliptic equations with right hand side f . We first define $\underline{\mathcal{S}}(\lambda, \Lambda, f)$ to be the space of continuous functions u in Ω such that

$$\mathcal{M}^+(D^2u, \lambda, \Lambda) \geq f(x)$$

in the viscosity sense in Ω . Similarly, $\overline{\mathcal{S}}(\lambda, \Lambda, f)$ denotes the space of continuous functions u in Ω such that

$$\mathcal{M}^-(D^2u, \lambda, \Lambda) \leq f(x)$$

in the viscosity sense in Ω . We also define

$$\mathcal{S}(\lambda, \Lambda, f) = \underline{\mathcal{S}}(\lambda, \Lambda, f) \cap \overline{\mathcal{S}}(\lambda, \Lambda, f).$$

We will denote $\underline{\mathcal{S}}, \overline{\mathcal{S}}, \mathcal{S}(\lambda, \Lambda, 0)$ by $\underline{\mathcal{S}}, \overline{\mathcal{S}}, \mathcal{S}(\lambda, \Lambda)$, or simply $\underline{\mathcal{S}}, \overline{\mathcal{S}}, \mathcal{S}$. We call them the *spaces of subsolutions, supersolutions and solutions*, respectively.

Given F as in Section 1, it is easy to check that

$$(2.2) \quad F(M + N) \leq F(M) + \Lambda\|N^+\| - \lambda\|N^-\| \quad \forall M, N \in \mathcal{S},$$

where $\|N^+\|$ is the maximum of the positive parts of the eigenvalues of N , and $\|N^-\| = \|(-N)^+\|$. This easily yields

PROPOSITION 2.1. *Let u be a viscosity subsolution (respectively, supersolution, solution) in Ω of*

$$F(D^2u) = f(x).$$

Then $u \in \underline{\mathcal{S}}(\lambda/n, \Lambda, f - F(0))$ (respectively, $\overline{\mathcal{S}}(\lambda/n, \Lambda, f - F(0)), \mathcal{S}(\lambda/n, \Lambda, f - F(0))$).

It is easy to check from the definition of viscosity subsolution that functions in $\underline{\mathcal{S}} = \underline{\mathcal{S}}(\lambda, \Lambda, 0)$ satisfy the maximum principle. That is, if $u \in \underline{\mathcal{S}}$ in Ω and $u \leq 0$ on $\partial\Omega$ then $u \leq 0$ in Ω . A more general maximum principle, the Aleksandrov estimate, also holds for the class $\underline{\mathcal{S}}(\lambda, \Lambda, f)$. The Aleksandrov estimate and the Krylov–Safonov Harnack inequality were adapted to viscosity solutions in [1]

(see also Chapters 3 and 4 in [2]). The following is the Krylov–Safonov Harnack inequality and related estimates.

THEOREM 2.2. *Let f be a continuous and bounded function in B_1 .*

(1) *Let $u \in \overline{S}(\lambda, \Lambda, f)$ in B_1 satisfy $u \geq 0$ in B_1 . Then*

$$\|u\|_{L^{p_0}(B_{1/4})} \leq C(\inf_{B_{1/2}} u + \|f\|_{L^n(B_1)}),$$

where $p_0 > 0$ and C are universal constants.

(2) *Let $u \in \underline{S}(\lambda, \Lambda, f)$ in B_1 . Then, for any $p > 0$,*

$$\sup_{B_{1/2}} u \leq C(p)(\|u^+\|_{L^p(B_{3/4})} + \|f\|_{L^n(B_1)}),$$

where $C(p)$ depends only on n, λ, Λ and p .

(3) *Let $u \in S(\lambda, \Lambda, f)$ in B_1 satisfy $u \geq 0$ in B_1 . Then*

$$\sup_{B_{1/2}} u \leq C(\inf_{B_{1/2}} u + \|f\|_{L^n(B_1)}),$$

where C is a universal constant.

(4) *Let $u \in S(\lambda, \Lambda, f)$ in B_1 . Then $u \in C^\alpha(\overline{B}_{1/2})$ and*

$$\|u\|_{C^\alpha(\overline{B}_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^n(B_1)}),$$

where $0 < \alpha < 1$ and $C > 0$ are universal constants.

(1) is called the *weak Harnack inequality for nonnegative supersolutions*. (2) is called (following [10]) the *local maximum principle for subsolutions*. (3) is the *Harnack inequality*; it follows from (1) and (2). Finally, (4) is an easy consequence of (3).

The following two theorems are proved using Jensen's approximate solutions (see Chapter 5 of [2] for the proofs). Note that only the second one requires the concavity of F .

THEOREM 2.3. *Let u be a viscosity subsolution of $F(D^2w) = 0$ in Ω and v be a viscosity supersolution of $F(D^2w) = 0$ in Ω . Then $u - v \in \underline{S}(\lambda/n, \Lambda)$ in Ω .*

THEOREM 2.4. *Let F be concave and let u and v be viscosity subsolutions of $F(D^2w) = 0$ in Ω . Let $\mu \in [0, 1]$. Then $\mu u + (1 - \mu)v$ is a viscosity subsolution of $F(D^2w) = 0$ in Ω .*

COROLLARY 2.5. *Let F be concave and suppose that u is a viscosity solution of $F(D^2u) = 0$ in B_1 . Let $e \in \mathbb{R}^n$ with $|e| = 1$ and $0 < h < 1/2$. Then*

$$(2.3) \quad \Delta_{he}^2 u(x) := \frac{1}{h^2}[u(x + he) + u(x - he) - 2u(x)] \in \underline{S}(\lambda/n, \Lambda) \quad \text{in } B_{1/2}.$$

Moreover, if $u \in C^2(B_1)$ then

$$(2.4) \quad u_{ee} := \partial^2 u / \partial e^2 \in \underline{S}(\lambda/n, \Lambda) \quad \text{in } B_1.$$

To see (2.3), we write

$$\frac{1}{2}[u(x + he) + u(x - he) - 2u(x)] = \frac{1}{2}[u(x + he) + u(x - he)] - u(x),$$

which is, by Theorem 2.4, the difference of a viscosity subsolution and supersolution of $F(D^2u) = 0$. (2.3) then follows from Theorem 2.3.

(2.4) follows from (2.3) and the fact that viscosity subsolutions, and hence the classes $\underline{\mathcal{S}}$, are closed under uniform limits in compact sets.

REMARK 2.6. We point out some facts about the previous results when dealing with smooth solutions u and smooth functionals F . The uniform ellipticity, condition (1.2), implies

$$(2.5) \quad \lambda|\xi|^2 \leq F_{i_j}(M)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^n \quad \forall M \in \mathcal{S},$$

where F_{i_j} denotes the partial derivative of F with respect to the ij th entry of M .

Let u be smooth and satisfy $F(D^2u) \geq f$ in Ω . We then have

$$(2.6) \quad f(x) - F(0) \leq [F(tD^2u(x))]_{t=0}^1 = \left(\int_0^1 F_{i_j}(tD^2u(x)) dt \right) \partial_{i_j}u(x).$$

Defining $a_{ij}(x) = \int_0^1 F_{i_j}(tD^2u(x)) dt$ and using (2.5), we see that u is a subsolution of a linear equation of the type (1.6). Proposition 2.1 (in this special smooth case) now follows easily.

Using the same idea as in (2.6), it is easy to prove Theorem 2.3 for smooth (sub, super) solutions. Note that Theorem 2.4 is obvious for C^2 subsolutions. The reader can also check (2.4) for smooth solutions by differentiating $F(D^2u) = 0$ twice.

The following result is the Aleksandrov–Buselman–Feller theorem (see Theorem 1 in Section 6.4 of [8]). It is used in the proof of Theorems 2.3 and 2.4 above. We will also need it in Section 5.

THEOREM 2.7. *Let u be a convex function in a ball B_d . Then for almost every point $x_0 \in B_d$ there is a polynomial P of degree at most 2 such that*

$$(2.7) \quad u(x) = P(x) + o(|x - x_0|^2) \quad \text{as } x \rightarrow x_0;$$

i.e., $|u(x) - P(x)| |x - x_0|^{-2} \rightarrow 0$ as $x \rightarrow x_0$. In this case, we define $D^2u(x_0)$ to be D^2P .

Using the definition of viscosity solution it is easy to check the following.

REMARK 2.8. Let u be a viscosity solution of (1.1) in Ω . Assume that (2.7) holds at $x_0 \in \Omega$. Then, if we define $D^2u(x_0) = D^2P$, we have $F(D^2u(x_0)) = 0$.

The following remarks imply that we may assume $F(0) = 0$ and $\|u\|_{L^\infty(B_1)} = 1$ in order to prove Theorems 1.1 and 1.2.

REMARK 2.9. Using the ellipticity condition for F , it is easy to see that there exists one $t \in \mathbb{R}$ such that $F(tI) = 0$ and $|t| \leq |F(0)|/\lambda$; here I denotes the identity matrix. Define, for $x \in \mathbb{R}^n$, the polynomial $P(x) = (t/2)|x|^2$. Then

$$F(D^2u) = F(D^2(u - P) + tI) =: G(D^2(u - P)).$$

We now have $G(0) = F(tI) = 0$ and G is uniformly elliptic.

REMARK 2.10. For any $t > 0$, the functional $t^{-1}F(tD^2w)$ is uniformly elliptic with ellipticity constants λ, Λ . Considering $w = u/t$ with $t = \|u\|_{L^\infty(B_1)}$, we may assume that $\|u\|_{L^\infty(B_1)} = 1$ in order to prove Theorems 1.1 and 1.2.

3. $C^{1,\alpha}$ regularity

Let u be a viscosity solution of $F(D^2u) = 0$ in B_1 . Let $0 < h < 1/8$ and $e \in \mathbb{R}^n$ with $|e| = 1$. Then

$$(3.1) \quad u(x + he) - u(x) \in S(\lambda/n, \Lambda) \quad \text{in } B_{7/8}.$$

This is an immediate consequence of Theorem 2.3 and the fact that $v(x) = u(x + he)$ is viscosity solution of $F(D^2v) = 0$ in $B_{7/8}$.

The Hölder regularity of functions in the class S , (3.1) and the following lemma (applied repeatedly) will give Theorem 1.1. In [26] this lemma about Hölder continuity (called Lipschitz continuity there) is already pointed out.

LEMMA 3.1. *Let $0 < \alpha < 1$, $0 < \beta \leq 1$ and $K > 0$ be constants. Let $u \in L^\infty([-1, 1])$ satisfy $\|u\|_{L^\infty([-1, 1])} \leq K$. Define, for $h \in \mathbb{R}$ with $0 < |h| \leq 1$,*

$$v_{\beta,h}(x) = \frac{u(x+h) - u(x)}{|h|^\beta}, \quad x \in I_h,$$

where $I_h = [-1, 1-h]$ if $h > 0$ and $I_h = [-1-h, 1]$ if $h < 0$. Assume that $v_{\beta,h} \in C^\alpha(I_h)$ and $\|v_{\beta,h}\|_{C^\alpha(I_h)} \leq K$, for any $0 < |h| \leq 1$. We then have:

- (1) If $\alpha + \beta < 1$ then $u \in C^{\alpha+\beta}([-1, 1])$ and $\|u\|_{C^{\alpha+\beta}([-1, 1])} \leq CK$;
- (2) If $\alpha + \beta > 1$ then $u \in C^{0,1}([-1, 1])$ and $\|u\|_{C^{0,1}([-1, 1])} \leq CK$,

where the constants C in (1) and (2) depend only on $\alpha + \beta$.

PROOF. By symmetry of the problem with respect to the change $x \rightarrow -x$, it is enough to bound $|u(x + \varepsilon) - u(x)|$ for

$$-1 \leq x \leq 0, \quad \varepsilon > 0 \quad \text{and} \quad x + \varepsilon \leq 1.$$

Let $i \geq 0$ be the integer such that $x + 2^i\varepsilon \leq 1 < x + 2^{i+1}\varepsilon$, and define $\tau_0 = 2^i\varepsilon$. Then $-1 \leq x < x + \tau_0 \leq 1$ and

$$(3.2) \quad 1/2 \leq \tau_0 \leq 2.$$

Define $w(\tau) = u(x + \tau) - u(x)$, $0 < \tau \leq \tau_0$. We have

$$\begin{aligned} |w(\tau) - 2w(\tau/2)| &= |u(x + \tau) - 2u(x + \tau/2) + u(x)| \\ &= (\tau/2)^\beta |v_{\beta, \tau/2}(x + \tau/2) - v_{\beta, \tau/2}(x)| \leq K(\tau/2)^{\alpha+\beta}, \end{aligned}$$

since $\|v_{\beta, \tau/2}\|_{C^\alpha([-1, 1-\tau/2])} \leq K$ by hypothesis. Hence

$$\begin{aligned} |w(\tau_0) - 2w(\tau_0/2)| &\leq CK\tau_0^{\alpha+\beta}, \\ |2w(\tau_0/2) - 2^2w(\tau_0/2^2)| &\leq CK2^{1-(\alpha+\beta)}\tau_0^{\alpha+\beta}, \dots, \\ |2^{i-1}w(\tau_0/2^{i-1}) - 2^iw(\tau_0/2^i)| &\leq CK2^{(i-1)(1-(\alpha+\beta))}\tau_0^{\alpha+\beta}, \end{aligned}$$

for some constant C which depends, as all C 's in the rest of this proof, only on $\alpha + \beta$. Adding all the inequalities, we get

$$|w(\tau_0) - 2^iw(\varepsilon)| = |w(\tau_0) - 2^iw(\tau_0/2^i)| \leq CK\tau_0^{\alpha+\beta} \sum_{j=0}^{i-1} 2^{j(1-(\alpha+\beta))}.$$

Since $2^{-i} = \tau_0^{-1}\varepsilon \leq 2\varepsilon$, by (3.2), and $\|u\|_{L^\infty([-1, 1])} \leq K$, we have

$$\begin{aligned} |w(\varepsilon)| &\leq 2^{-i}|w(\tau_0)| + CK2^{-i}\tau_0^{\alpha+\beta} \sum_{j=0}^{i-1} 2^{j(1-(\alpha+\beta))} \\ &\leq 4K\varepsilon + CK\varepsilon\tau_0^{\alpha+\beta-1} \sum_{j=0}^{i-1} 2^{j(1-(\alpha+\beta))}. \end{aligned}$$

If $\alpha + \beta < 1$, we get $|w(\varepsilon)| \leq 4K\varepsilon + CK\varepsilon\tau_0^{\alpha+\beta-1}2^{i(1-(\alpha+\beta))} = 4K\varepsilon + CK\varepsilon^{\alpha+\beta} \leq CK\varepsilon^{\alpha+\beta}$. If $\alpha + \beta > 1$, we get $|w(\varepsilon)| \leq 4K\varepsilon + CK\varepsilon\tau_0^{\alpha+\beta-1} \leq CK\varepsilon$. \square

PROOF OF THEOREM 1.1. We assume that $F(0) = 0$ and $\|u\|_{L^\infty(B_1)} = 1$ (see Remarks 2.9 and 2.10). We fix $e \in \mathbb{R}^n$ with $|e| = 1$ and $0 < h < 1/8$. By (3.1), we know that for $0 < \beta \leq 1$,

$$v_\beta(x) = \frac{1}{h^\beta}(u(x + he) - u(x)) \in S(\lambda/n, \Lambda) \quad \text{in } B_{7/8}.$$

Hence, by C^α interior estimates, Theorem 2.2(4) properly scaled, we have (here $C^{0, \beta}$ denotes C^β if $\beta < 1$)

$$(3.3) \quad \|v_\beta\|_{C^\alpha(\overline{B}_r)} \leq C(r, s)\|v_\beta\|_{L^\infty(B_{(r+s)/2})} \leq C(r, s)\|u\|_{C^{0, \beta}(\overline{B}_s)},$$

where $0 < r < s \leq 7/8$, $0 < h < (s-r)/2$, α is universal and $C(r, s)$ depends on n, λ, Λ, r and s .

By making α slightly smaller, we can assume that there is a universal integer i such that $i\alpha < 1$ and $(i+1)\alpha > 1$. By Proposition 2.1, we have $u \in S(\lambda/n, \Lambda)$ in B_1 . Hence, by Theorem 2.2(4),

$$\|u\|_{C^\alpha(\overline{B}_{7/8})} \leq C,$$

for some universal constant C . We apply (3.3) with $\beta = \alpha$ and $r = r_1 < s = 7/8$ to get

$$\|v_\alpha\|_{C^\alpha(\overline{B}_{r_1})} \leq C(r_1)\|u\|_{C^\alpha(\overline{B}_{7/8})} \leq C(r_1),$$

where $0 < h < (7/8 - r_1)/2$, and $C(r_1)$ depends only on n, λ, Λ and r_1 .

We can now apply, for any e as above, Lemma 3.1 (rescaled and with $\beta = \alpha$) on segments parallel to e and get

$$\|u\|_{C^{2\alpha}(\overline{B}_{r_2})} \leq C(r_1, r_2) \quad \text{for } r_2 < r_1.$$

We now use (3.3) and Lemma 3.1 with $\beta = 2\alpha$. We get $u \in C^{3\alpha}(\overline{B}_{r_4})$. We can repeat this process until $i\alpha < 1$, $(i+1)\alpha > 1$ and finally get, by Lemma 3.1(2),

$$\|u\|_{C^{0,1}(\overline{B}_{3/4})} \leq C.$$

We finally apply (3.3) with $\beta = 1$ to get

$$\|v_1\|_{C^\alpha(\overline{B}_{1/2})} \leq C\|u\|_{C^{0,1}(\overline{B}_{3/4})} \leq C,$$

for any e with $|e| = 1$ and any $h > 0$ universally small enough. Since v_1 is the difference quotient of u for h and e , we conclude that $u \in C^{1,\alpha}(\overline{B}_{1/2})$ and $\|u\|_{C^{1,\alpha}(\overline{B}_{1/2})} \leq C$. \square

4. Evans–Krylov theorem

In this section we prove the $C^{2,\alpha}$ *a priori* estimate

$$(4.1) \quad \|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq C(\|u\|_{C^2(\overline{B}_1)} + |F(0)|)$$

for C^2 solutions of concave equations $F(D^2u) = 0$, where C and $0 < \alpha < 1$ are universal constants; see also Section 17.4 of [10].

We use Corollary 2.5 and the weak Harnack inequality, Theorem 2.2(1). We also need the following lemma, which is a consequence of the uniform ellipticity of F ; it does not require the concavity of F . We denote by $\|N\|$ the (L^2, L^2) norm of $N \in \mathcal{S}$; it is therefore equal to $\max |e_i|$, where e_1, \dots, e_n are the eigenvalues of N . Recall that after (2.2) we defined $\|N^+\| = \max e_i^+$ and $\|N^-\| = \max e_i^-$, where $e_i^+ = \max(e_i, 0)$ and $e_i^- = -\min(e_i, 0)$.

LEMMA 4.1. *If $F(M_1) = F(M_2) = 0$ then*

$$c_0\|M_2 - M_1\| \leq \|(M_2 - M_1)^+\| = \sup_{e \in \mathbb{R}^n, |e|=1} (e^t(M_2 - M_1)e)^+,$$

where $c_0 = \lambda/(\Lambda + \lambda)$. Here the concavity of F is not needed.

PROOF. Using $\|M_2 - M_1\| \leq \|(M_2 - M_1)^+\| + \|(M_2 - M_1)^-\|$ and (2.2) we have

$$\begin{aligned} 0 &= F(M_2) \leq F(M_1) + \Lambda\|(M_2 - M_1)^+\| - \lambda\|(M_2 - M_1)^-\| \\ &= \Lambda\|(M_2 - M_1)^+\| - \lambda\|(M_2 - M_1)^-\| \\ &\leq (\Lambda + \lambda)\|(M_2 - M_1)^+\| - \lambda\|M_2 - M_1\|, \end{aligned}$$

which proves the first inequality of the lemma. The second one follows from our definition of $\|N^+\|$. \square

The following proof is a variation of Evans' original proof. Estimate (4.1) for C^2 solutions is an easy consequence of the next lemma.

LEMMA 4.2. *Under the hypothesis of Theorem 1.2 assume that $u \in C^2(\overline{B_1})$. Then there exists a universal constant $0 < \delta_0 < 1$ such that $\text{diam } D^2u(B_1) = 2$ implies $\text{diam } D^2u(B_{\delta_0}) \leq 1$.*

Estimate (4.1) follows immediately from Lemma 4.2 and scaling, using Lemma 8.23 in [10]. Note that it is no restriction to assume $\text{diam } D^2u(B_1) = 2$, again by Remark 2.10 applied with $t = \text{diam } D^2u(B_1)/2$.

The following is the main step towards Lemma 4.2.

LEMMA 4.3. *Under the hypothesis of Theorem 1.2, assume that $u \in C^2(\overline{B_1})$, $1 < \text{diam } D^2u(B_1) \leq 2$*

and that $D^2u(B_1)$ is covered by m balls B^1, \dots, B^m of radius ε (in the space \mathcal{S} of symmetric matrices) with $m \geq 1$ and $\varepsilon \leq \varepsilon_0$, where $\varepsilon_0 > 0$ is a universal constant. Then $D^2u(B_{1/2})$ is covered by $m - 1$ balls among B^1, \dots, B^m .

PROOF. Take c_0 universal as in Lemma 4.1. For $i = 1, \dots, m$, we take $x_i \in B_1$ such that $B^i \subset B_{2\varepsilon}(M_i)$, where

$$M_i = D^2u(x_i).$$

Hence, taking ε_0 such that $2\varepsilon \leq 2\varepsilon_0 \leq c_0/16$, we find that

$$B_{c_0/16}(M_1), \dots, B_{c_0/16}(M_m)$$

cover $D^2u(B_1)$. Since $D^2u(B_1)$ has diameter at most 2, every M_i belongs to one closed ball \overline{B} of radius 2 in \mathcal{S} . Let m' be the maximum number of points in the ball \overline{B} such that the distance between any two of them is at least $c_0/16$. Then m' depends only on n and c_0 . Therefore, we can assume that $\{B_{c_0/8}(M_i)\}_{i=1}^{m'}$ cover $D^2u(B_1)$, where m' is universal and $m' \leq m$. It follows that $\{(D^2u)^{-1}B_{c_0/8}(M_i)\}_{i=1}^{m'}$ cover B_1 and, therefore, there exists an M_i , say M_1 , such that

$$(4.2) \quad |(D^2u)^{-1}(B_{c_0/8}(M_1)) \cap B_{1/4}| \geq \eta > 0,$$

where η is universal.

Recall that $\text{diam } D^2u(B_1) > 1$ and take $2\varepsilon \leq 2\varepsilon_0 \leq 1/4$. Since $\{B_{2\varepsilon}(M_i)\}_{i=1}^m$ cover $D^2u(B_1)$, it follows that there is an M_i , say M_2 , such that $\|M_2 - M_1\| \geq 1/4$. Lemma 4.1 gives the existence of $e \in \mathbb{R}^n$ with $|e| = 1$ such that

$$(4.3) \quad u_{ee}(x_2) \geq u_{ee}(x_1) + c_0/4.$$

Define

$$K = \sup_{B_1} u_{ee} \quad \text{and} \quad v = K - u_{ee}.$$

We have $0 \leq v \in \overline{S}(\lambda/n, \Lambda)$ in B_1 , by Corollary 2.5. (4.2) and (4.3) imply that $|\{v \geq c_0/8\} \cap B_{1/4}| \geq \eta$. We can apply Theorem 2.2(1) to v and get, for a universal c_1 ,

$$(4.4) \quad \inf_{B_{1/2}} (K - u_{ee}) \geq c_1 > 0.$$

By the definition of K and again since $\{B_{2\varepsilon}(M_i)\}_{i=1}^m$ cover $D^2u(B_1)$, there is $j, 1 \leq j \leq m$, such that

$$(4.5) \quad K - u_{ee}(x_j) < 3\varepsilon.$$

If we finally take $5\varepsilon \leq 5\varepsilon_0 \leq c_1$, then (4.4) and (4.5) imply that $D^2u(B_{1/2}) \cap B_{2\varepsilon}(M_j) = \emptyset$. Hence $D^2u(B_{1/2}) \cap B^j = \emptyset$ and the lemma follows. \square

We can now give the

PROOF OF LEMMA 4.2. Since $\text{diam } D^2u(B_1) = 2$, we can cover $D^2u(B_1)$ by m balls of radius ε_0 , with ε_0 as in Lemma 4.3 and m universal. Lemma 4.3 shows that $D^2u(B_{1/2})$ is covered by $m - 1$ balls of radius ε_0 . Suppose that we have $\text{diam } D^2u(B_{1/2}) > 1$, and define

$$w(x) = 4u(x/2) \quad \text{for } x \in B_1.$$

Hence $D^2w(x) = D^2u(x/2)$, $F(D^2w) = 0$ in B_1 and $1 < \text{diam } D^2w(B_1) \leq 2$. Applying Lemma 4.3 to w , we deduce that $D^2u(B_{1/4}) = D^2w(B_{1/2})$ is covered by $m - 2$ balls.

Since we cannot run out of balls, it follows that there exists $k \leq m$ such that $\text{diam } D^2u(B_{1/2^k}) \leq 1$. Hence

$$\text{diam } D^2u(B_{1/2^m}) \leq 1;$$

that proves Lemma 4.2 with $\delta_0 = 1/2^m$, which is universal. \square

5. $C^{1,1}$ regularity

In this section we finish the proof of Theorem 1.2. We first show that viscosity solutions of concave equations (1.1) are $C^{1,1}$. The following is a sketch of the proof.

By the concavity of F , we will see that for a linear functional (that we may assume to be the Laplacian) we have $\Delta u \geq 0$ in the viscosity sense. It will follow that the function

$$u_h^*(x) = \frac{1}{h^2} \left(\int_{S_h(x)} u - u(x) \right)$$

is nonnegative in $B_{1/2}$, for any $0 < h < 1/2$; here $S_h(x) = \partial B_h(x)$ and \int denotes average. This will let us bound the L^1 norm of u_h^* uniformly in h . We will see that the u_h^* belong to the class \underline{S} of subsolutions. Theorem 2.2(2) then leads to an L^∞ bound for u_h^* uniform in h . Hence, making $h \rightarrow 0$, we will get $\Delta u \in L^\infty \subset L^2$. L^2 estimates for the Laplacian will give $u \in W^{2,2}$. This L^2 bound for the pure second incremental quotients $\Delta_{he}^2 u$ will lead to an L^∞ bound for $\Delta_{he}^2 u$, again by Theorem 2.2(2) applied to the subsolutions $\Delta_{he}^2 u$. We will therefore get $u \in C^{1,1}$. An adaptation of the proof in Section 4 to $C^{1,1}$ functions will finally give $C^{2,\alpha}$ regularity. The details of the proof go as follows.

PROOF OF THEOREM 1.2. We assume that $F(0) = 0$ and $\|u\|_{L^\infty(B_1)} = 1$ (see Remarks 2.9 and 2.10). We need to prove that $\|u\|_{C^{2,\alpha}(\bar{B}_{1/2})} \leq C$; here and in the rest of this proof, C denotes a universal constant.

Recall that F is a concave function on the space \mathcal{S} of symmetric matrices; it follows from the Hahn–Banach theorem that there is a supporting hyperplane (by above) to the graph of F at $0 \in \mathcal{S}$. That is, there is a linear functional L on the space of symmetric matrices such that

$$L(0) = 0 \quad \text{and} \quad L(M) \geq F(M)$$

for any $M \in \mathcal{S}$. Therefore, L is of the form

$$L(M) = a_{ij} m_{ij} = \text{Trace}(AM),$$

for some $A = [a_{ij}] \in \mathcal{S}$. We claim that all the eigenvalues of A belong to $[\lambda, \Lambda]$. To see this, let $\xi \in \mathbb{R}^n$ and let $\xi\xi^t$ denote the matrix with entries $\xi_i\xi_j$. We have

$$\begin{aligned} a_{ij}\xi_i\xi_j &= L(\xi\xi^t) \geq F(\xi\xi^t) \geq F(0) + \lambda\|\xi\xi^t\| = \lambda|\xi|^2 \quad \text{and} \\ a_{ij}\xi_i\xi_j &= -L(-\xi\xi^t) \leq -F(-\xi\xi^t) \leq -F(0) + \Lambda\|\xi\xi^t\| = \Lambda|\xi|^2. \end{aligned}$$

Hence, we can make a linear change of space variables such that in the new variables we have $L(M) = \text{Trace}(M)$, i.e. $L(D^2\varphi) = \Delta\varphi$ (see Lemma 6.1 of [10]). Using the fact that the eigenvalues of A are in $[\lambda, \Lambda]$, we see that $\|u\|_{C^{2,\alpha}}$ (computed in the old variables) is smaller than a universal constant times the

same norm computed in the new variables. Therefore we may (and do) assume that $L(D^2\varphi) = \Delta\varphi$ for any $\varphi \in C^2$.

Since $F(D^2u) = 0$ in the viscosity sense (we can still assume in B_1 , after rescaling) and $\Delta\varphi = L(D^2\varphi) \geq F(D^2\varphi)$ for any $\varphi \in C^2$, we immediately see that u also satisfies $\Delta u \geq 0$ in the viscosity sense in B_1 . From this, we get

$$(5.1) \quad u(x_0) \leq \int_{S_h(x_0)} u$$

for any $x_0 \in B_{1/2}$ and $0 < h < 1/2$. To see (5.1), one considers the harmonic function w in $B_h(x_0)$ equal to u on $S_h(x_0)$; the definition of viscosity subsolution implies that

$$u(x_0) \leq w(x_0) = \int_{S_h(x_0)} u.$$

We therefore conclude that the function

$$u_h^*(x) = \frac{1}{h^2} \left(\int_{S_h(x)} u - u(x) \right)$$

is continuous and nonnegative in $B_{1/2}$, for any $0 < h < 1/2$. Note that u_h^* is an approximation of $\Delta u/(2n)$, in the following sense. Using Taylor's (or Green's) formula, we see that if $\varphi \in C^\infty$ then φ_h^* converges as $h \rightarrow 0$ uniformly in compact sets to $\Delta\varphi/(2n)$; moreover, for a constant $C(n)$ depending only on n , $\|\varphi_h^*\|_{L^\infty(B_1)} \leq C(n)\|D^2\varphi\|_{L^\infty(B_2)}$.

Using the fact that $u_h^* \geq 0$, we now bound the L^1 norm of u_h^* as follows. Let $\varphi \geq 0$ be a C^∞ function with compact support in $B_{1/2}$ and $\varphi \equiv 1$ on $B_{1/3}$. Then, since $\|u\|_{L^\infty(B_1)} = 1$,

$$\int_{B_{1/3}} |u_h^*| = \int_{B_{1/3}} u_h^* \leq \int_{B_{1/2}} u_h^* \varphi = \int_{B_1} u \varphi_h^* \leq C(n)\|D^2\varphi\|_{L^\infty} \leq C(n).$$

The following step is to prove that

$$u_h^* \in \underline{S}(\lambda/n, \Lambda) \quad \text{in } B_{1/2}.$$

By Theorem 2.4 any convex combination $\lambda_1 u_1 + \dots + \lambda_k u_k$ (i.e. $\lambda_i \geq 0$, $\sum \lambda_i = 1$) of viscosity subsolutions of $F(D^2u) = 0$ is also a viscosity subsolution. It follows that

$$\int_{S_h(x)} u = \int_{S_h(0)} u(x+y) dy$$

is a viscosity subsolution of $F(D^2u) = 0$ in $B_{1/2}$, since $u(\cdot + y)$ is subsolution for any $y \in S_h(0)$ and the class of viscosity subsolutions of $F(D^2u) = 0$ is closed under uniform limits.

Therefore,

$$h^2 u_h^*(x) = \int_{S_h(x)} u - u(x)$$

is the difference of a viscosity subsolution and a viscosity supersolution of $F(D^2u) = 0$. Theorem 2.3 shows that $h^2u_h^*$, and hence u_h^* , belong to $\underline{S}(\lambda/n, \Lambda)$ in $B_{1/2}$.

We now apply Theorem 2.2(2) (rescaled) with $p = 1$ to u_h^* . We have $0 \leq u_h^* \in \underline{S}(\lambda/n, \Lambda)$ and $\|u_h^*\|_{L^1(B_{1/3})} \leq C(n)$. It follows that

$$(5.2) \quad \|u_h^*\|_{L^\infty(B_{1/4})} \leq C,$$

for a universal constant C (independent of h).

Let ψ be any C^∞ function with compact support in $B_{1/4}$. Since $2n\psi_h^* \rightarrow \Delta\psi$ as $h \rightarrow 0$ uniformly in $B_{1/4}$, we have

$$\int_{B_{1/4}} u \Delta\psi = 2n \lim_{h \rightarrow 0} \int_{B_{1/4}} u \psi_h^* = 2n \lim_{h \rightarrow 0} \int_{B_{1/4}} u_h^* \psi,$$

and hence, using (5.2),

$$\left| \int_{B_{1/4}} u \Delta\psi \right| \leq C \|\psi\|_{L^1(B_{1/4})}.$$

Therefore Δu (in the distributional sense) belongs to $L^\infty(B_{1/4})$ and $\|\Delta u\|_{L^\infty(B_{1/4})} \leq C$. In particular, $\|\Delta u\|_{L^2(B_{1/4})} \leq C$ and hence, by L^2 regularity theory, $u \in W^{2,2}(B_{1/5})$ and

$$(5.3) \quad \|D^2u\|_{L^2(B_{1/5})} \leq C.$$

By Corollary 2.5 we know that the pure second order incremental quotients of u , $\Delta_{he}^2 u(x)$, belong to $\underline{S}(\lambda/n, \Lambda)$ in $B_{1/2}$; here e is any vector with $|e| = 1$. (5.3) implies that $\|\Delta_{he}^2 u\|_{L^2(B_{1/10})} \leq C$, for a universal constant C independent of $h \in (0, 1/10)$. Theorem 2.2(2) used with $p = 2$ gives

$$(5.4) \quad \sup_{B_{1/11}} \Delta_{he}^2 u \leq C, \quad \forall 0 < h < 1/10, \quad \forall |e| = 1.$$

It follows that $v(x) = u(x) - (C/2)|x|^2$ is a concave function in $B_{1/11}$. The Aleksandrov–Buselman–Feller theorem, Theorem 2.7, implies that v , and therefore u , have second order tangent polynomials of degree 2 at every point in a set A with $|B_{1/11} \setminus A| = 0$. (5.4) implies that

$$(5.5) \quad u_{ee}(x) \leq C \quad \forall x \in A, \quad \forall |e| = 1.$$

Recall that we already know that u_{ee} is an L^2 function in $B_{1/11}$. Remark 2.8 gives that $F(D^2u(x)) = 0$ for any $x \in A$; we can therefore use Lemma 4.1 with $M_1 = 0$ to get

$$\|D^2u(x)\| \leq C \sup_{|e|=1} u_{ee}(x)^+ \quad \forall x \in A.$$

This, combined with (5.5) and $|B_{1/11} \setminus A| = 0$, implies that $D^2u \in L^\infty(B_{1/11})$ and $\|D^2u\|_{L^\infty(B_{1/11})} \leq C$.

We have thus proved that $u \in W^{2,\infty}(B_{1/11}) = C^{1,1}(\overline{B}_{1/11})$ and

$$(5.6) \quad \|u\|_{C^{1,1}(\overline{B}_{1/11})} \leq C;$$

moreover,

$$F(D^2u(x)) = 0 \quad \text{and} \quad u_{ee}(x) = \lim_{h \rightarrow 0} \Delta_{he}^2 u(x), \quad \forall x \in A, \quad \forall |e| = 1,$$

for a set A with $|B_{1/11} \setminus A| = 0$.

We finally show that the proof of Lemma 4.3 may be adapted to this situation (that is, when $u \in C^{1,1}$ instead of $u \in C^2$) and thus obtain $u \in C^{2,\alpha}(B_{1/12})$ and $\|u\|_{C^{2,\alpha}(\overline{B}_{1/12})} \leq C \|D^2u\|_{L^\infty(B_{1/11})} \leq C$, for a universal $\alpha \in (0, 1)$. A covering argument gives $\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq C$.

The proof of Lemma 4.3 proceeds in the same way; now B_1 is replaced by $B_{1/11}$ and sup, inf and diam (of second derivatives of u) are understood as essential supremum, essential infimum and essential diameter, respectively. $D^2u(B_{1/11})$ being covered by B^1, \dots, B^m is now understood as $D^2u(x) \in B^1 \cup \dots \cup B^m$ for a.e. $x \in B_{1/11}$.

We can take the points x_i in A . The only delicate point is to obtain (4.4) (recall that we still do not know that u_{ee} is continuous). We proceed as follows. We define $K = \sup_{B_{1/11}} u_{ee}$. Thus $\sup_{B_{1/12}} \Delta_{he}^2 u \leq K$ for h small enough, since we can write

$$\Delta_{he}^2 u(x_0) = \int_{-1}^1 u_{ee}(x_0 + the)(1 - |t|) dt.$$

Therefore $0 \leq v_h := K - \Delta_{he}^2 u \in C(B_{1/12})$ and $v_h \in \overline{S}(\lambda/n, \Lambda)$. Theorem 2.2(1) rescaled gives

$$\|K - \Delta_{he}^2 u\|_{L^{p_0}(B_{1/13})} \leq C(K - \Delta_{he}^2 u(x)) \quad \forall x \in B_{1/13}$$

for a universal $p_0 > 0$. Taking $\liminf_{h \rightarrow 0}$ and using Fatou's lemma, we get

$$(5.7) \quad \|K - u_{ee}\|_{L^{p_0}(B_{1/13})} \leq C(K - u_{ee}(x)) \quad \forall x \in B_{1/13} \cap A.$$

We now use the fact that $|\{v := K - u_{ee} \geq c_0/8\} \cap B_{1/13}| \geq \eta$ (in (4.2), $B_{1/4}$ is now replaced by $B_{1/13}$) to obtain

$$\|K - u_{ee}\|_{L^{p_0}(B_{1/13})} \geq \frac{c_0}{8} \eta^{1/p_0}.$$

This and (5.7) give $\inf_{B_{1/13}} (K - u_{ee}) \geq c_1 > 0$ for a universal c_1 .

We finally proceed as in the proof of Lemma 4.3 to conclude that $D^2u(x) \notin B_{2\varepsilon}(M_j)$ for almost every $x \in B_{1/13}$, for some index j . \square

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