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NONSELFADJOINT RESONANCE PROBLEMS WITH NONLINEARITIES OF SUPERLINEAR GROWTH

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Dedicated to Professor Ky Fan on his 80th birthday

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain with a C^2 boundary $\partial\Omega$, and

$$Au = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{j=1}^{N} a_j \frac{\partial u}{\partial x_j} + a_0 u, \tag{1}$$

where $\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j > 0$ for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N \setminus \{0\}$. We assume that $a_{ij}, a_j \in C^1(\overline{\Omega})$, $a_{ij} = a_{ji}$ in Ω for $1 \leq i,j \leq N$ and $a_0 \in L^\infty(\Omega)$. It follows from the Krein–Rutman theorem (see, for example, [9], p. 265) and the Bony maximal principle (see [2], Lemma 1) that the eigenvalue problem

$$Au + \lambda u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \tag{2}$$

has a real eigenvalue λ_1 of minimal modulus which is simple, and a corresponding eigenfunction $\phi \in C^1(\overline{\Omega})$ such that $\phi > 0$ in Ω and the outward normal derivative $\partial\phi/\partial\nu$ is negative on $\partial\Omega$. Moreover, λ_1 is also a simple eigenvalue of the adjoint eigenvalue problem

$$A^*u + \lambda u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \tag{3}$$

with a corresponding eigenfunction $\phi^* \in C^1(\overline{\Omega})$ such that $\phi^* > 0$ in Ω and $\partial\phi^*/\partial\nu < 0$ on $\partial\Omega$, where A^* denotes the adjoint operator of A. It follows from

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the L^p regularity of the linear Dirichlet problem that $\phi, \phi^* \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ for all $p \geq 2$. If the operator A is selfadjoint so that the coefficients a_j $(1 \leq j \leq N)$ in (1) vanish in Ω, then λ_1 is known as the first eigenvalue of (2) and in this case $\phi = \phi^*$.

In this paper we consider the Dirichlet problem

$$Au + \lambda_1 u + g(x,u) = h \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \tag{4}$$

where $h \in L^p(\Omega)$ $(p > N)$ is given and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, $g(x,u)$ is continuous in $u \in \mathbb{R}$ for almost all $x \in \Omega$ and is measurable in $x \in \Omega$ for all $u \in \mathbb{R}$. The solvability of the problem (4) has been extensively studied if the nonlinear term g is assumed to have linear growth in u. Let

$$\kappa_+(x) = \limsup_{u \to \infty} g(x,u)/u, \qquad \kappa_-(x) = \limsup_{u \to -\infty} g(x,u)/u$$

be nonnegative functions in $L^\infty(\Omega)$, $\kappa(x) = \max\{\kappa_+(x), \kappa_-(x)\}$. Existence theorems for a solution to (4) are proved in [1, 3, 6] if A is selfadjoint and $\kappa(x) \leq \lambda_2 - \lambda_1$ for almost all $x \in \Omega$, with the strict inequality holding on a subset of Ω of positive measure, where λ_2 denotes the second eigenvalue of (2). When A is nonselfadjoint, further results along these lines are obtained in which the norm in $L^\infty(\Omega)$ of one of κ_+ and κ_- can be arbitrary, provided that of the other is sufficiently small (see [6], Theorem 5, and also [3], Theorem 4). The purpose of this paper is to give solvability conditions for (4) when g is allowed to grow superlinear in u in one of the directions ∞ and $-\infty$, and is bounded in $L^p(\Omega)$ $(p > N)$ in the other. More precisely, we assume that $p > N$ and

(H) There exist $a > 0$ and $b \in L^p(\Omega)$ such that $b \geq 0$ in Ω and for almost all $x \in \Omega$ and $u \in \mathbb{R}$,

$$-b(x) \leq g(x,u) \leq a|u|^{p/(p-1)} + b(x). \tag{5}$$

Based on degree-theoretic arguments (see, for example, [4]) and the Borsuk odd mapping theorem, and similarly to an idea of [7], we obtain solvability results for (4) under assumptions either with or without a Landesman–Lazer condition (see (7) below) originated in [8]. The results, which remain valid when g is replaced by $-g$ in (5), complement those cited above. For the case $N = 1$, we refer to [5] for more general results.

In what follows we shall make use of the real Banach spaces $L^p(\Omega), C^1(\overline{\Omega})$ with the norms denoted by $\|u\|_{L^p}, \|u\|_{C^1}$, respectively, and of the Sobolev spaces $W^{2,p}(\Omega), H_0^1(\Omega)$. We note the continuous imbedding $C^1(\overline{\Omega}) \to L^p(\Omega)$ for $p \geq 1$ and the compact imbedding $W^{2,p}(\Omega) \to C^1(\overline{\Omega})$ for $p > N$. For a linear operator L, we denote by $D(L), N(L)$ and $R(L)$ the domain, the null space and the range of L, respectively.

2. Existence theorems

The main result is the following Theorem 1 which is an existence theorem for a solution to (4) under the Landesman–Lazer condition. By modifying the proof of Theorem 1, in Theorems 2 and 3 we obtain solvability conditions for (4) when the Landesman–Lazer condition is not satisfied.

THEOREM 1. *Let $p > N$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the condition* (H). *If there exists $c \in L^1(\Omega)$ such that for almost all $x \in \Omega$ and $u \le 0$,*

$$g(x,u) \le c(x), \tag{6}$$

then the problem (4) *has a solution $u \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ for any $h \in L^p(\Omega)$ such that*

$$\int g_-\phi^* < \int h\phi^* < \int g_+\phi^*, \tag{7}$$

where $g_+(x) = \liminf_{u\to\infty} g(x,u)$ and $g_-(x) = \limsup_{u\to-\infty} g(x,u)$.

PROOF. We divide the proof into four steps.

Step 1. We set $L = A + \lambda_1$ and consider L as a closed linear operator with the domain $D(L) = W^{2,p}(\Omega) \cap H_0^1(\Omega)$ which is dense in $L^p(\Omega)$ and range $R(L)$ closed in $L^p(\Omega)$. Let ϕ, $\phi^* \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ be the eigenfuctions corresponding to the eigenvalue λ_1 of the eigenvalue problems (2), (3), respectively, as described in §1. We assume $\int \phi^2 = \int (\phi^*)^2 = 1$, define P, $Q : L^p(\Omega) \to L^p(\Omega)$ by

$$Pu = \left(\int u\phi\right)\phi, \qquad Qu = \left(\int u\phi^*\right)\phi^*,$$

and define

$$jQu = \left(\int u\phi^*\right)\phi. \tag{8}$$

Obviously $N(L) = R(P)$. It follows from the L^p theory of the linear Dirichlet problem that $R(L) = N(Q)$. Moreover, the restriction of L to $D(L) \cap N(P)$ is one-one onto $R(L)$ and so has an inverse $L^{-1} : R(L) \to W^{2,p}(\Omega)$ which is a bounded linear operator. Hence by the compact imbedding of $W^{2,p}(\Omega)$ into $C^1(\overline{\Omega})$, $L^{-1}(I - Q) : L^p(\Omega) \to C^1(\overline{\Omega})$ is a compact linear operator, where I denotes the identity operator.

Let $f : \mathbb{R} \to \mathbb{R}$ be the continuous function defined by

$$f(u) = \begin{cases} u & \text{if } |u| \le 1, \\ 1 & \text{if } u > 1, \\ -1 & \text{if } u < -1. \end{cases}$$

For $0 \le t \le 1$ we define for $u \in C^1(\overline{\Omega})$,

$$(F_t u)(x) = th(x) - tg(x, u(x)) - (1-t)f(u(x))$$

and

$$T_t u = Pu + (jQ + L^{-1}(I - Q))F_t u,$$

where jQ is defined in (8). Since the map $[0,1] \times C^1(\overline{\Omega}) \to L^p(\Omega)$, $(t,u) \to F_t u$ is continuous and maps bounded sets to bounded sets, the map $[0,1] \times C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$, $(t,u) \to T_t u$ is compact. We consider the operator equations

$$u = T_t u \tag{9}$$

for $0 \le t \le 1$. By identifying $L^p(\Omega)$ with the direct sum $R(Q) \oplus N(Q)$ it follows that (9) are equivalent to the Dirichlet problems

$$\begin{aligned} Au + \lambda_1 u + (1-t)f(u) + tg(x,u) &= th \qquad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{10}$$

for $0 \le t \le 1$, which becomes the original problem (4) when $t = 1$.

We suppose for the moment that solutions to (9) for some $0 \le t \le 1$ are bounded in $C^1(\overline{\Omega})$, and use this to complete the proof of the theorem. Now for some $R > 0$ large enough the Leray–Schauder degree $\deg(I - T_t, B_R(0), 0)$ is defined for $0 \le t \le 1$ and does not depend on t, where $B_R(0) = \{u \in C^1(\overline{\Omega}) : \|u\|_{C^1} < R\}$. Obviously the map $T_0 : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ is odd and so by the Borsuk odd mapping theorem (see, for example, [4], Chap. 2), $\deg(I - T_0, B_R(0), 0)$ is odd. Hence $\deg(I - T_1, B_R(0), 0)$ is odd, which implies that the problem (4) has a solution in $B_R(0)$.

Step 2. We show that if $\{v_n\}$ is a sequence in $W^{2,p}(\Omega) \cap H_0^1(\Omega)$ such that $\{v_n\}$ is bounded in $C^1(\overline{\Omega})$ and $\{Lv_n\}$ is bounded in $L^p(\Omega)$, then $\{v_n\}$ has a subsequence convergent in $C^1(\overline{\Omega})$. Indeed, by the compactness of the map $L^{-1} : R(L) \to C^1(\overline{\Omega})$, $\{(I-P)v_n\}$ has a subsequence convergent in $C^1(\overline{\Omega})$. Since $R(P)$ is one-dimensional, $\{Pv_n\}$ has a subsequence convergent in $C^1(\overline{\Omega})$. Putting these together we obtain a subsequence of $\{v_n\}$ convergent in $C^1(\overline{\Omega})$.

Step 3. We shall prove some more preliminary results needed in the next step. We note first that there exist Carathéodory functions g_1, $g_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $g = g_1 + g_2$ and $0 \le g_1(x,u) \le a|u|^{p/(p-1)}$, $|g_2(x,u)| \le b(x)$ for almost all $x \in \Omega$ and $u \in \mathbb{R}$. This may be done by defining

$$g_1(x,u) = \min\{g(x,u) + b(x), a|u|^{p/(p-1)}\}$$

and $g_2 = g - g_1$.

Next we note by the properties of ϕ^* stated in §1 that there exists a constant $c_1 \ge 0$ such that

$$|u(x)| \le c_1 \|u\|_{C^1} \phi^*(x) \tag{11}$$

for $x \in \Omega$ is valid for all $u \in C^1(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$.

Now let u be a possible solution to (10) for some $0 \le t \le 1$. Then u satisfies (11) for $x \in \Omega$. By taking the inner product in $L^2(\Omega)$ of (10) with ϕ^*, we have

$$(1-t)\int f(u)\phi^* + t\int g(x,u)\phi^* = t\int h\phi^* \tag{12}$$

and so

$$\begin{aligned} t\int |g_1(x,u)|\phi^* &\le \int |g_2(x,u)|\phi^* + \int \phi^* + \int |h|\phi^* \\ &\le \int [b(x) + 1 + |h|]\phi^*. \end{aligned}$$

Thus by (11),

$$t^p \int |g_1(x,u)|^p \le a^{p-1}\, t\int |u|^p |g_1(x,u)| \le c_2 \|u\|_{C^1}^p\, t\int |g_1(x,u)|\phi^*$$

and hence

$$\|tg_1(x,u)\|_{L^p} \le c_3\|u\|_{C^1}$$

for some constants c_2, $c_3 \ge 0$ independent of u. Consequently, there exists a constant $c_0 \ge 0$ such that

$$\|F_t u\|_{L^p} \le c_0(1 + \|u\|_{C^1}) \tag{13}$$

for all possible solutions u to (10) for some $0 \le t \le 1$.

Step 4. It remains to show that solutions to (10) for some $0 \le t < 1$ have an a priori bound in $C^1(\overline{\Omega})$. If this were not the case, then there would exist sequences $\{u_n\}$ in $W^{2,p}(\Omega) \cap H_0^1(\Omega)$ and $\{t_n\}$ in $[0,1)$ such that u_n satisfies (10) when $t = t_n$ and $\|u_n\|_{C^1} \ge n$ for all $n \ge 1$. Let $v_n = u_n/\|u_n\|_{C^1}$. Then $\|v_n\|_{C^1} = 1$ and

$$Lv_n = (F_{t_n}u_n)/\|u_n\|_{C^1}. \tag{14}$$

By (13) the right hand side of (14) is bounded in $L^p(\Omega)$. It follows from Step 2 and the reflexivity of $L^p(\Omega)$ that $\{v_n\}$ has a subsequence convergent in $C^1(\overline{\Omega})$ and the sequence $\{g(x,u_n)/\|u_n\|_{C^1}\}$ has a subsequence weakly convergent in $L^p(\Omega)$. Without loss of generality we may assume that there exist $v \in C^1(\overline{\Omega})$, $m \in L^p(\Omega)$ and $t_0 \in [0,1]$ such that $v_n \to v$ in $C^1(\overline{\Omega})$, $\{g(x,u_n)/\|u_n\|_{C^1}\} \to m(x)$ weakly in $L^p(\Omega)$ and $t_n \to t_0$. Since L is also weakly closed, it follows that $v \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ and

$$Lv + t_0\, m = 0 \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0. \tag{15}$$

If $t_0 \ne 0$, we deduce using (5) that $m(x) \ge 0$ for almost all $x \in \Omega$ (see [3], Lemma 4). Then by (15), $\int m(x)\phi^* = 0$ and so $m(x) = 0$ for almost all $x \in \Omega$. Hence $v \in N(L) \setminus \{0\}$ which is obvious when $t_0 = 0$. Consequently, there is an alternative: either $v > 0$ in Ω and $\partial v/\partial\nu < 0$ on $\partial\Omega$, or $v < 0$ in Ω and $\partial v/\partial\nu > 0$ on $\partial\Omega$. If the first alternative holds, then $u_n > 0$ in Ω for n large enough and $\lim_{n\to\infty} u_n(x) = \infty$ for $x \in \Omega$. Writing (12) with $u = u_n$ and $t = t_n$ as

$$(1 - t_n) \int f(u_n)\phi^* + t_n \int g(x, u_n)\phi^* = t_n \int h\phi^*, \tag{16}$$

we have the following two cases to consider. In the case $t_0 = 0$, by (5)

$$(1 - t_n) \int f(u_n)\phi^* - t_n \int b\phi^* \leq t_n \int h\phi^*$$

and so by the Lebesgue theorem we would have $\int \phi^* \leq 0$, which is absurd. In the case $t_0 > 0$, applying the Fatou lemma to the left hand side of (16) we would have

$$(1 - t_0) \int \phi^* + t_0 \int g_+\phi^* \leq t_0 \int h\phi^*,$$

which contradicts the second inequality in (7). If the other alternative holds, then $u_n < 0$ in Ω for n large enough and $\lim_{n\to\infty} u_n(x) = -\infty$ for $x \in \Omega$. Again we have the following two cases to consider. In the case $t_0 = 0$, by (6) for n large enough

$$(1 - t_n) \int f(u_n)\phi^* + t_n \int c\phi^* \geq t_n \int h\phi^*$$

and so by the Lebesgue theorem we would have $-\int \phi^* \geq 0$, which is absurd. In the case $t_0 > 0$, applying the Fatou lemma to the left hand side of (16) we would have

$$-(1 - t_0) \int \phi^* + t_0 \int g_-\phi^* \geq t_0 \int h\phi^*,$$

which contradicts the first inequality in (7).

This completes the proof of Theorem 1.

An interesting case in which (7) is not satisfied is when $g_+(x) = g_-(x) = 0$ for almost all $x \in \Omega$ and $\int h\phi^* = 0$. By modifying the proof of Theorem 1, we obtain the following Theorem 2 which gives existence for a solution to (4) without assuming a Landesman–Lazer condition.

Theorem 2. *Let $p > N$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the condition* (H). *If for almost all $x \in \Omega$ and $u \in \mathbb{R}$,*

$$ug(x, u) \geq 0, \tag{17}$$

then the problem (4) *has a solution $u \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ for any $h \in L^p(\Omega)$ such that $\int h\phi^* = 0$.*

Proof. In proving Theorem 1 the condition (7) is used only in the final part of Step 4 to produce contradictions. Thus it suffices to substitute this part of the proof using (17) instead of (7). We suppose that the alternative for the sequence $\{u_n\}$ is established as in Step 4 of the proof of Theorem 1. Now (16) becomes

$$(1 - t_n) \int f(u_n)\phi^* + t_n \int g(x, u_n)\phi^* = 0. \tag{18}$$

If the first alternative holds, then $u_n > 0$ in Ω for n large enough. Since $\int f(u_n)\phi^* > 0$, it follows that $t_n \neq 0$. Then $0 < t_n < 1$ and so by (18) we would have $\int g(x, u_n)\phi^* < 0$, which contradicts (17). Likewise the other alternative leads to a contradiction. This completes the proof of Theorem 2.

We observe that the condition (7) is formed by two inequalities which are used separately in the proof of Theorem 1. To serve the same purpose as (7) in a parallel manner, the condition (17) can be written in the form of two inequalities, namely $g(x, u) \geq 0$ for $u \geq 0$ and $g(x, u) \leq 0$ for $u \leq 0$ for almost all $x \in \Omega$, which can be used separately in the proof of Theorem 2. Thus one in (7) may be combined with one in (17) to produce new solvability conditions. We obtain the following Theorem 3 by one of the combinations.

Theorem 3. *Let $p > N$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the condition* (H). *If for almost all $x \in \Omega$ and $u \leq 0$,*

$$g(x, u) \leq 0,$$

then the problem (4) *has a solution $u \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ for any $h \in L^p(\Omega)$ such that*

$$0 = \int h\phi^* < \int g_+\phi^*.$$

Finally, we note that by modifying the arguments in Step 4 of the proof of Theorem 1, we may obtain a generalization to the main result of [7], without assuming, say, that the nonlinear term g keeps one sign on $\Omega \times \mathbb{R}$.

References

[1] S. Ahmad, *Nonselfadjoint resonance problems with unbounded perturbations*, Nonlinear Anal. **10** (1986), 147–156.

[2] H. Amann and M. G. Crandall, *On some existence theorems for semilinear elliptic equations*, Indiana Univ. Math. J. **27** (1978), 779–790.

[3] H. Berestycki and D. G. de Figueiredo, *Double resonance in semilinear elliptic equations*, Comm. Partial Differential Equations **6** (1981), 91–120.

[4] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.

[5] C. W. Ha and W. B. Sung, *On a resonance problem with nonlinearities of arbitrary polynomial growth*, Bull. Austral. Math. Soc. **48** (1993), 435–440.

[6] R. Iannacci, M. N. Nkashama and J. R. Ward, Jr., *Nonlinear second order elliptic partial differential equations at resonance*, Trans. Amer. Math. Soc. **311** (1989), 711–726.

[7] R. Kannan and R. Ortega, *Landesman–Lazer conditions for problems with "one-side unbounded" nonlinearities*, Nonlinear Anal. **9** (1985), 1313–1317.

[8] E. M. Landesman and A. C. Lazer, *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech. **19** (1970), 609–623.

[9] H. H. Schaefer, *Topological Vector Spaces*, Springer-Verlag, New York, 1971.

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