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NONLINEAR ERGODIC THEOREMS FOR ALMOST NONEXPANSIVE CURVES OVER COMMUTATIVE SEMIGROUPS

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Dedicated to Ky Fan on the occasion of his 80th birthday

1. Introduction

Let S be a commutative semigroup with identity, and let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

We also denote by $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$ and $\mathbb{R}^+$ the sets of all integers, nonnegative integers, real numbers and nonnegative real numbers, respectively. Let C be a subset of H. Then a mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. The first nonlinear ergodic theorem for nonexpansive mappings (in a Hilbert space) was established by Baillon [1]: Let C be a nonempty closed convex subset of H and let T be a nonexpansive mapping of C into itself. If T has a fixed point, then the Cesàro means $(1/n)\sum_{k=0}^{n-1} T^k x$ converge weakly as $n \to \infty$ to a fixed point y of T. In this case, put $y = Px$ for each $x \in C$. Then P is a nonexpansive retraction of C onto the set $\mathrm{Fix}(T)$ of fixed points of T such that $PT^n = T^n P = P$ for all $n \in \mathbb{Z}^+$, and $Px \in \mathrm{clco}\{T^n x : n \in \mathbb{Z}^+\}$ for each $x \in C$, where $\mathrm{clco}\, A$ is the closure of the convex hull of A. In [33, 34], Takahashi proved the existence of such an ergodic retraction for an amenable semigroup of nonexpansive mappings in a Hilbert space. And also Rodé [30] found a sequence of means on the semigroup, generalizing the Cesàro means on the positive integers, such that the corresponding sequence of mappings converges to an ergodic

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retraction onto the set of common fixed points. Recently Takahashi [36] proved a nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings without convexity in a Hilbert space; see also [17]. On the other hand, Bruck [10, 11] and Miyadera and Kobayasi [22] introduced the notion of an almost orbit of a nonexpansive semigroup on C and studied the weak and strong convergence theorems of such an almost orbit. Then Rouhani [31, 32] introduced the notions of almost nonexpansive sequences and curves in a Hilbert space, and proved weak and strong convergence theorems for such sequences and curves.

This paper is organized as follows: In Section 2, we give some definitions and elementary results. In Section 3, we introduce the notion of an almost nonexpansive curve over a commutative semigroup S which generalizes the notions given in [31, 32] and give some examples of such curves. We also define the generalized fixed point sets $F(u)$ and $F_\mu(u)$, where u is an almost nonexpansive curve and μ is an invariant mean. Then, using the metric projection P onto $F_\mu(u)$, we prove (Theorem 3.8) that $\{Pu(s) : s \in S\}$ converges strongly to $u(\mu)$, where $u(\mu)$ is the asymptotic center of u for μ. We also know that $u(\mu)$ is an element of H such that for any $y \in H, \langle u(\mu), y\rangle = \mu_t\langle u(t), y\rangle$. This result is an extension of Baillon [1], Baillon and Brezis [3], Moroşanu [24] and Rouhani [31]. In Section 4, we prove nonlinear ergodic theorems for almost nonexpansive curves over commutative semigroups. First we prove (Theorem 4.5) that for an asymptotically invariant net $\{\mu_\alpha : \alpha \in A\}$ of means which generalizes a sequence of Cesàro means, $u(r_s^*\mu_\alpha)$ converges weakly to $u(\mu)$. Further for a strongly regular net $\{\mu_\alpha : \alpha \in A\}$, we prove (Theorem 4.7) that $u(r_s^*\mu_\alpha)$ converges weakly to $u(\mu)$ uniformly in $s \in S$. These results generalize the results of Rodé [30], Takahashi [36] and Rouhani [31]. In Section 5, we find necessary and sufficient conditions for an almost nonexpansive curve to be weakly convergent.

2. Preliminaries

Let S be a commutative semigroup with identity and let $l^\infty(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm. Then for each $s \in S$ and $f \in l^\infty(S)$, we can define an element $r_s f$ in $l^\infty(S)$ by $(r_s f)(t) = f(t+s)$ for all $t \in S$. Let X be a subspace of $l^\infty(S)$ containing the constant functions on S. Then an element μ of X^*, where X^* is the dual space of X, is called a *mean* on X if $\|\mu\| = \mu(1) = 1$. As is known, μ is a mean on X if and only if

$$\inf_{s\in S} f(s) \le \mu(f) \le \sup_{s\in S} f(s)$$

for each $f \in X$. A real-valued function μ on X is called a *submean* on X if the following conditions are satisfied:

(i) $\mu(f+g) \le \mu(f) + \mu(g)$ for every $f, g \in X$;
(ii) $\mu(\alpha f) = \alpha\mu(f)$ for every $f \in X$ and $\alpha \ge 0$;
(iii) for $f, g \in X$, $f \le g$ implies $\mu(f) \le \mu(g)$;
(iv) $\mu(c) = c$ for every constant function c.

Clearly every mean on X is a submean. The notion of a submean was first introduced by Mizoguchi and Takahashi in [23]; see also [19].

Let X be a subspace of $l^\infty(S)$ containing the constant functions on S which is invariant under r_s, $s \in S$. A submean (or mean) μ on X is *invariant* if $\mu(r_s f) = \mu(f)$ for all $s \in S$ and $f \in X$. In the case when S is commutative, we know that there exists an invariant mean μ on $l^\infty(S)$; see Day [12]. For an invariant mean μ on $l^\infty(S)$, the restriction of μ to X is an invariant mean on X. Sometimes, the value of a submean (or mean) μ at $f \in X$ will also be denoted by $\mu(f)$ or $\mu_s(f(s))$. A commutative semigroup S is a directed system when the binary relation is defined by $s \le t$ if and only if $\{s\} \cup (S+s) \supset \{t\} \cup (S+t)$.

Throughout this paper, we denote by C a nonempty subset of a real Hilbert space H, by S a commutative semigroup with identity, and by X a subspace of $l^\infty(S)$ containing the constant functions on S which is invariant under r_s, $s \in S$. Furthermore, an order "$\le$" on S is defined as above. We also denote by $C_{\mathrm{b}}(S)$ and $M_{\mathrm{b}}((\mathbb{R}^+)^n)$ all bounded continuous functions on a semitopological semigroup S and all bounded Lebesgue measurable functions on $(\mathbb{R}^+)^n$, respectively. And also we write $x_n \rightharpoonup x$ (or w-$\lim x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors converges weakly to x; similarly $x_n \to x$ and $x_n \xrightarrow{w^*} x$ (or w^*-$\lim x_n = x$) will symbolize strong convergence and w^*-convergence, respectively. We also denote by $\mathrm{cl}\, A$ and $\mathrm{co}\, A$ the closure of A and the convex hull of A, respectively.

The following definition which was introduced by Takahashi [33] (see also Day [13]) is crucial in the nonlinear ergodic theory for abstract semigroups. Let u be a bounded function from S to H such that $\langle u(\cdot), y\rangle \in X$ for every $y \in H$, and let μ be an element of X^*. Then the function g from H into $\mathbb{R}$ given by

$$g(y) = \mu_s\langle u(s), y\rangle \qquad \text{for every } y \in H$$

is linear and continuous. So by the Riesz theorem, there exists an element $u(\mu)$ in H such that $\langle u(\mu), y\rangle = \mu_s\langle u(s), y\rangle$ for all $y \in H$. Let u be a bounded function from S to C such that for any $x \in C$, $\|u(\cdot) - x\|^2 \in X$. Then as in [14, 15, 20], for a submean μ on X, we define the *μ-asymptotic center* μ-$\mathrm{AC}(u, C)$ of u in C as follows:

$$\mu\text{-AC}(u, C) = \{x \in C : \mu_s\|u(s) - x\|^2 = \inf_{y \in C} \mu_s\|u(s) - y\|^2\}.$$

REMARK 2.1. Let u be a bounded function from S to C such that for any $x \in C$, $\|u(\cdot) - x\|^2 \in X$. Then it follows that for any $x \in C$, $\langle u(\cdot), x\rangle \in X$. If C is closed and convex, we also know that μ-$\mathrm{AC}(u, C)$ is nonempty [23].

We give two results which are used in Sections 3, 4 and 5.

Lemma 2.2 [34]. *Let μ be an invariant submean on X. Then*

$$\varliminf_s f(s) \le \mu(f) \le \varlimsup_s f(s)$$

for every $f \in X$, where $\varliminf_s f(s) = \sup_s \inf_{t\ge s} f(t)$ and $\varlimsup_s f(s) = \inf_s \sup_{t\ge s} f(t)$.

Lemma 2.3. *Let u be a bounded function from S into H with the property that $\|u(\cdot) - y\|^2 \in X$ for all $y \in H$. Then for any mean μ on X, the μ-asymptotic center μ-AC(u,H) of u in H consists of a single point $u(\mu)$. If μ is an invariant mean on X, then $u(\mu) \in \bigcap_s \operatorname{clco}\{u(t) : t \ge s\}$.*

Proof. Since, for each $y \in H$ and $t \in S$,

$$\|u(\mu) - y\|^2 = \|u(t) - y\|^2 - \|u(t) - u(\mu)\|^2 - 2\langle u(t) - u(\mu), u(\mu) - y\rangle,$$

we have

$$\begin{aligned}
0 &\le \|u(\mu) - y\|^2 \\
&= \mu_t\|u(t) - y\|^2 - \mu_t\|u(t) - u(\mu)\|^2 - 2\mu_t\langle u(t) - u(\mu), u(\mu) - y\rangle \\
&= \mu_t\|u(t) - y\|^2 - \mu_t\|u(t) - u(\mu)\|^2 - 2\langle u(\mu) - u(\mu), u(\mu) - y\rangle \\
&= \mu_t\|u(t) - y\|^2 - \mu_t\|u(t) - u(\mu)\|^2.
\end{aligned}$$

This implies that μ-AC(u,H) consists of a single point $u(\mu)$.

Assume $u(\mu) \notin \bigcap_{s\in S} \operatorname{clco}\{u(t) : t \ge s\}$. Then $u(\mu) \notin \operatorname{clco}\{u(t) : t \ge s\}$ for some $s \in S$. By the separation theorem, there exists a y_0 in H such that

$$\langle u(\mu), y_0\rangle < \inf\{\langle z, y_0\rangle : z \in \operatorname{clco}\{u(t) : t \ge s\}\}.$$

As

$$\begin{aligned}
\langle u(\mu), y_0\rangle &< \inf\{\langle z, y_0\rangle : z \in \operatorname{clco}\{u(t) : t \ge s\}\} \\
&\le \inf_{t\in S}\langle u(s+t), y_0\rangle \le \mu_t\langle u(s+t), y_0\rangle = \mu_t\langle u(t), y_0\rangle = \langle u(\mu), y_0\rangle,
\end{aligned}$$

we have a contradiction. Therefore $u(\mu) \in \bigcap_{s\in S} \operatorname{clco}\{u(t) : t \ge s\}$.

3. Almost nonexpansive curves

In this section, we introduce the notion of an almost nonexpansive curve over a commutative semigroup and prove some results for such curves.

Let u be a function from S into H. Then u is said to be an *almost nonexpansive curve* if there exists a real-valued function $\varepsilon(\cdot,\cdot)$ on $S\times S$ such that

$$\|u(s+h) - u(t+h)\|^2 \le \|u(s) - u(t)\|^2 + \varepsilon(s,t)$$

for every s, t and h in S and $\lim_{s,t\to\infty} \varepsilon(s,t) = 0$, where $\lim_{s,t\to\infty} \varepsilon(s,t) = 0$ means that, for any $\delta > 0$, there exists $s_0 \in S$ such that $\varepsilon(s,t) \le \delta$ for every $s,t \in S$ with $s,t \ge s_0$. In the case when $\varepsilon(s,t) = 0$ for every $s,t \in S$, u is said to be a *nonexpansive curve*.

REMARK 3.1. Let u be a bounded function from S to H such that

$$\|u(s+h) - u(t+h)\| \le \|u(s) - u(t)\| + \varepsilon_1(s,t)$$

for every s,t and h in S and $\lim_{s,t\to\infty} \varepsilon_1(s,t) = 0$. Then it is obvious that u is an almost nonexpansive curve with $\varepsilon(s,t) = 4(\sup_{r\in S} \|u(r)\|)\varepsilon_1(s,t) + \varepsilon_1(s,t)^2$.

We give some examples of almost nonexpansive curves.

EXAMPLE 3.2. Consider the initial value problem

$$(*) \qquad \frac{du}{dt}(t) + Au(t) \ni f(t), \quad t > 0, \qquad u(0) = x,$$

where A is a maximal monotone operator in H, $f \in L^1(0,\infty;H)$ and $x \in \operatorname{cl} D(A)$. Then it is well known that $(*)$ has a unique integral solution $u(t)$; see [4, 5]. We also know that if $v(t)$ is another integral solution of $(*)$ corresponding to $g \in L^1(0,\infty;H)$ and $y \in \operatorname{cl} D(A)$, then

$$\|u(t_2) - v(t_2)\| \le \|u(t_1) - v(t_1)\| + \int_{t_1}^{t_2} \|f(\theta) - g(\theta)\|\, d\theta,$$

whenever $0 \le t_1 \le t_2 < \infty$. Putting $v(t) = u(t-r+s)$, $g(t) = f(t-r+s)$, $t_1 = r$ and $t_2 = r+h$, we get

$$\begin{aligned}\|u(r+h) - u(s+h)\| &\le \|u(r) - u(s)\| + \int_r^{r+h} \|f(\theta - r + s) - f(\theta)\|\, d\theta \\ &\le \|u(r) - u(s)\| + \int_r^{\infty} \|f(\theta - r + s)\|\, d\theta + \int_r^{\infty} \|f(\theta)\|\, d\theta \\ &= \|u(r) - u(s)\| + \int_s^{\infty} \|f(\tau)\|\, d\tau + \int_r^{\infty} \|f(\theta)\|\, d\theta.\end{aligned}$$

So put $\varepsilon_1(r,s) = \int_s^\infty \|f(\tau)\|\, d\tau + \int_r^\infty \|f(\theta)\|\, d\theta$, and $\varepsilon(r,s) = 4(\sup_{t\in S} \|u(t)\|) \times \varepsilon_1(r,s) + \varepsilon_1(r,s)^2$. If $A^{-1}(0) \ne \emptyset$, by Remark 3.1, u is a bounded almost nonexpansive curve from $\mathbb{R}^+$ to H.

EXAMPLE 3.3. Let S be a commutative semitopological semigroup with identity, i.e. a commutative semigroup with a Hausdorff topology such that for each $t \in S$, the mapping $s \mapsto s + t$ from S to S is continuous, and let $\mathcal{T} = \{T(s) : s \in S\}$ be a family of nonexpansive mappings from C into itself such that $T(s+t) = T(s)T(t)$ for all $s,t \in S$ and $s \mapsto T(s)x$ is continuous for each $x \in C$. Such a family $\mathcal{T} = \{T(s) : s \in S\}$ is called a *nonexpansive semigroup* on C. We denote by $\operatorname{Fix}(\mathcal{T})$ the set of common fixed points of $T(s)$, $s \in S$. Assume that $\langle T(\cdot)x, y\rangle \in X$ for all $x \in C$ and $y \in H$. Then for any mean μ on

X, we define a unique element $\mathcal{T}(\mu)x$ of C such that $\langle \mathcal{T}(\mu)x, y\rangle = \mu_t\langle T(t)x, y\rangle$ for all $y \in H$. A continuous function $u : S \to C$ is said to be an *almost orbit* of $\mathcal{T} = \{T(s) : s \in S\}$ if

$$\lim_t \sup_{s\in S} \|u(s+t) - T(s)u(t)\| = 0.$$

If u is a bounded almost orbit of $\mathcal{T}$, then we have

$$\begin{aligned}&\|u(s+h) - u(t+h)\| \\ &\leq \|u(s+h) - T(h)u(s)\| + \|u(t+h) - T(h)u(t)\| + \|T(h)u(s) - T(h)u(t)\|.\end{aligned}$$

So, putting $\varepsilon_1(s,t) = \sup_{h\in S} \|u(s+h)-T(h)u(s)\|+\sup_{h\in S} \|u(t+h)-T(h)u(t)\|$, and $\varepsilon(s,t) = 4(\sup_{r\in S} \|u(r)\|)\varepsilon_1(s,t) + \varepsilon_1(s,t)^2$, by Remark 3.1, we see that u is an almost nonexpansive curve from S into C.

Example 3.4. Let $u : \mathbb{Z}^+ \to \mathbb{R}$ be the function given by $u(n) = 1/n$ and $\varepsilon_1(n,m) = 1/n + 1/m$. Then by Remark 3.1, u is an almost nonexpansive curve from $\mathbb{Z}^+$ to $\mathbb{R}$. More generally, let $u(n_1,\dots,n_k) = 1/(n_1\dots n_k)$ and $\varepsilon_1((n_1,\dots,n_k),(m_1,\dots,m_k)) = 1/(n_1\dots n_k) + 1/(m_1\dots m_k)$, where $(n_1,\dots,n_k)$ and $(m_1,\dots,m_k)$ are elements of $(\mathbb{Z}^+)^k$. Then u is an almost nonexpansive curve from $(\mathbb{Z}^+)^k$ to $\mathbb{R}$.

Example 3.5. Let $u : \mathbb{R}^+ \to \mathbb{R}^2$ be the function given by $u(s) = (\cos s + 1/(s+1), \sin s)$ and $\varepsilon_1(s,t) = 3(1/(s+1) + 1/(t+1))$. Then u is an almost nonexpansive curve from $\mathbb{R}^+$ to $\mathbb{R}^2$. More generally, put

$$u(s_1,\dots,s_k) = \left(\cos\left(\sum_{i=1}^k s_i\right) + 1/\prod_{i=1}^k (s_i+1),\ \sin\left(\sum_{i=1}^k s_i\right)\right).$$

Then u is an almost nonexpansive curve from $(\mathbb{R}^+)^k$ to $\mathbb{R}^2$ with $\varepsilon_1((s_i),(t_i)) = 3(1/\prod_{i=1}^k(s_i+1) + 1/\prod_{i=1}^k(t_i+1))$.

Let u be a function from S to H. Then we denote by $F_1(u)$ and $F(u)$ the subsets of H defined by $q \in F_1(u)$ if and only if $\|u(t)-q\| \leq \|u(s)-q\|$ for every $t,s \in S$ with $t \geq s$, and $q \in F(u)$ if and only if $\lim_{s\to\infty} \|u(s) - q\|$ exists.

Let u be an almost nonexpansive curve from S to H with $\varepsilon(\,\cdot\,,\,\cdot\,)$ such that $\varepsilon(s,\,\cdot\,) \in X$ for all $s \in S$. Let μ be an invariant mean on X and put $\varepsilon(s) = \mu_t\varepsilon(s,t)$, $s \in S$. Then $\lim_{s\to\infty} \varepsilon(s) = 0$. In fact, for any $\delta > 0$, there exists s_0 such that for any $s,t \geq s_0$, $\varepsilon(s,t) \leq \delta$. Then for any $s \geq s_0$, $\varepsilon(s) = \mu_t\varepsilon(s,t) = \mu_t\varepsilon(s,s_0+t) \leq \delta$. This implies $\lim_{s\to\infty} \varepsilon(s) = 0$. So, for an almost nonexpansive curve u from S to H with $\varepsilon(\,\cdot\,,\,\cdot\,)$ and an invariant mean μ on X, we denote by $F_\mu(u)$ the subset of H given by $q \in F_\mu(u)$ if and only if $\|u(t)-q\|^2 \leq \|u(s)-q\|^2 + \varepsilon(s)$ for every $t,s \in S$ with $t \geq s$, where $\varepsilon(s) = \mu_t\varepsilon(s,t)$.

LEMMA 3.6. *Let u be an almost nonexpansive curve from S to H with $\varepsilon(\,\cdot\,,\,\cdot\,)$ such that $\|u(\,\cdot\,)-y\|^2$ and $\varepsilon(s,\,\cdot\,)$ are in X for all $y \in H$ and $s \in S$. Let μ be an invariant mean on X. Then:*

(i) *$F(u)$, $F_1(u)$ and $F_\mu(u)$ are closed convex subsets of H;*
(ii) *$F_1(u) \subset F_\mu(u) \subset F(u)$. In particular, $u(\mu) \in F_\mu(u)$.*

PROOF. (i) We use the methods of [31] and [32]. Let $\{q_n\} \subset F(u)$ and $q_n \to q$. Then, from

$$\begin{aligned} &|\,\|u(s)-q\|-\|u(t)-q\|\,| \\ &\qquad \le |\,\|u(s)-q\|-\|u(s)-q_n\||+|\,\|u(s)-q_n\|-\|u(t)-q_n\|\,| \\ &\qquad\quad +|\,\|u(t)-q_n\|-\|u(t)-q\|\,| \\ &\qquad \le 2\|q-q_n\|+|\,\|u(s)-q_n\|-\|u(t)-q_n\|\,|, \end{aligned}$$

it follows that $\{\|u(s)-q\| : s \in S\}$ is a Cauchy net. So, we have $q \in F(u)$. This implies $F(u)$ is closed. Let $q_1, q_2 \in F(u)$. Then, from

$$\|u(s)-(1/2)(q_1+q_2)\|^2 = (1/2)\|u(s)-q_1\|^2+(1/2)\|u(s)-q_2\|^2-(1/4)\|q_1-q_2\|^2,$$

we have $(1/2)(q_1+q_2) \in F(u)$ and hence $F(u)$ is convex. Let $s \in S$. Then, putting

$$F_s = \{q \in H : \forall t \ge s,\ \|u(t)-q\|^2 \le \|u(s)-q\|^2+\varepsilon(s)\},$$

we have

$$F_s = \{q \in H : \forall t \ge s,\ 2\langle u(s)-u(t), q\rangle \le \|u(s)\|^2-\|u(t)\|^2+\varepsilon(s)\}.$$

This implies F_s is closed and convex. Since $F_\mu(u) = \bigcap_{s\in S} F_s$, $F_\mu(u)$ is closed and convex. Similarly, $F(u)$ and $F_1(u)$ are closed and convex.

(ii) For any $s, t, a \in S$, we have

$$\begin{aligned} &2\langle u(s)-u(t+s), u(a)-u(\mu)\rangle \\ &\quad = \|u(s)-u(\mu)\|^2-\|u(t+s)-u(\mu)\|^2+\|u(t+s)-u(a)\|^2-\|u(s)-u(a)\|^2, \end{aligned}$$

and hence

$$\begin{aligned} 0 = 2\langle u(s)-u(t+s), u(\mu)-u(\mu)\rangle &= \|u(s)-u(\mu)\|^2-\|u(t+s)-u(\mu)\|^2 \\ &\quad +\mu_a\|u(t+s)-u(a)\|^2-\mu_a\|u(s)-u(a)\|^2. \end{aligned}$$

On the other hand, since $\|u(t+s)-u(t+a)\|^2 \le \|u(s)-u(a)\|^2+\varepsilon(s,a)$, we have

$$\begin{aligned} \mu_a\|u(t+s)-u(a)\|^2 = \mu_a\|u(t+s)-u(t+a)\|^2 &\le \mu_a(\|u(s)-u(a)\|^2+\varepsilon(s,a)) \\ &= \mu_a(\|u(s)-u(a)\|^2+\varepsilon(s). \end{aligned}$$

Therefore,

$$\|u(t+s)-u(\mu)\|^2-\|u(s)-u(\mu)\|^2 = \mu_a\|u(t+s)-u(a)\|^2-\mu_a\|u(s)-u(a)\|^2 \le \varepsilon(s),$$

and hence $\|u(t+s)-u(\mu)\|^2 \le \|u(s)-u(\mu)\|^2 + \varepsilon(s)$. So, $u(\mu) \in F_\mu(u)$.

Let $q \in F_\mu(u)$. Since $\lim_{s\to\infty} \varepsilon(s) = 0$, for any $\varepsilon > 0$ there exists s_0 such that for any $s \ge s_0$ and $t \in S$,

$$\|u(t+s)-q\|^2 \le \|u(s)-q\|^2 + \varepsilon.$$

So, we get

$$\overline{\lim_t}\, \|u(t)-q\|^2 \le \|u(s)-q\|^2 + \varepsilon,$$

and hence

$$\overline{\lim_t}\, \|u(t)-q\|^2 \le \underline{\lim_s}\, \|u(s)-q\|^2 + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\lim_t \|u(t)-q\|^2$ must exist. So, $F_\mu(u) \subset F(u)$.

The following lemma is a modification of [36].

Lemma 3.7. *Let μ be an invariant mean on X, and let u be a bounded almost nonexpansive curve from S to H with $\varepsilon(\,\cdot\,,\,\cdot\,)$ such that $\|u(\,\cdot\,)-y\|^2$ and $\varepsilon(s,\,\cdot\,)$ are in X for all $y \in H$ and $s \in S$. Then*

$$\overline{\lim}\,\text{-AC}(u,H) = \mu\text{-AC}(u,H) = \{u(\mu)\},$$

where $\overline{\lim}\text{-AC}(u,H) = \{x \in H : \overline{\lim}_s \|u(s)-x\|^2 = \inf_{y\in H} \overline{\lim}_s \|u(s)-y\|^2\}$. *Consequently, if μ and λ are invariant means on X, then $u(\mu) = u(\lambda)$.*

Proof. As in the proof of Lemma 2.3, for any $t \in S$ and $y \in H$,

$$(*) \qquad 0 \le \|u(\mu)-y\|^2 = \mu_t\|u(t)-y\|^2 - \mu_t\|u(t)-u(\mu)\|^2.$$

This implies μ-AC$(u,H) = \{u(\mu)\}$. Since $u(\mu) \in F(u)$, by Lemma 2.2 we have $\mu_t\|u(t)-u(\mu)\|^2 = \lim_t \|u(t)-u(\mu)\|^2$ and $\mu_t\|u(t)-y\|^2 \le \overline{\lim}_t \|u(t)-y\|^2$. Then from $(*)$ we get

$$0 \le \|u(\mu)-y\|^2 \le \overline{\lim_t}\, \|u(t)-y\|^2 - \overline{\lim_t}\, \|u(t)-u(\mu)\|^2.$$

Therefore $\overline{\lim}$-AC$(u,H) = \{u(\mu)\}$. So, the first assertion follows. From this, it is obvious that $u(\mu) = u(\lambda)$.

The following theorem is an extension of Baillon [1], Baillon and Brezis [3], Moroşanu [24] and Rouhani [32].

Theorem 3.8. *Let u be an almost nonexpansive curve from S to H with $\varepsilon(\,\cdot\,,\,\cdot\,)$ such that $\|u(\,\cdot\,)-y\|^2$ and $\varepsilon(s,\,\cdot\,)$ are in X for all $y \in H$ and $s \in S$, and let μ be an invariant mean on X. Let P be the metric projection of H onto $F_\mu(u)$. Then $Pu(s)$ converges strongly to $u(\mu)$, which is the μ-asymptotic center of u in H.*

Proof. Let $\varepsilon > 0$. Then there exists $s_0 \in S$ such that for any $s \ge s_0$, $\varepsilon(s) = \mu_t\varepsilon(s,t) \le \varepsilon$. Put $\varphi(s) = \|u(s)-Pu(s)\|^2$ for all $s \in S$. Then, for any $s \ge s_0$, we have

$$\begin{aligned}\varphi(t+s) = \|u(t+s) - Pu(t+s)\|^2 &\leq \|u(t+s) - Pu(s)\|^2 \\ &\leq \|u(s) - Pu(s)\|^2 + \varepsilon(s) \leq \varphi(s) + \varepsilon,\end{aligned}$$

and hence

$$\overline{\lim_t}\, \varphi(t) = \overline{\lim_t}\, \varphi(t+s) \leq \varphi(s) + \varepsilon.$$

So, we have

$$\overline{\lim_t}\, \varphi(t) = \underline{\lim}_s\, \varphi(s) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $\overline{\lim}_t\, \varphi(t) \leq \underline{\lim}_s\, \varphi(s)$. This implies that $\lim_s \varphi(s)$ exists. Since P is the metric projection of H onto $F_\mu(u)$, we have, for any $t, s \in S$,

$$\langle u(t+s) - Pu(t+s), Pu(s) - Pu(t+s)\rangle \leq 0.$$

Hence, from

$$\begin{aligned}\|Pu(t+s) - Pu(s)\|^2 + \|u(t+s) - Pu(t+s)\|^2 - \|u(t+s) - Pu(s)\|^2 \\ = 2\langle u(t+s) - Pu(t+s), Pu(s) - Pu(t+s)\rangle,\end{aligned}$$

we have

$$\begin{aligned}\|Pu(t+s) - Pu(s)\|^2 &\leq \|u(t+s) - Pu(s)\|^2 - \|u(t+s) - Pu(t+s)\|^2 \\ &\leq \|u(s) - Pu(s)\|^2 + \varepsilon(s) - \|u(t+s) - Pu(t+s)\|^2 \\ &= \varphi(s) - \varphi(t+s) + \varepsilon(s).\end{aligned}$$

This implies $Pu(s)$ is a Cauchy net. Let q be a point of H such that $Pu(s) \to q$. Then we show $q = u(\mu)$. Since $u(\mu) \in F_\mu(u)$, we have, for any $s \in S$,

$$\langle u(s) - Pu(s), u(\mu) - Pu(s)\rangle \leq 0.$$

Then we obtain

$$\begin{aligned}\langle u(s) - Pu(s), u(\mu)\rangle - \langle u(s) - Pu(s), q\rangle &\leq \langle u(s) - Pu(s), Pu(s) - q\rangle \\ &\leq K\|Pu(s) - q\|,\end{aligned}$$

where $K = \sup_{s\in S} \|u(s) - Pu(s)\|$. So, we have

$$\langle u(\mu) - q, u(\mu)\rangle - \langle u(\mu) - q, q\rangle \leq K\|q - q\| = 0.$$

This implies $\langle u(\mu) - q, u(\mu) - q\rangle \leq 0$ and hence $q = u(\mu)$. Therefore $Pu(s)$ converges strongly to $u(\mu)$.

Remark 3.9. Let P be the metric projection of H onto $F(u)$. Then there is an example that $Pu(s)$ does not converge; see Rouhani [32, Example 3.5].

From Theorem 3.8, we obtain the following two results.

Corollary 3.10 [32]. *Let $\{x(n) : n \in \mathbb{Z}^+\}$ be a bounded nonexpansive sequence in H. Then, for P being the metric projection of H onto $F_1(x(\cdot))$, $Px(n)$ converges strongly to the* ($\overline{\lim}$-)*asymptotic center of* $\{x(n)\}$.

Proof. Take $S = \mathbb{Z}^+$, $u = x(\cdot)$ and $X = l^\infty(\mathbb{Z}^+)$. Then $F_\mu(x(\cdot)) = F_1(x(\cdot))$, and by Lemma 3.7, μ-asymptotic center = $\overline{\lim}$-asymptotic center. So, Corollary 3.10 is obvious from Theorem 3.8.

Corollary 3.11. *Let $u : S \to C$ be a bounded almost orbit of a nonexpansive semigroup $\mathcal{T} = \{T(s) : s \in S\}$ on C, let μ be an invariant mean on $l^\infty(S)$, and assume $\bigcap_{s\in S} \operatorname{clco}\{u(t) : t \geq s\} \subset C$. Let P be the metric projection of H onto $F_\mu(u)$. Then $\operatorname{Fix}(\mathcal{T}) \neq \emptyset$, and $Pu(s)$ converges strongly to $u(\mu)$, which is the μ-asymptotic center of u in C and also a point of* $\operatorname{Fix}(\mathcal{T})$.

Proof. By Theorem 3.8 and Example 3.3, it is obvious that $Pu(s)$ converges strongly to $u(\mu)$, which is an element of C by Lemma 2.3 and the assumption. Next, we show μ-$\operatorname{AC}(u, C)$ is $T(t)$-invariant for all $t \in S$. Let $y \in \operatorname{AC}(u, C)$. For any $\varepsilon > 0$, there exists s_0 such that for any $s \geq s_0$ and $t \in S$,

$$\|u(t+s) - T(t)u(s)\| \leq \varepsilon.$$

Then we have

$$\|u(t+s)-T(t)y\| \leq \|u(t+s)-T(t)u(s)\| + \|T(t)u(s)-T(t)y\| \leq \varepsilon + \|u(s)-y\|.$$

Putting $K = 2\sup_{s\in S} \|u(s) - y\|$, we have

$$\|u(t+s) - T(t)y\|^2 \leq \|u(s) - y\|^2 + K\varepsilon + \varepsilon^2,$$

and hence

$$\mu_s\|u(s) - T(t)y\|^2 = \mu_s\|u(t+s) - T(t)y\|^2 \leq \mu_s\|u(s) - y\|^2 + K\varepsilon + \varepsilon^2.$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\mu_s\|u(s) - T(t)y\|^2 \leq \mu_s\|u(s) - y\|^2.$$

Thus $T(t)y \in \mu$-$\operatorname{AC}(u, C)$. This implies μ-$\operatorname{AC}(u, C)$ is $T(t)$-invariant for all $t \in S$. Since μ-$\operatorname{AC}(u, C)$ consists of one point $u(\mu)$, we have $u(\mu) = T(t)u(\mu)$ for all $t \in S$. Therefore $\operatorname{Fix}(\mathcal{T}) \neq \emptyset$.

4. Nonlinear ergodic theorems

In this section, we prove nonlinear ergodic theorems for almost nonexpansive curves. Let u be a bounded function from S to H. We define $W(u)$ = the set of all weak limit points of subnets of the net $\{u(s) : s \in S\}$. As in the proof of Bruck [9, Lemma 1.2], we have the following.

LEMMA 4.1. *Let E be a reflexive Banach space and let u be a bounded function from S to E. Then* $\operatorname{clco} W(u) = \bigcap_{t\in S} \operatorname{clco}\{u(s) : s \geq t\}$.

Using Lemma 4.1, we have the following.

LEMMA 4.2. *Let u be a bounded function from S to H. Then $F(u)$ is orthogonal to* $\operatorname{clco} W(u)$, *i.e. for any $p_1, p_2 \in F(u)$ and $q_1, q_2 \in \operatorname{clco} W(u)$,*

$$\langle p_1 - p_2, q_1 - q_2\rangle = 0.$$

In particular, $F(u) \cap \operatorname{clco} W(u)$ consists of at most one point.

PROOF. We use the method of Brezis [6] and Rouhani [31]. It is sufficient to give the proof for $q_1, q_2 \in W(u)$. Let $u(s_\alpha) \rightharpoonup q_1$ and $u(t_\beta) \rightharpoonup q_2$. For any $s \in S$,

$$\|u(s) - p_2\|^2 = \|u(s) - p_1\|^2 + \|p_1 - p_2\|^2 + \langle u(s) - p_1, p_1 - p_2\rangle.$$

So, we get

$$\lim_s \|u(s) - p_2\|^2 = \lim_s \|u(s) - p_1\|^2 + \|p_1 - p_2\|^2 + \langle q_1 - p_1, p_1 - p_2\rangle$$

and

$$\lim_s \|u(s) - p_2\|^2 = \lim_s \|u(s) - p_1\|^2 + \|p_1 - p_2\|^2 + \langle q_2 - p_1, p_1 - p_2\rangle.$$

Therefore, we get $\langle p_1 - p_2, q_1 - q_2\rangle = 0$.

Using Lemmas 4.1 and 4.2, we have the following.

THEOREM 4.3. *Let u be an almost nonexpansive curve from S to H with $\varepsilon(\cdot\,,\cdot)$ such that $\|u(\cdot) - y\|^2$ and $\varepsilon(s,\cdot)$ are in X for all $y \in H$ and $s \in S$, and let μ be an invariant mean on X. Then*

$$F(u) \cap \bigcap_{s\in S} \operatorname{clco}\{u(t) : t \geq s\} = \{u(\mu)\}.$$

PROOF. By Lemma 2.3, $u(\mu) \in \bigcap_{s\in S} \operatorname{clco}\{u(t) : t \geq s\}$ and by Lemma 3.6, $u(\mu) \in F(u)$. Thus, by Lemmas 4.1 and 4.2, the proof is complete.

Let $\{\mu_\alpha : \alpha \in A\}$ be a net of means on X. Then $\{\mu_\alpha : \alpha \in A\}$ is said to be *asymptotically invariant* on X (cf. [30]) if for any $s \in S$ and $f \in X$,

$$\mu_\alpha(f) - \mu_\alpha(r_s f) \to 0.$$

Let $\{\mu_\alpha : \alpha \in A\}$ be a net of continuous linear functionals on X. Then $\{\mu_\alpha : \alpha \in A\}$ is said to be *strongly regular* on X (cf. [16]) if the following conditions are satisfied:

(a) $\sup_\alpha \|\mu_\alpha\| < \infty$;
(b) $\lim_\alpha \mu_\alpha(1) = 1$;
(c) $\lim_\alpha \|\mu_\alpha - r_s^* \mu_\alpha\| = 0$ for every $s \in S$.

We give some examples of asymptotically invariant nets and strongly regular nets; see Hirano, Kido and Takahashi [16] and Takahashi [35].

Example 4.4. (i) Let $S = \mathbb{Z}^+$ and $X = l^\infty(\mathbb{Z}^+)$ $(= C_{\rm b}(\mathbb{Z}^+))$. Put $\mu_n(f) = (1/n)\sum_{k=0}^{n-1} f(k)$ for $f \in X$. Then $\{\mu_n : n \in \mathbb{Z}^+ \setminus \{0\}\}$ is an asymptotically invariant and strongly regular net.

(ii) Let $S = \mathbb{Z}^+$ and $X = l^\infty(\mathbb{Z}^+)$ $(= C_{\rm b}(\mathbb{Z}^+))$. Put $\mu_s(f) = (1-s) \times \sum_{k=0}^\infty s^k f(k)$ for $f \in X$. Then $\{\mu_s : s \in (0,1)\}$ is an asymptotically invariant and strongly regular net.

(iii) Let $S = \mathbb{Z}^+ \times \mathbb{Z}^+$ and $X = l^\infty(\mathbb{Z}^+ \times \mathbb{Z}^+)$ $(= C_{\rm b}(\mathbb{Z}^+ \times \mathbb{Z}^+))$. Put $\mu_n(f) = (1/n^2)\sum_{i,j=0}^{n-1} f(i,j)$ for $f \in X$. Then $\{\mu_n : n \in \mathbb{Z}^+ \setminus \{0\}\}$ is an asymptotically invariant and strongly regular net.

(iv) Let $S = \mathbb{R}^+$ and $X = M_{\rm b}(\mathbb{R}^+)$, or $X = C_{\rm b}(\mathbb{R}^+)$. Put $\mu_s(f) = (1/s) \times \int_0^s f(t)\,dt$ for $f \in X$. Then $\{\mu_s : s \in \mathbb{R}^+ \setminus \{0\}\}$ is an asymptotically invariant and strongly regular net.

(v) Let $S = \mathbb{R}^+$ and $X = M_{\rm b}(\mathbb{R}^+)$, or $X = C_{\rm b}(\mathbb{R}^+)$. Put $\mu_s(f) = s\int_0^\infty e^{-st} f(t)\,dt$ for $f \in X$. Then $\{\mu_s : s \in \mathbb{R}^+ \setminus \{0\}\}$ is an asymptotically invariant and strongly regular net.

(vi) Let $S = \mathbb{Z}^+$ and $X = l^\infty(\mathbb{Z}^+)$ $(= C_{\rm b}(\mathbb{Z}^+))$. Put $\mu_n(f) = \sum_{m=0}^\infty q_{n,m} f(m)$ for $f \in X$, where $\{q_{n,m}\}_{n,m\in\mathbb{Z}^+}$ is a strongly regular matrix. Then $\{\mu_n : n \in \mathbb{Z}^+\}$ is a strongly regular net. Here $\{q_{n,m}\}_{n,m\in\mathbb{Z}^+}$ is called a *strongly regular matrix* [21] if it satisfies the following conditions:

(a) $\sup_{n\in\mathbb{Z}^+} \sum_{m=0}^\infty |q_{n,m}| < \infty$;
(b) $\lim_{n\to\infty} \sum_{m=0}^\infty q_{n,m} = 1$;
(c) $\lim_{n\to\infty} \sum_{m=0}^\infty |q_{n,m+1} - q_{n,m}| = 0$.

(vii) Let $S = \mathbb{R}^+$ and $X = M_{\rm b}(\mathbb{R}^+)$, or $C_{\rm b}(\mathbb{R}^+)$. Put $\mu_s(f) = \int_0^\infty Q(s,t)u(t)\,dt$ for $f \in X$, where $Q(\,\cdot\,,\,\cdot\,)$ is a strongly regular kernel. Then $\{\mu_s : s \in \mathbb{R}^+\}$ is a strongly regular net. Here a function $Q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called a *strongly regular kernel* if it satisfies the following conditions:

(a) $\sup_{s\in\mathbb{R}^+} \int_0^\infty |Q(s,t)|\,dt < \infty$;
(b) $\lim_{s\to\infty} \int_0^\infty Q(s,t)\,dt = 1$;
(c) $\lim_{s\to\infty} \int_0^\infty |Q(s,t+h) - Q(s,t)|\,dt = 0$ for every $h \in \mathbb{R}^+$.

Now we prove a generalized mean convergence theorem for almost nonexpansive curves.

Theorem 4.5. *Let u be an almost nonexpansive curve from S to H with $\varepsilon(\,\cdot\,,\,\cdot\,)$ such that $\|u(\,\cdot\,) - y\|^2$ and $\varepsilon(s,\,\cdot\,)$ are in X for all $y \in H$ and $s \in S$, and let $\{\mu_\alpha : \alpha \in A\}$ be an asymptotically invariant net of means on X. Then for any $s \in S$, $u(r_s^*\mu_\alpha)$ converges weakly to $x_0 \in F(u) \cap \bigcap_{s\in S} \operatorname{clco}\{u(t) : t \geq s\}$, where $x_0 = u(\mu)$ is the μ-asymptotic center of u in H for any invariant mean μ on X.*

PROOF. By Lemma 3.7, we know that if λ and μ are invariant means on X, then $u(\mu) = u(\lambda)$. Let $s \in S$ and assume that $u(r_s^*\mu_\alpha)$ does not converge weakly to $u(\mu)$. Then there exists a subnet $\{u(r_s^*\mu_{\alpha_\beta}) : \beta \in B\}$ of $\{u(r_s^*\mu_\alpha) : a \in A\}$ such that for any subnet $\{u(r_s^*\mu_{\alpha_{\beta_\gamma}}) : \gamma \in \Gamma\}$ of $\{u(r_s^*\mu_{\alpha_\beta}) : \beta \in B\}$,

$$u(r_s^*\mu_{\alpha_{\beta_\gamma}}) \not\rightharpoonup u(\mu).$$

Since $\{r_s^*\mu_{\alpha_\beta} : \beta \in B\} \subset B(X^*)$, where $B(X^*)$ is the closed unit ball of X^*, there exists a subnet $\{r_s^*\mu_{\alpha_{\beta_\gamma}} : \gamma \in \Gamma\}$ of $\{r_s^*\mu_{\alpha_\beta} : \beta \in B\}$ such that

$$r_s^*\mu_{\alpha_{\beta_\gamma}} \xrightarrow{w^*} \lambda.$$

Then λ is an invariant mean on X. Indeed, since it is obvious that μ is a mean, we show that λ is invariant. For simplicity, put $\{\alpha_{\beta_\gamma} : \gamma \in \Gamma\} = \{\alpha_\beta : \beta \in B\} = \{\alpha : \alpha \in A\}$. For any $t \in S$, $f \in X$ and $\varepsilon > 0$, there exists $\alpha \in A$ such that

$$|r_s^*\mu_\alpha(r_t f) - \lambda(r_t f)| \le \varepsilon, \qquad |r_s^*\mu_\alpha(f) - \lambda(f)| \le \varepsilon,$$

and

$$|\mu_\alpha(r_s f) - r_t^*\mu_\alpha(r_s f)| \le \varepsilon.$$

Then

$$\begin{aligned} &|\lambda(r_t f) - \lambda(f)| \\ &\quad \le |\lambda(r_t f) - r_s^*\mu_\alpha(r_t f)| + |r_t^*\mu_\alpha(r_s f) - \mu_\alpha(r_s f)| + |r_s^*\mu_\alpha(f) - \lambda(f)| \le 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$, $f \in X$ and $t \in S$ are arbitrary, this implies λ is an invariant mean on X. Since $r_s^*\mu_{\alpha_{\beta_\gamma}} \xrightarrow{w^*} \lambda$, for any $y \in H$,

$$\langle u(r_s^*\mu_{\alpha_{\beta_\gamma}}), y\rangle = (r_s^*\mu_{\alpha_{\beta_\gamma}})_t\langle u(t), y\rangle \to \lambda_t\langle u(t), y\rangle = \langle u(\lambda), y\rangle,$$

which implies $u(r_s^*\mu_{\alpha_{\beta_\gamma}}) \rightharpoonup u(\lambda) = u(\mu)$ by Lemma 3.7. This is a contradiction. Therefore $u(r_s^*\mu_\alpha)$ converges weakly to $u(\mu)$, where $\{u(\mu)\} = F(u) \cap \bigcap_{s\in S} \operatorname{clco}\{u(t) : t \ge s\} = \mu\text{-AC}(u, H)$ by Theorem 4.3 and Lemma 3.7.

As a direct consequence of Theorem 4.5, we have the following.

COROLLARY 4.6. *Let $u : \mathbb{R}^+ \to C$ be a bounded almost orbit of a nonexpansive semigroup $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ on C, and assume $\bigcap_{s\in S} \operatorname{clco}\{u(t) : t \ge s\} \subset C$. Then $\operatorname{Fix}(\mathcal{T}) \ne \emptyset$, and for an asymptotically invariant net $\{\mu_\alpha : \alpha \in A\}$ on $M_b(\mathbb{R}^+)$, where $M_b(\mathbb{R}^+)$ is the set of all bounded Lebesgue measurable functions on $\mathbb{R}^+$, for any $s \in S$, $u(r_s^*\mu_\alpha)$ converges weakly to $x_0 \in \operatorname{Fix}(\mathcal{T}) \cap \bigcap_{s\in S} \operatorname{clco}\{u(t) : t \ge s\}$, which is the μ-asymptotic center of u in C for any invariant mean μ on $M_b(\mathbb{R}^+)$.*

PROOF. Take $S = \mathbb{R}^+$ and $X = M_b(\mathbb{R}^+)$ in Theorem 4.5. Then as u is continuous, $\varepsilon(\cdot\,,\cdot) \in M_b(\mathbb{R}^+ \times \mathbb{R}^+)$ by Example 3.3, and $u(\mu) \in \operatorname{Fix}(\mathcal{T})$ by Corollary 3.11. So the assertion follows.

When $\{\mu_\alpha : \alpha \in A\}$ is a strongly regular net, the convergence is uniform. For the proof, we use the method of Hirano, Kido and Takahashi [16, Theorem 2].

Theorem 4.7. *Let u be an almost nonexpansive curve from S to H with $\varepsilon(\,\cdot\,,\,\cdot\,)$ such that $\|u(\,\cdot\,) - y\|^2$ and $\varepsilon(s,\,\cdot\,)$ are in X for all $y \in H$ and $s \in S$, and let $\{\mu_\alpha : \alpha \in A\}$ be a strongly regular net of continuous linear functionals on X. Then $u(r_s^*\mu_\alpha)$ converges weakly to $y_0 \in F(u) \cap \bigcap_{s\in S} \operatorname{clco}\{u(t) : t \geq s\}$ uniformly in $s \in S$, which is the unique point of the μ-asymptotic center of u in H, where μ is any invariant mean on X.*

For the proof, we show the following lemma, which is a partial extension of Hirano, Kido and Takahashi [16, Theorem 1]. See also Oka [26, Lemma 9].

Lemma 4.8. *Let $\{\lambda_\alpha : \alpha \in A\}$ be a net of means on X such that for any $s \in S$, $\lim_\alpha \|\lambda_\alpha - r_s^*\lambda_\alpha\| = 0$, and let u be an almost nonexpansive curve from S to H with $\varepsilon(\,\cdot\,,\,\cdot\,)$ such that $\|u(\,\cdot\,) - y\|^2$ and $\varepsilon(s,\,\cdot\,)$ are in X for all $y \in H$ and $s \in S$. Let λ be an invariant mean on X. Then $u(r_s^*\lambda_\alpha)$ converges weakly to $u(\lambda)$ uniformly in $s \in S$.*

Proof. We prove that for any net $\{s_\beta : \beta \in B\} \subset S$, $u(r_{s_\beta}^*\lambda_\alpha) \rightharpoonup u(\lambda)$. Assume this does not hold. Then there exists a subnet $\{\beta'\} \times \{\alpha'\}$ of $B \times A$ such that for any subnet $\{\beta''\} \times \{\alpha''\}$, $u(r_{s_{\beta''}}^*\lambda_{\alpha''}) \not\rightharpoonup u(\lambda)$. Since $\{r_{s_{\beta'}}^*\lambda_{\alpha'}\}_{(\beta',\alpha')}$ is bounded, there exists a subnet $\{\beta''\} \times \{\alpha''\}$ of $\{\beta'\} \times \{\alpha'\}$ such that

$$r_{s_{\beta''}}^*\lambda_{\alpha''} \xrightarrow{w^*} \mu.$$

Then μ is an invariant mean. Indeed, put $\{\beta''\} \times \{\alpha''\} = \{\beta'\} \times \{\alpha'\} = B \times A$ for simplicity. For any $s \in S$, $f \in X$ and $\varepsilon > 0$, there exists $(\beta,\alpha) \in B \times A$ such that

$$|r_{s_\beta}^*\lambda_\alpha(f) - \mu(f)| \leq \varepsilon, \qquad |r_{s_\beta}^*\lambda_\alpha(r_sf) - \mu(r_sf)| \leq \varepsilon,$$

and

$$\|\lambda_\alpha - r_s^*\lambda_\alpha\| \leq \varepsilon.$$

Then we obtain

$$\begin{aligned}|\mu(r_sf) - \mu(f)| &\leq |\mu(r_sf) - r_{s_\beta}^*\lambda_\alpha(r_sf)| + |r_s^*\lambda_\alpha(r_{s_\beta}f) - \lambda_\alpha(r_{s_\beta}f)| \\ &\quad + |r_{s_\beta}^*\lambda_\alpha(f) - \mu(f)| \\ &\leq \varepsilon + \|r_s^*\lambda_\alpha - \lambda_\alpha\|\|f\| + \varepsilon \leq (2 + \|f\|)\varepsilon.\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, for any $s \in S$ and $f \in X$, we have $\mu(r_sf) = \mu(f)$, and hence μ is an invariant mean on X.

Since $r_{s\beta''}^*\lambda_{\alpha''} \xrightarrow{w^*} \mu$, for any $z \in H$,

$$\langle u(r_{s_{\beta''}}\lambda_{\alpha''}), z\rangle = (r_{s_{\beta''}}\lambda_{\alpha''})_t\langle u(t), z\rangle \to \mu_t\langle u(t), z\rangle = \langle u(\mu), z\rangle,$$

and hence $u(r_{s_{\beta''}}\lambda_{\alpha''}) \rightharpoonup u(\mu) = u(\lambda)$ by Lemma 3.7. This is a contradiction. This completes the proof.

PROOF OF THEOREM 4.7. By Day [12] or Namioka [25], we know that there exists a net $\{\lambda_\beta : \beta \in B\}$ of finite means such that for any $s \in S$, we have $\lim_\beta \|\lambda_\beta - r_s^*\lambda_\beta\| = 0$. Then clearly $\lim_\beta \|\lambda_\beta - r_s^*\lambda_\beta\|_{X^*} = 0$, so that for any $y \in H$ and $\varepsilon > 0$, by Lemma 4.8 there exists $\beta \in B$ such that

$$\sup_{t\in S} |\langle u(r_t^*\lambda_\beta) - u(\mu), y\rangle| \le \varepsilon.$$

Put $\lambda_\beta = \sum_{i=1}^n a_i\delta(t_i)$, where $a_1, \dots, a_n \ge 0$ with $\sum_{i=1}^n a_i = 1$ and $\delta(t)(f) = f(t)$ for all $f \in X$. Then since $\{\mu_\alpha\}$ is strongly regular, there exists $\alpha_0 \in A$ such that for any $\alpha \ge \alpha_0$,

$$|1 - \mu_\alpha(1)| \le \varepsilon \quad \text{and} \quad \|\mu_\alpha - r_{t_i}^*\mu_\alpha\| \le \varepsilon \quad \text{for all } i \in \{1, \dots, n\}.$$

Therefore, for any $s \in S$ and $\alpha \ge \alpha_0$,

$$\begin{aligned}
&\langle u(r_s^*\mu_\alpha) - u(\mu), y\rangle \\
&\quad\le \Big|(\mu_\alpha)_t\langle u(t+s), y\rangle - (\mu_\alpha)_t\Big\langle \sum_i a_i u(t_i+t+s), y\Big\rangle\Big| \\
&\qquad + \Big|(\mu_\alpha)_t\Big\langle \sum_i a_i u(t_i+t+s), y\Big\rangle - \mu_\alpha\langle u(\mu), y\rangle\Big| + |\mu_\alpha\langle u(\mu), y\rangle - \langle u(\mu), y\rangle| \\
&\quad\le \sum_i a_i|(\mu_\alpha)_t\langle u(t+s) - u(t_i+t+s), y\rangle| \\
&\qquad + \|\mu_\alpha\| \sup_{t\in S}\Big|\Big\langle \sum_i a_i u(t_i+t+s) - u(\mu), y\Big\rangle\Big| + |\langle u(\mu), y\rangle(\mu_\alpha(1) - 1)| \\
&\quad\le \sum_i a_i|(\mu_\alpha - r_{t_i}^*\mu_\alpha)_t\langle u(t+s), y\rangle| \\
&\qquad + \|\mu_\alpha\| \sup_{t\in S} |\langle u(r_{t+s}^*\lambda_\beta) - u(\mu), y\rangle| + |\langle u(\mu), y\rangle|\varepsilon \\
&\quad\le \sum_i a_i\|\mu_\alpha - r_{t_i}^*\mu_\alpha\| K\|y\| + L\varepsilon + |\langle u(\mu), y\rangle|\varepsilon \le (K\|y\| + L + |\langle u(\mu), y\rangle|)\varepsilon,
\end{aligned}$$

where $K = \sup_{t\in S}\|u(t)\|$ and $L = \sup_{\alpha\in A}\|\mu_\alpha\|$. As $\varepsilon > 0$, $s \in S$ and $y \in H$ are arbitrary, this implies $u(r_s^*\mu_\alpha) \rightharpoonup u(\mu)$ uniformly in $s \in S$.

As direct consequences of Theorem 4.7, we have the following:

COROLLARY 4.9 (Rouhani [31]). *Let $\{x(n) : n \in \mathbb{Z}^+\}$ be a bounded almost nonexpansive sequence in H. Then $(1/n)\sum_{i=0}^{n-1} x(i+k)$ converges weakly to the* ($\overline{\lim}$-)*asymptotic center of $x(\cdot)$ in H as $n \to \infty$, uniformly in $k \in \mathbb{Z}^+$.*

Corollary 4.10 (Rouhani [31]). *Let $\{u(t) : t \in \mathbb{R}^+\}$ be a bounded continuous almost nonexpansive curve in H. Then $(1/s)\int_0^s u(t+h)\,dt$ converges weakly to the the ($\overline{\lim}$-)asymptotic center of $u(\,\cdot\,)$ in H as $s \to \infty$, uniformly in $h \in \mathbb{R}^+$.*

Corollary 4.11. *Let $\mathcal{T} = \{T(s) : s \in S\}$ be a nonexpansive semigroup on C and assume $\{T(s)x_0 : s \in S\}$ is bounded and $\bigcap_{s\in S} \operatorname{clco}\{T(t)x_0 : t \geq s\} \subset C$ for some $x_0 \in C$. Then $\operatorname{Fix}(\mathcal{T}) \neq \emptyset$, and for a strongly regular net $\{\mu_\alpha : \alpha \in A\}$ on X, for each $x \in C$, the net $\{\mathcal{T}(r_s^*\mu_\alpha)x : \alpha \in A\}$ converges weakly to a point $y_0 \in \operatorname{Fix}(\mathcal{T})$ uniformly in $s \in S$, where $y_0 = \mathcal{T}(\mu)x_0$ for any invariant mean μ on X.*

Proof. By Corollary 3.11, taking $u = T(\,\cdot\,)x$, we obtain $u(\mu) = \mathcal{T}(\mu)x \in \operatorname{Fix}(\mathcal{T})$.

From Example 4.4, we also get the following corollaries.

Corollary 4.12. *Let C be a closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. Then:*

- (i) (Baillon [1]) *Assume that $\{T^i x_0 : i \in \mathbb{Z}^+\}$ is bounded for some $x_0 \in C$. Then for each $x \in C$, $(1/n)\sum_{i=0}^{n-1} T^{i+k}x$ converges weakly to some point of $\operatorname{Fix}(T)$ as $n \to \infty$, uniformly in $k \in \mathbb{Z}^+$.*
- (ii) (Rodé [30]) *For each $x \in C$, $(1-r)\sum_{i=0}^{\infty} r^k T^{i+k}x$ converges weakly to some point of $\operatorname{Fix}(T)$ as $r \uparrow 1$, uniformly in $k \in \mathbb{Z}^+$.*
- (iii) (Brezis and Browder [7]) *Let $\{q_{n,m}\}_{n,m\in\mathbb{Z}^+}$ be a strongly regular matrix. Then for any $x \in C$, $\sum_{m=0}^{\infty} q_{n,m}T^{m+k}x$ converges weakly to some point of $\operatorname{Fix}(T)$ as $n \to \infty$, uniformly in $k \in \mathbb{Z}^+$.*

Corollary 4.13 (Hirano, Kido and Takahashi [16]). *Let C be a closed convex subset a Hilbert space, let T and S be nonexpansive mappings of C into itself with $TS = ST$ and assume $\{S^iT^jx_0 : i,j \in \mathbb{Z}^+\}$ is bounded for some $x_0 \in C$. Then for any $x \in C, (1/n^2)\sum_{i,j=0}^{n-1} S^{i+k}T^{j+h}x$ converges weakly to an element of $\operatorname{Fix}(T) \cap \operatorname{Fix}(S)$ as $n \to \infty$, uniformly in $k, h \in \mathbb{Z}^+$.*

Corollary 4.14. *Let C be a closed convex subset of a Hilbert space and let $\mathcal{T} = \{T(s) : s \in S\}$ be a nonexpansive semigroup on C. Then:*

- (i) (Baillon [2], Miyadera and Kobayashi [22]) *Let $u : \mathbb{R}^+ \to C$ be a bounded almost orbit of $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$. Then $(1/\lambda)\int_0^\lambda u(t+h)\,dt$ converges weakly to some point of $\operatorname{Fix}(\mathcal{T})$ as $\lambda \to \infty$, uniformly in $h \in \mathbb{R}^+$. In particular, let $A^{-1}(0) \neq \emptyset$, and let u be a solution of $(*)$ in Example 3.2. Then $(1/\lambda)\int_0^\lambda u(t+h)\,dt$ converges weakly to some point of $A^{-1}(0)$ as $\lambda \to \infty$, uniformly in $h \in \mathbb{R}^+$.*

(ii) (Hirano, Kido and Takahashi [16]) *Let u be a bounded almost orbit of $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$. Then $r\int_0^\infty e^{-rt}u(t+h)\,dt$ converges weakly to some point of* $\mathrm{Fix}(\mathcal{T})$ *as $r \downarrow 0$, uniformly in $h \in \mathbb{R}^+$.*

(iii) (Reich [29]) *Let $u : \mathbb{R}^+ \to C$ be a bounded almost orbit of $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$, and let $Q(\,\cdot\,,\,\cdot\,)$ be a strongly regular kernel. Then $\int_0^s Q(s,t)u(t+h)\,dt$ converges weakly to some point of* $\mathrm{Fix}(\mathcal{T})$ *as $s \to \infty$, uniformly in $h \in \mathbb{R}^+$.*

PROOF. We only prove (i). The proofs of (ii) and (iii) are similar. Take $S = \mathbb{R}^+$, $X = M_{\mathrm{b}}(\mathbb{R}^+)$ and $\mu_\lambda(f) = (1/\lambda)\int_0^\lambda u(t+h)\,dt$ for $f \in M_{\mathrm{b}}(\mathbb{R}^+)$. Then, as in the proof of Corollary 4.6, we get the first assertion. As $A^{-1}(0) = \mathrm{Fix}(\mathcal{T})$, $A^{-1}(0) \neq \emptyset$ means that $\lim_s \|u(s) - y\|$ exists for $y \in \mathrm{Fix}(\mathcal{T})$, and this implies u is bounded. So, the second assertion follows.

5. Weak asymptotic regularity

In this section, we give an extension of Bruck [9, 10] and Takahashi and Park [37]. See also Browder and Petryshyn [8], Opial [27], Lau [18] and Oka [26].

THEOREM 5.1. *Let u be a bounded almost nonexpansive curve from S to H. Then the following are equivalent:*

(i) $w\text{-}\lim_s u(s) = y$ *for some* $y \in H$;
(ii) $w\text{-}\lim_s(u(s+t) - u(s)) = 0$ *uniformly in* $t \in S$;
(iii) $w\text{-}\lim_s(u(s+t) - u(s)) = 0$ *for all* $t \in S$;
(iv) $F(u) = H$.

In this case, $y = u(\mu)$ for any invariant mean μ on $l^\infty(S)$, which is the μ-asymptotic center of u in H.

PROOF. (i)⇒(ii)⇒(iii) is evident. We prove (iii)⇒(i). Let λ be an invariant mean on $l^\infty(S)$ and let $\{\lambda_\alpha : \alpha \in A\}$ be a net of finite means on S such that for any $s \in S$, $\lim_\alpha \|\lambda_\alpha - r_s^*\lambda_\alpha\| = 0$; see Day [12] or Namioka [25]. Let $z \in H$ and $\varepsilon > 0$. Then by Lemma 4.8, there exists α such that for any $s \in S$,

$$|\langle u(r_s^*\lambda_\alpha) - u(\lambda), z\rangle| \leq \varepsilon.$$

Put $\lambda_\alpha = \sum_{i=1}^n a_i\delta(t_i)$, where $a_1, \dots, a_n \geq 0$ with $\sum_{i=1}^n a_i = 1$. Then by (iii), there exists s_0 such that for any $s \geq s_0$ and $i \in \{1, \dots, n\}$,

$$|\langle u(s) - u(s+t_i), z\rangle| \leq \varepsilon.$$

Therefore, for any $s \geq s_0$,

$$\begin{aligned} |\langle u(s) - u(\lambda), z\rangle| &\leq |\langle u(s) - u(r_s^*\lambda_\alpha), z\rangle| + |\langle u(r_s^*\lambda_\alpha) - u(\lambda), z\rangle| \\ &= \left|\left\langle u(s) - \sum_i a_i u(t_i + s), z \right\rangle\right| + \varepsilon \\ &\leq \sum_i a_i |\langle u(s) - u(t_i + s), z\rangle| + \varepsilon \leq 2\varepsilon. \end{aligned}$$

As $\varepsilon > 0$ and $z \in H$ are arbitrary, this implies $w\text{-}\lim_s u(s) = u(\lambda)$.

Next we prove (ii)⇒(iv). Let λ be an invariant mean on $l^\infty(S)$ and let $z \in H$. Then, for any $t, s \in S$ with $t \geq s$,

$$\begin{aligned} &| \|u(t) - z\|^2 - \|u(s) - z\|^2| \\ &\qquad \leq | \|u(t) - u(\lambda)\|^2 - \|u(s) - u(\lambda)\|^2| + 2|\langle u(t) - u(s), z - u(\lambda)\rangle|. \end{aligned}$$

So, using $u(\lambda) \in F(u)$ and (ii), it follows that $\{\|u(s) - z\|^2 : s \in S\}$ is a Cauchy net, and hence $z \in F(u)$. Therefore $F(u) = H$.

Finally, we prove (iv)⇒(i). Assume $u(s)$ does not converge weakly to $u(\mu)$. Then there exists a subnet $\{u(s_\alpha)\}$ of $\{u(s)\}$ such that no subnet $\{u(s_{\alpha_\beta})\}$ converges weakly to $u(\mu)$. As $\{u(s_\alpha)\}$ is bounded, there exists a subnet $\{u(s_{\alpha_\beta})\}$ which converges weakly to some point y_0. From Lemma 4.1, Theorem 4.3 and (iv), $y_0 = u(\mu)$. This is a contradiction.

As direct consequences of Theorem 5.1, we have the following results.

Corollary 5.2 (Oka [26]). *Let $u : S \to C$ be a bounded almost orbit of a nonexpansive semigroup $\mathcal{T} = \{T(s) : s \in S\}$, and assume that $\bigcap_{s\in S} \operatorname{clco}\{u(t) : t \geq s\} \subset C$. Then the following are equivalent:*

(i) $w\text{-}\lim_s u(s) = y$ *for some* $y \in H$;
(ii) $w\text{-}\lim_s(u(s + t) - u(s)) = 0$ *uniformly in* $t \in S$;
(iii) $w\text{-}\lim_s(u(s + t) - u(s)) = 0$ *for all* $t \in S$;
(iv) $W(u) \subset \operatorname{Fix}(\mathcal{T})$;
(v) $W(u) \subset F(u)$;
(vi) $F(u) = H$.

In this case, $y = u(\mu)$, for any invariant mean μ on $l^\infty(S)$.

Proof. As $\operatorname{Fix}(\mathcal{T}) \subset F(u)$, (iv)⇒(v) is evident, and from Lemma 4.2, (v)⇒(i) follows. By Theorem 5.1 and by Corollary 3.11, $y = u(\mu) \in \operatorname{Fix}(\mathcal{T})$. So, (i)⇒(iv) follows.

Corollary 5.3 (Pazy [28], Bruck [9, 10]). *Let T be a nonexpansive mapping from C into itself, and assume $\operatorname{Fix}(T) \neq \emptyset$ and $\bigcap_{s\in S} \operatorname{clco}\{T(t)x : t \geq s\} \subset C$. Then for any $x \in C$, the following are equivalent:*

(i) $w\text{-}\lim_n T^n x = y$ *for some* $y \in H$;
(ii) $w\text{-}\lim_n (T^{n+k}x - T^n x) = 0$ *uniformly in* $k \in \mathbb{Z}^+$;
(iii) $w\text{-}\lim_n (T^{n+1}x - T^n x) = 0$;
(iv) $\omega_w(x) \subset \text{Fix}(T)$;
(v) $\omega_w(x) \subset F(T)$;
(vi) $F(T) = H$;

here $\omega_w(x)$ *is the set of all weak limit points of subsequences of* $\{T^n x : n \in \mathbb{Z}^+\}$, *and* $F(T) = \{q \in H : \exists \lim_{n\to\infty} \|T^n x - q\|\}$. *In this case,* y *is an element of* $\text{Fix}(T)$.

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